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Vague BF-algebras

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Abstract. In this paper, by using the concept of vague sets and BF-algebra we introduce the notions of vague BF-algebra. After that we state and prove some theorems in vague BF-algebras, α-cut and vague-cut. The relationship between these notions and crisp subalgebras are studied.

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1 Introduction

It is known that mathematical logic is a discipline used in sciences and humanities with different point of view. Non-classical logic takes the advantage of the classical logic (two-valued logic) to handle information with various facts of uncertainty. The non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information.

Y. Imai and K. Iseki [7] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Recently, Andrzej Walendziak defined a BF-algebra [12].

The notion of vague set theory was introduced by W. L. Gau and D. J. Buehrer [3], as a generalizations of Zadeh's fuzzy set theory [13]. In [1], R. Biswas applied the notion to group theory and introduced vague groups.

Now, in this note we use the notion of vague set to establish the notions of vague BF-algebras; then we obtain some related results which have been mentioned in the abstract.

2 Preliminaries

In this section, we present now some preliminaries on the theory of vague sets (VS). In his pioneer work [13], Zadeh proposed the theory of fuzzy sets. Since then it has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems, etc. to list a few only.

Let $U = \{u_1, u_2, ..., u_n\}$ be the universe of discourse. The membership function for fuzzy sets can take any value from the closed interval $[0; 1]$. An fuzzy set A is defined as the set of ordered pairs $A = \{(u; \mu_A(u)) \mid u \in U\}$ where $\mu_A(u)$ is the grade of membership of element u in set A. The greater $\mu_A(u)$, the greater is the truth of the statement that 'the element u belongs to the set A'. But Gau and Buehrer [3] pointed out that this single value combines the 'evidence for u' and the 'evidence against u'. It does not indicate the 'evidence for u' and the 'evidence against u', and it does not also indicate how much there is of each. Consequently, there is a genuine necessity of a different kind of fuzzy sets which could be treated as a generalization of Zadeh's fuzzy sets [13].

Definition 1. A vague set A in the universe of discourse U is characterized by two membership functions given by:

1. A truth membership function $t_A : U \to [0,1]$,
2. A false membership function $f_A : U \to [0,1]$,

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the 'evidence for u', and $f_A(u)$ is a lower bound of the negation of u derived from the 'evidence against u' and $t_A(u) + f_A(u) \leq 1$. Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1-f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{(u, [t_A(u), f_A(u)]) \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the 'vague value' of u in A and is denoted by $V_A(u)$.

It is worth to mention here that interval-valued fuzzy sets (i-v fuzzy sets) [14] are not vague sets. In i-v fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the 'evidence for u' only, without considering 'evidence against u'. In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.

Definition 2. (see [1]). A vague set A of a set U is called

1) the zero vague set of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,
2) the unit vague set of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$,
3) the α-vague set of U if $t_A(u) = \alpha$ and $f_A(u) = 1 - \alpha$ for all $u \in U$, where $\alpha \in (0,1)$.

Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. Now we define refined minimum (briefly, $rmin$) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}],$$

$$D_1 \leq D_2 \Longleftrightarrow a_1 \leq a_2 \quad \wedge \quad b_1 \leq b_2.$$

Similarly we can define $\geq$, $=$ and $rmax$. Then the concept of $rmin$ and $rmax$ could be extended to define $rinf$ and $rsup$ of infinite number of elements of $D[0,1]$.

It is that $L = \{D[0,1], rinf, rsup, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

For $\alpha, \beta \in [0,1]$ we now define (α,β)-cut and α-cut of a vague set.

Definition 3. (see [1]). Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α,β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha,\beta]\},$$

where $\alpha \leq \beta$.

Clearly $A(0,0) = X$. The (α,β)-cuts are also called vague-cuts of the vague set A.

Definition 4. (see [1]). The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\beta,\alpha)} = A_\alpha$. Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

Definition 5. Let f be a mapping from the set X to the set Y and let B be a vague set of Y. The inverse image of B, denoted by $f^{-1}(B)$, is a vague set of X which is defined by $V_{f^{-1}(B)}(x) = V_B(f(x))$ for all $x \in X$.

Conversely, let A be a vague set of X. Then the image of A, denoted by $f(A)$, is a vague set of Y such that:

$$V_{f(A)}(y) = \begin{cases} rsup_{z \in f^{-1}(y)} V_A(z) & \text{if} \quad f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ [0,0] & \text{otherwise.} \end{cases}$$

Definition 6. A vague set A of BF-algebra X is said to have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$V_A(x_0) = rsup_{t \in T} V_A(t).$$

Definition 7. (see [12]). A BF-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $0 * (x * y) = (y * x)$,

for all $x, y \in X$.

Example 1. (see [12]). (a) Let $\mathbf{R}$ be the set of real numbers and let $A = (\mathbf{R}; *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x & \text{if} \quad y = 0, \\ y & \text{if} \quad x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then A is a BF-algebra.

(b) Let $A = [0; \infty)$. Define the binary operation $*$ on A as follows: $x*y = |x-y|$, for all $x, y \in A$. Then $(A; *, 0)$ is a BF-algebra.

Proposition 1. *(see [12]). Let X be a BF-algebra. Then for any x and y in X, the following hold:*

(a) $0 * (0 * x) = x$ *for all* $x \in A$;

(b) if $0 * x = 0 * y$, *then* $x = y$ *for any* $x, y \in A$;

(c) if $x * y = 0$, *then* $y * x = 0$ *for any* $x, y \in A$.

Definition 8. (see [13]). A non-empty subset S of a BF-algebra X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \longrightarrow Y$ of BF-algebras is called a BF-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 9. (see [2]). Let μ be a fuzzy set in a BF-algebra X. Then μ is called a fuzzy BF-subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3 Vague BF-algebras

From now on $(X, *, 0)$ is a BF-algebra, unless otherwise is stated.

Definition 10. A vague set A of X is called a vague BF-algebra of X if it satisfies the following condition:

$$V_A(x * y) \geq rmin\{V_A(x), V_A(y)\}$$

for all $x, y \in X$, that is

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\},$$

$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}.$$

Example 2. Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Then $(X,*,0)$ is a BF-algebra, but is not a $BCH/BCI/BCK$-algebra.

Define

$$t_A(x) = \begin{cases} 0.7 & \text{if} \quad x = 0, \\ 0.3 & \text{if} \quad x \neq 0 \end{cases}$$

and

$$f_A(x) = \begin{cases} 0.2 & \text{if} \quad x = 0, \\ 0.4 & \text{if} \quad x \neq 0. \end{cases}$$

It is routine to verify that $A = \{(x, [t_A(x), f_A(x)]) \mid x \in X\}$ is a vague BF-algebra of X.

Lemma 1. *If A is a vague BF-algebra of X, then $V_A(0) \geq V_A(x)$, for all $x \in X$.*

Proof. For all $x \in X$, we have $x * x = 0$, hence

$$V_A(0) = V_A(x * x) \geq rmin\{V_A(x), V_A(x)\} = V_A(x).$$ □

Proposition 2. *Let A be a vague BF-algebra of X and let $n \in \mathcal{N}$. Then:*

(i) $V_A(\prod^n x * x) \geq V_A(x)$, *for any odd number* n,

(ii) $V_A(\prod^n x * x) = V_A(x)$, *for any even number* n,

where $\prod^n x * x = \overbrace{x * x * \ldots * x}^{n-times}$.

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. We prove by induction, definition and above lemma imply that $V_A(x*x) = V_A(0) \geq V_A(x)$. Now suppose that $V_A(\prod^{2k-1} x * x) \geq V_A(x)$. Then by assumption

$$\begin{aligned} V_A(\prod^{2(k+1)-1} x * x) &= V_A(\prod^{2k+1} x * x) = \\ &= V_A(\prod^{2k-1} x * (x * (x * x))) = \\ &= V_A(\prod^{2k-1} x * x) \geq V_A(x). \end{aligned}$$

Which proves (i). Similarly we can prove (ii). □

Theorem 1. *Let A be a vague BF-algebra of X. If there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n \to \infty} V_A(x_n) = [1, 1],$$

then $V_A(0) = [1, 1]$.

Proof. By Lemma 1, we have $V_A(0) \geq V_A(x)$, for all $x \in X$, thus $V_A(0) \geq V_A(x_n)$, for every positive integer n. Since $t_A(0) \leq 1$ and $1 - f_A(0) \leq 1$, then we have $V_A(0) = [t_A(0), 1 - f_A(0)] \leq [1, 1]$. Consider

$$V_A(0) \geq \lim_{n\to\infty} V_A(x_n) = [1, 1].$$

Hence $V_A(0) = [1, 1]$. □

μ is called an antifuzzy BF-subalgebra of X if $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

In the next proposition we state the relationship between vague BF-algebra and fuzzy BF-algebras.

Proposition 3. *A vague set $A = \{(u, [t_A(u), f_A(u)]) \mid u \in X\}$ of X is a vague BF-algebra of X if and only if t_A be a fuzzy BF-subalgebra of X and f_A be an antifuzzy BF-subalgebra of X.*

Proof. The proof is straightforward. □

Theorem 2. *The family of vague BF-algebras forms a complete distributive lattice under the ordering of vague set.*

Proof. Let $\{V_i \mid i \in I\}$ be a family of vague BF-algebra of X. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigcap V_i = [\bigwedge t_i, \bigvee f_i]$ is a vague BF-algebra. Let $x, y \in X$. Then

$$\begin{aligned}
(\bigwedge t_i)(x * y) &= \inf\{t_i(x * y) \mid i \in I\} \geq \\
&\geq \inf\{\min\{t_i(x), t_i(y)\} \mid i \in I\} = \\
&= \min(\inf\{t_i(x) \mid i \in I\}, \inf\{t_i(y) \mid i \in I\}) = \\
&= \min(\bigwedge t_i(x), \bigwedge t_i(y)),
\end{aligned}$$

also we have

$$\begin{aligned}
(\bigvee f_i)(x * y) &= \sup\{f_i(x * y) \mid i \in I\} \leq \\
&\leq \sup\{\max\{f_i(x), f_i(y)\} \mid i \in I\} = \\
&= \max(\sup\{f_i(x) \mid i \in I\}, \sup\{f_i(y) \mid i \in I\}) = \\
&= \max(\bigvee f_i(x), \bigvee f_i(y)).
\end{aligned}$$

Hence $\bigcap V_i = [\bigwedge t_i, \bigvee f_i]$ is a vague BF-algebra. which proves the theorem. □

Proposition 4. *Zero vague set, unit vague set and α-vague set of X are trivial vague BF-algebras of X.*

Proof. Let A be a α-vague set of X. For $x, y \in X$ we have

$$t_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{t_A(x), t_A(y)\},$$

$$1 - f_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{1 - f_A(x), 1 - f_A(y)\}.$$

By above proposition it is clear that A is a vague BF-algebra of X. The proof of other cases is similar. □

Theorem 3. *Let A be a vague BF-algebra of X. Then for $\alpha \in [0, 1]$, the α-cut A_α is a crisp subalgebra of X.*

Proof. Let $x, y \in A_\alpha$. Then $t_A(x), t_A(y) \geq \alpha$, and so $t_A(x*y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$. Thus $x * y \in A_\alpha$. □

Theorem 4. *Let A be a vague BF-algebra of X. Then for all $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a (crisp) subalgebra of X.*

Proof. Let $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x), V_A(y) \geq [\alpha, \beta]$, and so $t_A(x), t_A(y) \geq \alpha$ and $1 - f_A(x), 1 - f_A(y) \geq \beta$. Then $t_A(x * y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$, and $1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta$. Thus $x * y \in A_{(\alpha,\beta)}$. □

The subalgebra $A_{(\alpha,\beta)}$ is called vague-cut subalgebra of X.

Proposition 5. *Let A be a vague BF-algebra of X. Two vague-cut subalgebras $A_{(\alpha,\beta)}$ and $A_{(\delta,\epsilon)}$ with $[\alpha, \beta] < [\delta, \epsilon]$ are equal if and only if there is no $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$.*

Proof. In contrary, let $A_{(\alpha,\beta)} = A_{(\delta,\epsilon)}$ where $[\alpha, \beta] < [\delta, \epsilon]$ and there exists $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$. Then $A_{(\delta,\epsilon)}$ is a proper subset of $A_{(\alpha,\beta)}$, which is a contradiction.

Conversely, suppose that there is no $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$. Since $[\alpha, \beta] < [\delta, \epsilon]$, then $A_{(\delta,\epsilon)} \subseteq A_{(\alpha,\beta)}$. If $x \in A_{(\alpha,\beta)}$, then $V_A(x) \geq [\alpha, \beta]$ by hypothesis we get that $V_A(x) \geq [\delta, \epsilon]$. Therefore $x \in A_{(\delta,\epsilon)}$, then $A_{(\alpha,\beta)} \subseteq A_{(\delta,\epsilon)}$. Hence $A_{(\delta,\epsilon)} = A_{(\alpha,\beta)}$. □

Theorem 5. *Let $\mid X \mid < \infty$ and A be a vague BF-algebra of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) \mid x \in X\}.$$

Then $A_{(\alpha,\beta)}$ are the only vague-cut subalgebras of X, where $(\alpha, \beta) \in V(A)$.

Proof. Let $[a_1, a_2] \notin V(A)$, where $[a_1, a_2] \in D[0, 1]$. If $[\alpha, \beta] < [a_1, a_2] < [\delta, \epsilon]$, where $[\alpha, \beta], [\delta, \epsilon] \in V(A)$, then $A_{(\alpha,\beta)} = A_{(a_1,a_2)} = A_{(\delta,\epsilon)}$. If $[a_1, a_2] < [a_1, b]$ where

$$[a_1, b] = rmin\{(x, y) \mid (x, y) \in V(A)\},$$

then $A_{(a_1,a_2)} = X = A_{(a_1,b)}$. Hence for any $[a_1, a_2] \in D[0, 1]$, the vague-cut subalgebra $A_{(a_1,b)}$ is one of the $A_{(\alpha,\beta)}$ for $(\alpha, \beta) \in V(A)$. □

Theorem 6. *Any subalgebra S of X is a vague-cut subalgebra of some vague BF-algebra of X.*

Proof. Define

$$t_A(x) = \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_A(x) = \begin{cases} 1-\alpha & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that

$$V_A(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in S, \\ [0,0] & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$. It is clear that $S = A_{(\alpha,\alpha)}$. Let $x, y \in X$. We consider the following cases:

1) If $x, y \in S$, then $x * y \in S$ therefore

$$V_A(x * y) = [\alpha, \alpha] = rmin\{V_A(x), V_A(y)\}.$$

2) If $x, y \notin S$, then $V_A(x) = [0,0] = V_A(y)$ and so

$$V_A(x * y) \geq [0,0] = rmin\{V_A(x), V_A(y)\}.$$

3) If $x \in S$ and $y \notin S$, then $V_A(x) = [\alpha, \alpha]$ and $V_A(y) = [0,0]$. Thus

$$V_A(x * y) \geq [0,0] = rmin\{[\alpha, \alpha], [0,0]\} = rmin\{V_A(x), V_A(y)\}.$$

Therefore A is a vague BF-algebra of X. □

Theorem 7. *Let S be a subset of X and A be a vague set of X which is given in the proof of above theorem. If A is a vague BF-algebra of X, then S is a (crisp) subalgebra of X.*

Proof. Let A be a vague BF-algebra of X and $x, y \in S$. Then $V_A(x) = [\alpha, \alpha] = V_A(y)$, thus

$$V_A(x * y) \geq rmin\{V_A(x), V_A(y)\} = rmin\{[\alpha, \alpha], [\alpha, \alpha]\} = [\alpha, \alpha].$$

which implies that $x * y \in S$. □

Theorem 8. *Let A be a vague BF-algebra of X. Then the set*

$$X_{V_A} := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a (crisp) subalgebra of X.

Proof. Let $a, b \in X_{V_A}$. Then $V_A(a) = V_A(b) = V_A(0)$, and so

$$V(a * b) \geq rmin\{V_A(a), V_A(b)\} = V_A(0).$$

Then X_{V_A} is a subalgebra of X. □

Theorem 9. *Let N be the vague set of X which is defined by:*

$$V_N(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in N, \\ [\beta, \beta] & \text{otherwise,} \end{cases}$$

for $\alpha, \beta \in [0, 1]$ with $\alpha \geq \beta$. Then N is a vague BF-algebra of X if and only if N is a (crisp) subalgebra of X. Moreover, in this case $X_{V_N} = N$.

Proof. Let N be a vague BF-algebra of X. Let $x, y \in X$ be such that $x, y \in N$. Then

$$V_N(x * y) \geq rmin\{V_N(x), V_N(y)\} = rmin\{[\alpha, \alpha], [\alpha, \alpha]\} = [\alpha, \alpha]$$

and so $x * y \in N$.

Conversely, suppose that N is a (crisp) subalgebra of X, let $x, y \in X$.

(i) If $x, y \in N$ then $x * y \in N$, thus

$$V_N(x * y) = [\alpha, \alpha] = rmin\{V_N(x), V_N(y)\}.$$

(ii) If $x \notin N$ or $y \notin N$, then

$$V_N(x * y) \geq [\beta, \beta] = rmin\{V_N(x), V_N(y)\}.$$

This shows that N is a vague BF-algebra of X.

Moreover, we have

$$X_{V_N} := \{x \in X \mid V_N(x) = V_N(0)\} = \{x \in X \mid V_N(x) = [\alpha, \alpha]\} = N. \qquad \square$$

Proposition 6. *Let X and Y be BF-algebras and f be a BF-homomorphism from X into Y and G be a vague BF-algebra of Y. Then the inverse image $f^{-1}(G)$ of G is a vague BF-algebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} V_{f^{-1}(G)}(x * y) &= V_G(f(x * y)) = \\ &= V_G(f(x) * f(y)) \geq \\ &\geq rmin\{V_G(f(x)), V_G(f(y))\} = \\ &= rmin\{V_{f^{-1}(G)}(x), V_{f^{-1}(G)}(y)\}. \qquad \square \end{aligned}$$

Proposition 7. *Let X and Y be BF-algebras and f be a BF-homomorphism from X onto Y and D be a vague BF-algebra of X with the* sup *property. Then the image $f(D)$ of D is a vague BF-algebra of Y.*

Proof. Let $a, b \in Y$, let $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ such that

$$V_D(x_0) = rsup_{t \in f^{-1}(a)} V_D(t), \qquad V_D(y_0) = rsup_{t \in f^{-1}(b)} V_D(t).$$

Then by the definition of $V_{f(D)}$, we have

$$\begin{aligned} V_{f(D)}(x * y) &= rsup_{t \in f^{-1}(a * b)} V_D(t) \geq \\ &\geq V_D(x_0 * y_0) \geq \\ &\geq rmin\{V_D(x_0), V_D(y_0) = \\ &= rmin\{rsup_{t \in f^{-1}(a)} V_D(t), rsup_{t \in f^{-1}(b)} V_D(t)\} = \\ &= rmin\{V_{f(D)}(a), V_{f(D)}(b)\}. \qquad \square \end{aligned}$$

4 Artinian and Noetherian BF-algebras

Definition 11. A BF-algebra X is said to be *Artinian* if it satisfies the descending chain condition on subalgebras of X (simply written as DCC), that is, for every chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of subalgebras of X, there is a natural number i such that $I_i = I_{i+1} = \cdots$.

Theorem 10. *Let X be a BF-algebra. Then each vague BF-algebra of X has finite values if and only if X is Artinian.*

Proof. Suppose that each vague BF-algebra of X has finite values. If X is not Artinian, then there is a strictly descending chain

$$G = I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

of subalgebras of X, where $I_i \supset I_j$ expresses $I_i \supseteq I_j$ but $I_i \neq I_j$. We now construct the vague set $B = [t_A, f_A]$ of X by

$$t_A(x) := \begin{cases} \dfrac{n}{n+1} & \text{if} \quad x \in I_n \setminus I_{n+1},\ n = 1, 2, \cdots, \\ 1 & \text{if} \quad x \in \bigcap_{n=1}^{\infty} I_n, \end{cases}$$
$$f_A(x) := 1 - t_A(x).$$

We first prove that B is a vague BF-algebra of X. For this purpose, we need to verify that t_A is a fuzzy subalgebra of X. We assume that $x, y \in X$. Now, we consider the following cases:

Case 1: $x, y \in I_n \setminus I_{n+1}$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$t_A(x * y) \geq \frac{n}{n+1} = \min\{t_A(x),\, t_A(y)\}.$$

Case 2: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n < m)$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$t_A(x * y) \geq \frac{n}{n+1} = \min\{t_A(x),\, t_A(y)\}.$$

Case 3: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n > m)$. In this case, $x, y \in I_m$, and $x * y \in I_m$. Thus

$$t_A(x * y) \geq \frac{m}{m+1} = \min\{t_A(x),\, t_A(y)\}.$$

Therefore t_A is a fuzzy subalgebra of X. This shows that B is a vague BF-algebra of X, but the values of B are infinite, which is a contradiction. Thus X is Artinian.

Conversely, suppose that X is Artinian. If there is a vague BF-algebra $B = [t_A, f_A]$ of X with $|Im(B)| = +\infty$, then $|Im(t_A)| = +\infty$ or $|Im(f_A)| = +\infty$. Without loss of generality, we may assume that $Im(t_A) = +\infty$. Select $s_i \in Im(t_A)$ $(i = 1,\, 2 \cdots)$ and $s_1 < s_2 < \cdots$. Then $U(t_A;\, s_i)(i = 1,\, 2,\, \cdots)$ are subalgebras of X and $U(t_A;\, s_1) \supseteq U(t_A;\, s_2) \supseteq \cdots$ with $U(t_A;\, s_i) \neq U(t_A;\, s_{i+1})\, (i = 1,\, 2,\, \cdots)$, a contradiction. Similar for $\mathrm{Im}(f_A)$. The proof is completed. $\square$

Definition 12. A BF-algebra X is said to be *Noetherian* if every subalgebra of X is finitely generated. X is said to satisfy *the ascending chain condition (briefly, ACC)* if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of subalgebras of X there is a natural number n such that $I_i = I_n$, for all $i \geq n$.

Theorem 11. *X is Noetherian if and only if for any vague BF-algebra A, the set $Im(B)$ is a well ordered subset, that is, $(Im(t_A), \leq)$ and $(Im(f_A), \geq)$ are well ordered subsets of $[0, 1]$, respectively.*

Proof. ($\Rightarrow$) Suppose that X is Noetherian. For any chain $t_1 > t_2 > \cdots$ of $Im(t_A)$, let $t_0 = \inf\{t_i | \, i = 1, 2, \cdots\}$. Then $I := \{x \in X | \, t_A(x) > t_0\}$ is a subalgebra of X, and so I is finitely generated. Let $I = (a_1, \cdots, a_k]$. Then $t_A(a_1) \wedge \cdots \wedge t_A(a_k)$ is the least element of the chain $t_1 > t_2 > \cdots$. Thus $(Im(t_A), \leq)$ is a well ordered subset of $[0, 1]$. By using the same argument as above, we can easily show that $(Im(f_A), \geq)$ is a well ordered subset of $[0, 1]$. Therefore, $Im(B)$ is a well ordered subset.

($\Leftarrow$) Let $Im(B)$ be a well ordered subset. If X is not Noetherian, then there is a strictly ascending sequence of subalgebras of X such that $I_1 \subset I_2 \subset \cdots$. We construct the bipolar fuzzy set $B = [t_A, f_A]$ of X by

$$\begin{aligned} t_A(x) &:= \begin{cases} \dfrac{1}{n} & \text{if} \quad x \in I_n - I_{n-1}, \; n = 1, 2, \cdots, \\ 0 & \text{if} \quad x \notin \bigcup\limits_{n=1}^{\infty} I_n, \end{cases} \\ f_A(x) &:= \; 1 - t_A(x) \end{aligned}$$

where $I_0 = \emptyset$. By using similar method as the necessity part of Theorem 18, we can prove that B is a vague BF-algebra of X. Because $Im(B)$ is not well ordered, which is a contradiction. This completes the proof. $\square$

5 Conclusions

In the present paper, we have introduced the concept of vague BF-algebras and investigated some of their useful properties.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis [11].

In future work the vague ideals and quotient of BF-algebras by using these vague ideals will be presented.

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