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On spaces related to the Navier-Stokes equations

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Abstract. Some examples of multidimensional Riemann metrics related to the Navier-Stokes and the Euler equations are constructed. Their properties are discussed.

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1 6*D*-metrics and the NS-equations

With the Navier-Stokes equations

$$\frac{\partial}{\partial t}\vec{V} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = \mu\Delta\vec{V} + \vec{\nabla}f, \quad \vec{\nabla}\cdot\vec{V} = 0, \tag{1}$$

where $\vec{V} = (U(\vec{x},t), V(\vec{x},t), W(\vec{x},t))$ is the fluid velocity, $P(\vec{x},t)$ is the pressure and μ is the viscosity, $\vec{x} = (x,y,z)$ presented as the conditions of compatibility

$$H_y(\vec{x},t) - E_x(\vec{x},t) = 0, \quad H_z(\vec{x},t) - B_x(\vec{x},t) = 0, \; E_z(\vec{x},t) - B_y(\vec{x},t) = 0, \tag{2}$$

where the functions $H(\vec{x},t),\ E(\vec{x},t),\ B(\vec{x})$ have the form

$$H(\vec{x},t) = f_x(\vec{x},t),\ E(\vec{x},t) = f_y(\vec{x},t),\ B(\vec{x},t) = f_z(\vec{x},t),$$

we can associate 6D-metrics

$$ds^2 = -2\,B(\vec{x},t)dt\,dv + 2\,E(\vec{x},t)dt\,dw + 2\,H(\vec{x},t)dv\,dw -$$

$$-2\left(\int \frac{\partial}{\partial y}H(\vec{x},t)dz\right)dw^2 + dt\,dx + dv\,dy + dw\,dz \tag{3}$$

having fifteen components.

Nine of them are equal to zero if the functions $H(\vec{x},t)$, $E(\vec{x},t)$, $B(\vec{x},t)$ satisfy the conditions (2). The remaining six components $R_{vv},\ R_{vw},\ R_{ww},\ R_{tt},\ R_{tv},\ R_{tw}$ are expressed in terms of the functions $H(\vec{x},t)$, $E(\vec{x},t)$, $B(\vec{x},t)$ and their derivatives.

Properties of the 6-dimensional space with the metrics (3) can be used for the study of properties of the N-S-equations.

Example. The metrics (3) is Ricci-flat $R_{ik}=0$ if the velocity components of fluid have the form

$$U(\vec{x},t)=-1/2\,x\frac{\partial}{\partial z}F(z,t),\;\; V(\vec{x},t)=-1/2\,\left(\frac{\partial}{\partial z}F(z,t)\right)y,\;\; W(\vec{x},t)=F(z,t),$$

and the function $F(z,t)$ satisfies the equation (see [1])

$$-F_{zzt}(z,t)-F(z,t)F_{zzz}(z,t)+\mu\,F_{zzzz}(z,t)=0. \tag{4}$$

In this case the functions $H(\vec{x},t), E(\vec{x},t)$ and $B(\vec{x},t)$ have the form

$$B(\vec{x},t)=F_t(z,t)+F(z,t)F_z(z,t)-\mu\,F_{zz}(z,t),$$

$$H(\vec{x},t)=-1/2\,xF_{zt}(z,t)+1/4\,xF_z(z,t)^2-1/2\,xF(z,t)F_{zz}(z,t)+1/2\,\mu\,xF_{zzz}(z,t),$$

$$E(\vec{x},t)=-1/2\,yF_{zt}(z,t)+1/4\,yF_z(z,t)^2-1/2\,yF(z,t)F_{zz}(z,t)+1/2\,\mu\,yF_{zzz}(z,t).$$

2 12*D*-metrics and the Euler equations

The Euler system of equations, which is the limit case $\mu=0$ of the NS-equations, is considered as a part of conditions $R_{ik}=0$ on the Ricci tensor of the Riemann metrics of the $12D$ space in local coordinates $(x,y,z,t,u,v,w,p,\xi,\eta,\chi,\rho)$:

$$ds^2=dv\,d\rho+c(\vec{x},t)d\rho^2+du\,d\chi+a(\vec{x},t)d\chi^2+$$
$$+\Big((V(\vec{x},t))^2-f(\vec{x},t)\Big)\,dp^2+dz\,d\xi+dy\,dp+dy\,d\chi+dt\,d\eta+2\,\beta(\vec{x})d\eta\,d\rho+$$
$$+2\,V(\vec{x},t)dp\,d\rho+2\,W(\vec{x},t)V(\vec{x},t)dp\,d\xi+2\,U(\vec{x},t)W(\vec{x},t)dw\,d\xi+$$
$$+2\,U(\vec{x},t)V(\vec{x},t)dw\,dp+2\,U(\vec{x},t)dw\,d\eta+2\,V(\vec{x},t)dp\,d\chi+2\,U(\vec{x},t)dw\,d\rho+$$
$$+2\,\epsilon(\vec{x})d\eta\,d\chi+2\,W(\vec{x},t)d\xi\,d\chi+\Big((U(\vec{x},t))^2-f(\vec{x},t)\Big)\,dw^2+2\,b(\vec{x},t)d\chi\,d\rho+$$
$$+2\,U(\vec{x},t)dw\,d\chi+2\,V(\vec{x},t)dp\,d\eta+2\,W(\vec{x},t)d\xi\,d\rho+2\,W(\vec{x},t)d\xi\,d\eta+\alpha(\vec{x})d\eta^2+$$
$$+dx\,dw+\Big((W(\vec{x},t))^2-f(\vec{x},t)\Big)\,d\xi^2+dx\,d\rho,$$

where $f(\vec{x},t)$ is the pressure of fluid.

Such a metric has 45 components of the Ricci tensor. 21 components from 45 are equal to zero on solutions of the Euler equations.

Let us consider some examples of solutions of the Euler equations defined by the condition $T=R_{ik}\cdot R^{ik}=0$ on scalar invariant of the metric.

Proposition. On the solutions of the Euler equations

1. (Shanko [2])

$$U(x,y,z,t)=\frac{y-x+tx}{t^2},\quad V(x,y,z,t)=\frac{ty+y-2\,x}{t^2},$$

$$W(x,y,z,t) = -2\,\frac{z}{t}, \quad f(x,y,z,t) = -P_0 - 1/2\,\frac{x^2+y^2}{t^4} + 3\,\frac{z^2}{t^2};$$

2.

$$U(x,y,z,t) = -\cos(x)\cos(z), \quad V(x,y,z,t) = -2\,\cos(x)\sin(z),$$

$$W(x,y,z,t) = -\sin(x)\sin(z), \quad f(x,y,z,t) = 1/4\,\cos(2\,x) - 1/4\,\cos(2\,z) + P_0(t);$$

3. (Aristov, Polyanin [1])

$$U(x,y,z,t) = \frac{\sin(kx)}{A\,(\cos(kx)\cos(ky)-1)}, \; V(x,y,z,t) = \frac{\sin(ky)}{A\,(\cos(kx)\cos(ky)-1)},$$

$$f(x,y,z,t) = -\frac{1}{A^2\,(\cos(kx)\cos(ky)-1)}, \quad W(x,y,z,t) = 0$$

the invariant $T = 0$.

3 14D-metrics and the NS-equations

Proposition. With the full system of the NS-equations we can associate the 14D space in local coordinates

$$\mathtt{x}^a = [x,y,z,t,u,v,w,p,\xi,\eta,\chi,\rho,q,\delta]$$

equipped with the Riemann metrics of the form

$$\begin{gathered}
ds^2 = -\int V_t(\vec{x},t)dy\,d\xi^2 - \int W_t(\vec{x},t)dz\,d\eta^2 + dv\,dq + s(\vec{x},t)dq^2 + \\
+2\,\kappa(\vec{x},t)d\delta^2 + c(\vec{x},t)d\rho^2 + 2\,\Big((U(\vec{x},t))^2 - \mu\,U_x(\vec{x},t) - f(\vec{x},t)\Big)\,dp\,d\rho + \\
+2\,U(\vec{x},t)dp\,d\chi - \int U_t(\vec{x},t)dx\,dp^2 + 2\,\epsilon(\vec{x},t)dq\,d\delta + 2\,h(\vec{x},t)d\rho\,d\delta + \\
+2\,\tau(\vec{x},t)d\rho\,dq + 2\,W(\vec{x},t)d\chi\,d\delta + 2\,V(\vec{x},t)d\chi\,dq + 2\,U(\vec{x},t)d\chi\,d\rho + \\
+2\,\big(W(\vec{x},t)^2 - \mu\,W_z(\vec{x},t) - f(\vec{x},t)\big)\,d\eta\,d\delta + 2\,\left(V(\vec{x},t)W(\vec{x},t) - \mu\,V_z(\vec{x},t)\right)\,d\eta\,dq + \\
+2\,\left(U(\vec{x},t)W(\vec{x},t) - \mu\,U_z(\vec{x},t)\right)\,d\eta\,d\rho + 2\,\left(V(\vec{x},t)W(\vec{x},t) - \mu\,W_y(\vec{x},t)\right)\,d\xi\,d\delta + \\
+2\,\big(V(\vec{x},t)^2 - \mu\,V_y(\vec{x},t) - f(\vec{x},t)\big)\,d\xi\,dq + 2\,\left(U(\vec{x},t)V(\vec{x},t) - \mu\,U_y(\vec{x},t)\right)\,d\xi\,d\rho + \\
+2\,\left(U(\vec{x},t)W(\vec{x},t) - \mu\,W_x(\vec{x},t)\right)\,dp\,d\delta + 2\,\left(U(\vec{x},t)V(\vec{x},t) - \mu\,V_x(\vec{x},t)\right)\,dp\,dq + \\
+2\,W(\vec{x},t)d\eta\,d\chi + dy\,d\xi + 2\,V(\vec{x},t)d\xi\,d\chi + dt\,d\chi + dw\,d\delta + \\
+dx\,dp + dz\,d\eta + du\,d\rho.
\end{gathered} \tag{5}$$

It has 56 components of the Ricci tensor. 28 from them are equal to zero on solutions of the NS-equations.

Proposition. Due to the condition (2) the simplest scalar invariant of the metric (5) $T = R_{ij} \cdot R^{ij} = 0$, but the invariant $S = R_{ijkl} \cdot R^{ijkl}$ of the metrics (5) is not equal to zero and has the form

$$S = 5\,U_{xt}(\vec{x},t)^2 - 4\,U_{xx}(\vec{x},t)U_{tt}(\vec{x},t) + 2\int V_{xxt}(\vec{x},t)dy \int U_{yyt}(\vec{x},t)dx -$$

$$-4\,V_{xx}(\vec{x},t)\int U_{tt}(\vec{x},t)dx - 4\int V_{xtt}(\vec{x},t)dy U_{yy}(\vec{x},t) + 8\,V_{xt}(\vec{x},t)U_{yt}(\vec{x},t) +$$

$$+2\int W_{xxt}(\vec{x},t)dz \int U_{zzt}(\vec{x},t)dx - 4\,W_{xx}(\vec{x},t)\int U_{ztt}(\vec{x},t)dx -$$

$$-4\int W_{xtt}(\vec{x},t)dz U_{zz}(\vec{x},t) + 8\,W_x(\vec{x},t)U_z(\vec{x},t) + 5\,V_{yt}(\vec{x},t)^2 - 4\,V_{yy}(\vec{x},t)V_{tt}(\vec{x},t) +$$

$$+2\int W_{yyt}(\vec{x},t)dz \int V_{zzt}(\vec{x},t)dy - 4\,W_{yy}(\vec{x},t)\int V_{ztt}(\vec{x},t)dy -$$

$$-4\int W_{ytt}(\vec{x},t)dz V_{zz}(\vec{x},t) + 8\,W_{yt}(\vec{x},t)V_{zt}(\vec{x},t) + 5\,W_{zt}(\vec{x},t)^2 - 4\,W_{zz}(\vec{x},t)W_{tt}(\vec{x},t).$$

On solution of the NS-equations

$$U(x,y,z,t) = \frac{y - x + tx - K(z,t)t\sin(t^{-1})}{t^2},$$

$$V(x,y,z,t) = \frac{y + ty - 2\,x - K(z,t)t\sin(t^{-1}) - \cos(t^{-1})K(z,t)t}{t^2},$$

$$W(x,y,z,t) = -2\,\frac{z}{t}, \quad f(x,y,z,t) = -P_0 - 1/2\,\frac{x^2+y^2}{t^4} + 3\,\frac{z^2}{t^2},$$

where the function $K(z,t)$ satisfies the equation

$$\frac{\partial}{\partial t}K(z,t) = 2\,\frac{z\frac{\partial}{\partial z}K(z,t)}{t} + \mu\,\frac{\partial^2}{\partial z^2}K(z,t) + J(t)$$

with arbitrary $J(t)$, the invariant S takes the form $S = 96\dfrac{5t^2-4}{t^6}$ and the space with the metrics (5) is not Ricci-flat, but its scalar curvature $R = 0$ and invariant $T = 0$.

References

[1] Aristov S. N., Polyanin A.D. *Exact solutions of unsteady three-dimensional Navier-Stokes equations*. Doklady Physics, 2009, **54**, No. 7, 316–321 (see also arXiv:0909.0446v1 [physics.flu-dyn]).

[2] Shanko Yu. *Some exact solutions of 3-dimensional non-stationary Euler equations*. International Conference "Differential equations, functions theory and applications", Novosibirsk, Abstracts, 2007, p. 377 (in Russian).

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