BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(63), 2010, Pages 67–83
ISSN 1024–7696

On Solvability of one Class of Hammerstein Nonlinear Integral Equations

A. Kh. Khachatryan, Kh. A. Khachatryan

Abstract. The article deals with one class of Hammerstein nonlinear integral equations with kernel depending on the sum and the difference of arguments. In the particular case of basic nonlinear equation the existence of one parameter family of solutions is proved. Using special solution of this family the solution of basic nonlinear equation is constructed and asymptotic behavior at infinity is investigated. At the end of the work some of examples are given.

Mathematics subject classification: 45Gxx, 45G05.
Keywords and phrases: Hammerstein equation, factorization, set of parameters, iteration, Caratheodory condition.

1 Introduction

We consider the following Hammerstein nonlinear integral equation:

$$F(x)=\mu(x)\int_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)]Q(t,F(t))dt, \quad x\geq 0 \tag{1}$$

with respect to unknown function $F(x)$, where $\varepsilon \geq 0$, $p>0$ are parameters. Here $\mu(x)$ is a defined on $[0,+\infty)$ measurable function, satisfying the following conditions:

- $\mu(x)\uparrow$ in x on $[0,+\infty)$, (2)
- $0<\varepsilon_0\leq \mu(x)\leq 1, \quad x\in[0,+\infty)$, (3)
- $(1-\mu(x))x^j\in L_1(0,+\infty), \quad j=0,1.$ (4)

The kernels K and $\overset{\circ}{K}$ are defined on the sets $(-\infty,+\infty)$ and $(0,+\infty)$ respectively, and have the following properties:

- $0<K(x)=\int\limits_{a>0}^{b} e^{-|x|s}d\sigma(s), \quad x\in(-\infty,+\infty),$ (5)

where

- $\sigma\uparrow[a,b), \quad 0<a<b<+\infty, \quad 2\int\limits_a^b \frac{d\sigma(s)}{s}=1,$ (6)

- $$\overset{\circ}{K}(\tau) \geq 0, \quad \tau \in (0, +\infty), \quad m_j \equiv \int\limits_0^\infty x^j \overset{\circ}{K}(x)dx < +\infty, \quad j = 0, 1, 2. \tag{7}$$

The function $Q(t, z)$ is a real and measurable function, which is defined on the set $(0, +\infty) \times (-\infty, +\infty)$ and satisfies conditions below:

- there exists a number $\delta > 0$ such that
$$z - w(t, z) \leq Q(t, z) \leq z, \quad (t, z) \in [0, +\infty) \times [\delta, +\infty) \tag{8}$$

where $w(t, z)$ is a real function on $(0, +\infty) \times (-\infty, +\infty)$ possessing the following properties:

- $$w(t, z) \geq 0, \quad (t, z) \in [0, +\infty) \times [\delta, +\infty) \equiv \Omega_\delta, \tag{9}$$
- $$w(t, z) \downarrow \text{ in } z \text{ on } [\delta, +\infty) \text{ for each } t > 0, \tag{10}$$
- $$w(t, z) \in \mathrm{Carat}(\Omega_\delta) \tag{11}$$

i.e. $w(t, z)$ satisfies Caratheodory condition on the set Ω_δ [1].

The last condition means that the function $w(t, z)$ for each fixed $z \in [\delta, +\infty)$ is measurable in $t > 0$, and for almost all $t > 0$ the function $w(t, z)$ is continuous by z in $[\delta, +\infty)$

- there exists a measurable function
$$w_0 \in L_1(0, +\infty) \cap C_0[0, +\infty), \quad m_1(w_0) = \int\limits_0^\infty x w_0(x)dx < +\infty,$$
$$0 \leq w_0(x) \downarrow \text{ in } x \text{ on } [\delta, +\infty), \text{ such that}$$
$$w(t, z) \leq w_0(t + z) \tag{12}$$

- $Q(t, z) \uparrow$ in z on $[\delta_0, +\infty)$ for each $t > 0$ and for some $\delta_0 \geq \delta$,
$$Q(t, z) \in \mathrm{Carat}\,(\Omega_\delta) \tag{13}$$

The equation (1) with conditions (2)–(13) is not only of pure mathematical interest, but it also has application in radiative transfer theory [2].

In the particular case when $Q(t, z) = z - w(t, z)$ and $\mu(x) \equiv 1$, the equation (1) was studied by Kh. A. Khachatryan [3].

In the present work the existence of solution of nonlinear equation (1) is proved, as well as the asymptotic property of solution is investigated. At the end of the work some of examples are given.

2 Corresponding linear equation

Step. I. First we consider the following linear homogeneous Wiener-Hopf equation

$$S^*(x) = \int\limits_0^\infty K(x - t)S^*(t)dt, \quad x > 0 \tag{14}$$

with respect to unknown function $S^*(x)$, where the kernel $K(x)$ is given by (5).

We rewrite the equation (14) in the operator form

$$(I-\mathcal{K})S^*=0 \tag{15}$$

where I is the unit operator, and $\mathcal{K}$ is the Winer-Hopf integral operator with the kernel $K(x)$. Let E be one of the following Banach spaces: $L_p(0,+\infty)$, $p\geq 1$, $M(0,+\infty)$, $C_M(0,+\infty)$, $C_0(0,+\infty)$. It is known that the operator $I-\mathcal{K}$ permits the following factorization [4]:

$$I-\mathcal{K}=(I-V_-)(I-H)(I-V_+) \tag{16}$$

where

$$(V_-f)(x)=\beta\int\limits_x^\infty e^{-\beta(t-x)}f(t)dt, \quad x\in(0,+\infty), \tag{17}$$

$$(V_+f)(x)=\beta\int\limits_0^x e^{-\beta(x-t)}f(t)dt, \quad x\in(0,+\infty), \tag{18}$$

$f\in E$, $\beta>0$ is a parameter and

$$(Hf)(x)=\int\limits_0^\infty h(x-t)f(t)dt, \quad x>0, \tag{19}$$

$$h(x)=\int\limits_a^b\left(1-\frac{\beta^2}{s^2}\right)e^{-|x|s}d\sigma(s), \quad x\in(-\infty,+\infty). \tag{20}$$

Using factorization (16) we rewrite the equation (15) in the following form

$$(I-V_-)(I-H)(I-V_+)S^*=0. \tag{21}$$

The solution of equation (21) is equivalent to the solution of the following coupled equations:

$$(I-V_-)S_0^*=0. \tag{22}$$

$$(I-H)S_1^*=S_0^*, \tag{23}$$

$$(I-V_+)S^*=S_1^*. \tag{24}$$

From (17) it follows that the function $S_0^*=c_0^*=const>0$ satisfies the equation (22). Substituting S_0^* in (23) we get the following integral equation

$$S_1^*(x)=c_0^*+\int\limits_0^\infty h(x-t)S_1^*(t)dt, \quad x>0. \tag{25}$$

From (20) it follows that for $\beta \in (0, a]$ the kernel $h(x) \geq 0$. Therefore from (19) we obtain

$$\|h\|_{L_1} = \int_{-\infty}^{+\infty} h(x)dx = \rho^* = 1 - 2\beta^2 \int_a^b \frac{d\sigma(s)}{s^3} < 1. \tag{26}$$

On the other hand in each space of E for the norm of Wiener-Hopf operator the following estimation takes place

$$\|H\|_E \leq \|h\|_{L_1}. \tag{27}$$

Taking into account (26) from (27) we conclude that the operator H in each space of E is contractive with coefficient ρ^*. Therefore the equation (25) in the space of bounded functions has a unique solution which satisfies the double inequalities:

$$c_0^* \leq S_1^* \leq c_0^*(1 - \rho^*)^{-1}. \tag{28}$$

Solving equations (24) we obtain

$$S^*(x) = S_1^*(x) + \beta \int_0^x S_1^*(t)dt. \tag{29}$$

Step. II. The following more general linear equation is considered

$$\Phi^*(x) = \mu(x) \int_0^\infty K(x-t)\Phi^*(t)dt, \quad x > 0. \tag{30}$$

Arabadjyan [5] proved that equation (30) by conditions (3), (4) has nonnegative and nontrivial solution with asymptotic $\Phi^*(x) = O(x), \quad x \to +\infty$. Moreover the solution is represented in the form of

$$0 \leq \Phi^*(x) = S^*(x) - \varphi^*(x), \quad x > 0 \tag{31}$$

where $\varphi^*(x) \geq 0$ is the solution of the equation

$$\varphi^*(x) = (1 - \mu(x))S^*(x) + \mu(x) \int_0^\infty K(x-t)\varphi^*(t)dt, \quad x > 0 \tag{32}$$

and has the following asymptotic behavior (see [6])

$$\int_0^x \varphi^*(\tau)d\tau = o(x), \quad x \to +\infty. \tag{33}$$

Consider the following iteration

$$\Phi^{(n+1)}(x) = \mu(x) \int_0^\infty K(x-t)\Phi^{(n)}(t)dt, \quad \Phi^{(0)}(x) \equiv S^*(x) > 0 \\ n = 0, 1, 2, \ldots, \quad x \in (0, +\infty). \tag{34}$$

Using (2)–(4), (31), by induction it is easy to check the truth of the following facts

$$\bullet \quad \Phi^{(n)}(x) \downarrow \text{ in } n, \tag{35}$$

$$\bullet \quad \Phi^{(n)}(x) \geq \Phi^{*}(x), \quad n = 0, 1, 2 \tag{36}$$

$$\bullet \quad \Phi^{(n)}(x) \uparrow \text{ in } x, \quad n = 0, 1, 2. \tag{37}$$

Therefore the sequence of functions $\{\Phi^{(n)}(x)\}_0^\infty$ has the limit

$$\lim_{n\to\infty} \Phi^{(n)}(x) = \Phi(x) \tag{38}$$

and the function $\Phi(x)$ satisfies the equation (30), moreover

$$\Phi^{*}(x) \leq \Phi(x) \leq S^{*}(x), \quad x > 0. \tag{39}$$

From (37) it follows that

$$\Phi(x) \uparrow \text{ in } x, \tag{40}$$

Now we show that

$$\alpha = \inf_{x>0} ess\, \Phi(x) > 0. \tag{41}$$

As $\Phi(x) \geq 0$ and $\Phi(x) \not\equiv 0$, then even if one point $x_0 \geq 0$ there exists such that $\Phi(x_0) > 0$. From (30) we have

$$\Phi(x) \geq \varepsilon_0 \int_{x_0}^{\infty} K(x-t)\Phi(t)dt \geq \varepsilon_0 \Phi(x_0) \int_{-\infty}^{-x_0} K(\tau)d\tau > 0.$$

Therefore the statement (41) is true.

Step. III. Now we consider the following linear integral equation with the kernel depending on the sum and difference of arguments:

$$S(x) = \int_0^\infty K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)]S(t)dt, \quad x > 0. \tag{42}$$

We rewrite the equation (42) in the operator form

$$(I - \mathcal{K} - \varepsilon \overset{\circ}{\mathcal{K}})S = 0$$

where $\overset{\circ}{\mathcal{K}}$ is the Hankell integral operator

$$(\overset{\circ}{\mathcal{K}} f)(x) = \int_0^\infty \overset{\circ}{K}(x+pt)f(t)dt, \quad f \in E \tag{43}$$

with kernel $\overset{\circ}{K}(x)$.

By Ω_0 we denote the class of Hankel integral operators: $\widehat{T}_0 \in \Omega_0$ if

$$(\widehat{T}_0 f)(x) = \int_0^\infty T_0(x+pt)f(t)dt, \quad f \in E, \quad p > 0, \qquad 0 \le T_0 \in L_1(0,+\infty). \tag{44}$$

Let $I + R_\pm$ be resolvent operators for Volterra-type operators $I - V_\pm$ (see formulae (17) and (18)).

It is easy to check that

$$(R_- f)(x) = \beta \int_x^\infty f(t)dt, \quad \beta > 0,\ x > 0, \tag{45}$$

$$(R_+ f)(x) = \beta \int_0^x f(t)dt, \quad \beta > 0,\ x > 0, \quad f \in L_1(0,+\infty). \tag{46}$$

Using (45) and (46) and taking into account (16) we have

$$I - \mathcal{K} - \varepsilon \mathring{\mathcal{K}} = (I - V_-)(I - H)(I - V_+) - \varepsilon \mathring{\mathcal{K}} =$$
$$= (I - V_-)[I - H - \varepsilon (I + R_-) \mathring{\mathcal{K}} (I + R_+)](I - V_+) =$$
$$= (I - V_-)(I - H - \varepsilon \widehat{T}_0)(I - V_+).$$

From (7), (45), (46) and Fubin's theorem it follows that the kernel of the operator

$$\widehat{T}_0 = (I + R_-) \mathring{\mathcal{K}} (I + R_+) \in \Omega_0$$

has the form of

$$T_0(x) = K_0(x) + \beta \int_x^\infty \mathring{K}(\tau)d\tau + \frac{\beta}{p} \int_x^\infty \mathring{K}(\tau)d\tau + \frac{\beta^2}{p} \int_x^\infty \int_y^\infty \mathring{K}(u)dudy. \tag{47}$$

Finally we come to the following factorization

$$I - \mathcal{K} - \varepsilon \mathring{\mathcal{K}} = (I - V_-)(I - \widehat{T})(I - V_+) \tag{48}$$

where $\widehat{T} = H + \varepsilon \widehat{T}_0$. Thus the Lemma holds

Lemma 1. *Let the condition*

$$0 \le \varepsilon < 4\beta^2 p^2 (\beta^2 m_2 + (2p+2)\beta m_1 + 2m_0 p)^{-1} \int_a^b \frac{1}{s^3} d\sigma(s)$$

be fulfilled. Then the equation (42) *has a positive solution with asymptotic* $S(x) = O(x)$ *as* $x \to +\infty$. *Moreover* $S(x) \ge S^*(x)$, *where* $S^*(x)$ *is given by (29).*

Proof. From (47) and (20) it follows that

$$T(x,\tau) \geq 0, \quad (x,\tau) \in (0,+\infty) \times (0,+\infty),$$

moreover

$$\begin{gathered}\int\limits_0^\infty T(x,\tau)d\tau \leq \int\limits_{-\infty}^{+\infty} h(x)dx + \frac{\varepsilon}{p}\int\limits_0^\infty T_0(\tau)d\tau = \\ = \rho^* + \frac{\varepsilon m_0}{p} + \frac{\varepsilon\beta m_1}{p} + \frac{\varepsilon\beta m_1}{p^2} + \frac{\varepsilon\beta m_2}{2p^2} < \rho^* + 2\beta^2 \int\limits_a^b \frac{1}{s^3} d\sigma(s) = 1,\end{gathered} \tag{49}$$

$$\int\limits_0^\infty T(x,\tau)dx \leq \int\limits_{-\infty}^{+\infty} h(x)dx + \varepsilon\int\limits_0^\infty T_0(\tau)d\tau = \rho^* + \varepsilon m_0 + \varepsilon\beta m_1 + \frac{\varepsilon\beta m_1}{p} + \frac{\varepsilon\beta^2 m_2}{2p} < +\infty. \tag{50}$$

The factorization (48) reduces the solution of equation (42) to the solution of the following coupled equations:

$$(I - V_-)S_0 = 0, \tag{51}$$

$$(I - \widehat{T})S_1 = S_0, \tag{52}$$

$$(I - V_+)S = S_1. \tag{53}$$

Note that an arbitrary constant satisfies the equation (51). As S_0 we take

$$S_0(x) = c_0^*(1-\rho^*)^{-1}. \tag{54}$$

Inserting (54) in (52) and using (50) we come to the conclusion that equation (52) has a unique, positive and bounded solution $S_1(x)$, and moreover

$$c_0^*(1-\rho^*)^{-1} \leq S_1(x) \leq \frac{c_0^*}{(1-\rho^*)(1-\widetilde{\rho})} \tag{55}$$

where

$$\widetilde{\rho} = \rho^* + \frac{\varepsilon}{2p^2}(\beta^2 m_2 + (2p+2)\beta m_1 + 2\rho m_0) < 1. \tag{56}$$

Solving equation (53) we obtain

$$S(x) = S_1(x) + \beta\int\limits_0^x S_1(t)dt. \tag{57}$$

From (55), (57) it follows that

$$S(x) = O(x), \quad x \to \infty.$$

To finalize the proof of Lemma it is necessary to show that

$$S(x) \geq S^*(x). \tag{58}$$

Really we have

$$S(x) \geq c_0^*(1-\rho^*)^{-1}(1+\beta x) \geq S_1^*(x) + \beta \int\limits_0^x S_1^*(t)dt = S^*(x). \qquad \square$$

Step. IV. Finally we consider the linear equation corresponding to nonlinear equation (1)

$$B(x) = \mu(x) \int\limits_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)]B(t)dt, \quad x \in (0, +\infty) \tag{59}$$

and the following iteration process.

$$B^{(n+1)}(x) = \mu(x) \int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)]B^{(n)}(t)dt, \quad B^{(0)}(x) = S(x), \quad n = 0, 1, 2. \tag{60}$$

By induction it is easy to check that

- $B^{(n)}(x) \downarrow$ in n (61)
- $B^{(n)}(x) \geq \Phi(x), \quad n = 0, 1, 2,$ (62)

where $\Phi(x)$ satisfies the equation (30) and possesses properties (39)–(41).

Therefore there exists

$$\lim_{n\to\infty} B^{(n)}(x) = B(x). \tag{63}$$

Note that $B(x)$ satisfies the equation (59) and the double inequalities hold

$$\Phi(x) \leq B(x) \leq S(x), \quad x > 0. \tag{64}$$

From (64) we have

$$\underset{x>0}{inf ess}\, B(x) \equiv \beta_0 \geq \alpha > 0. \tag{65}$$

The (65) inequality will be of essential use in future.

Thus the following lemma is true:

Lemma 2. *Let the conditions of lemma 1 be fulfilled. Then the equation* (59) *has the nontrivial solution with asymptotic* $B(x) = O(x), \quad x \to \infty$. *Moreover the estimation holds*

$$\beta_0 \equiv \underset{x>0}{inf ess}\, B(x) > 0.$$

Step. V. Consider the following nonhomogeneous equations with the sum-difference kernel:

$$f(x) = 2\overset{\circ}{w}(x+\delta_0) + \mu(x) \int\limits_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)]f(t)dt, \quad x \in (0, +\infty), \tag{66}$$

$$\widetilde{f}(x) = 2\,\overset{\circ}{w}(x+\delta_0) + \int\limits_0^\infty [K(x-t) + \varepsilon\,\overset{\circ}{K}(x+pt)]\widetilde{f}(t)dt, \quad x \in (0,+\infty), \tag{67}$$

with respect to unknown functions $f(x)$ and $\widetilde{f}(x)$. Using factorization (48) the solution of equation (67) may be reduced to solutions of the following coupled equations

$$(I - V_-)\widetilde{f}_0 = g, \tag{68}$$

$$(I - \widehat{T})\widetilde{f}_1 = \widetilde{f}_0, \tag{69}$$

$$(I - V_+)\widetilde{f} = \widetilde{f}_1 \tag{70}$$

where

$$g(x) = 2\,\overset{\circ}{w}(x+\delta_0). \tag{71}$$

From (68) by direct checking we obtain

$$\widetilde{f}_0(x) = g(x) + \beta\int\limits_x^\infty g(t)dt. \tag{72}$$

It is obvious that

$$\widetilde{f}_0 \in L_1(0,+\infty) \cap C_0[0,+\infty). \tag{73}$$

Now pass to equation (69). We introduce the following simple iterations:

$$\widetilde{f}_1^{(n+1)}(x) = \widetilde{f}_0(x) + \int\limits_0^\infty [h(x-t) + \varepsilon T_0(x+pt)]\widetilde{f}_1^{(n)}(t)dt, \tag{74}$$

$$\widetilde{f}_1^{(0)}(x) \equiv \widetilde{f}_0(x), \quad n = 0,1,2,\ldots$$

Note that

$$f_1^{(n)}(x) \text{ in } n. \tag{75}$$

On the other hand, if

$$0 \le \varepsilon < 4\beta^2 p^j \int\limits_a^b \frac{1}{s^3} d\sigma(s)(\beta^2 m_2 + (2p+2)m_1\beta + 2m_0\beta)^{-1}, \quad j = 1,2, \tag{76}$$

then from (49), (50) it follows that

$$\widetilde{f}_1^{(n)}(x) \in L_1(0,+\infty) \cap M(0,+\infty), \quad n = 0,1,2\ldots \tag{77}$$

Moreover

$$\widetilde{f}_1^{(n)}(x) \le \frac{\underset{x>0}{supess}\,\widetilde{f}_0(x)}{1-q_1}, \tag{78}$$

$$\int_0^\infty \widetilde{f}_1^{(n)}(x)dx \leq \frac{\int_0^\infty \widetilde{f}_0(x)dx}{1-q_2} \tag{79}$$

where

$$q_j = \rho^* + \frac{1}{2p^j}\varepsilon(\beta^2 m_2 + (2p+2)\beta m_1 + 2m_0 p), \quad j = 1, 2. \tag{80}$$

Taking into consideration B. Levi's theorem (see [7]), from (75), (78), (79) we conclude that:

i) there exists

$$\lim_{n\to\infty} \widetilde{f}_1^{(n)}(x) = \widetilde{f}_1(x) \in L_1(0,+\infty) \cap M(0,+\infty), \tag{81}$$

ii) the function $\widetilde{f}_1(x)$ satisfies equation (69).

Finally solving equation (70) we obtain

$$\widetilde{f}(x) = \widetilde{f}_1(x) + \beta \int_0^x \widetilde{f}_1(t)dt \in M(0,+\infty), \tag{82}$$

because $\widetilde{f}_1(x) \in L_1(0,+\infty) \cap M(0,+\infty)$.

We consider the following iteration

$$f^{(n+1)}(x) = 2\overset{\circ}{w}(x+\delta_0) + \mu(x)\int_0^\infty [K(x-t) + \varepsilon\overset{\circ}{K}(x+pt)]f^{(n)}(t)dt, \tag{83}$$

$$f^{(0)}(x) = 2\overset{\circ}{w}(x+\delta_0), \quad n = 0, 1, 2, \ldots, \quad x \in (0,+\infty). \tag{84}$$

By induction we obtain

- $f^{(n)}(x) \uparrow$ by n,
- $f^{(n)}(x) \leq \widetilde{f}(x), \quad n = 0, 1, 2, \ldots$

Therefore there exists

$$\lim_{n\to\infty} f^{(n)}(x) = f(x) \leq \widetilde{f}(x) \tag{85}$$

which satisfies the equation (66).

From (85) and (82) it follows that $f \in M(0,+\infty)$.

Step. VI. Let $\lambda(x)$ be a defined in $(0,+\infty)$ measurable function of the form

$$\lambda(x) = 1 - \frac{w_0(x + B_\gamma(x))}{B_\gamma(x)} \tag{86}$$

where

$$B_\gamma(x) \equiv \gamma B(x), \tag{87}$$

$\gamma \in \triangle \equiv \left[\frac{max(æ, \gamma_0)}{\beta_0}, +\infty\right)$ is an arbitrary number. Here $\gamma_0 \in [\delta_0, +\infty)$ is the first root when $w_0(\gamma_0) < \gamma_0$ takes place and $æ = \underset{x>0}{supess}\, f(x)$.

We have $B_\gamma \geq \gamma\beta_0 \geq \max(æ, \gamma_0) \geq \gamma_0 \geq \delta_0$, therefore

$$0 < 1 - \frac{w_0(\gamma_0)}{\gamma_0} \leq \lambda(x) \leq 1, \quad x \in (0, +\infty), \tag{88}$$

$$1 - \lambda(x) = \frac{w_0(x + B_\gamma(x))}{B_\gamma(x)} \leq \frac{1}{\gamma_0} w_0(x + \delta_0) \in L_1(0, +\infty),$$

$$(1 - \lambda(x))x \leq \frac{x + \delta_0}{\gamma_0} w_0(x + \delta_0) \in L_1(0, +\infty),$$

i.e.

$$(1 - \lambda(x))x^j \in L_1(0, +\infty), \quad j = 0, 1, \tag{89}$$

$$|1 - \lambda(x)| \leq \frac{1}{\gamma_0} w_0(x + B_\gamma(x)) \leq \frac{1}{\gamma_0} w_0(x + \delta_0) \to 0$$

when $x \to \infty$.

It is easy also to check that $\lambda(x) \uparrow$ in x. Now we consider the following nonhomogeneous integral equation

$$\varphi(x) = 2\,\overset{\circ}{w}(x + B_\gamma(x)) + \lambda(x)\mu(x) \int_0^\infty [K(x - t) + \varepsilon\, \overset{\circ}{K}(x + pt)]\varphi(t)dt \tag{90}$$

with respect to the function $\varphi(x)$.

Introduce the following simple iteration

$$\varphi^{(n+1)}(x) = 2\,\overset{\circ}{w}(x + B_\gamma(x)) + \lambda(x)\mu(x) \int_0^\infty [K(x - t) + \varepsilon\, \overset{\circ}{K}(x + pt)]\varphi^{(n)}(t)dt,$$
$$n = 0, 1, 2, \ldots, \quad \varphi^{(0)}(x) = 2\,\overset{\circ}{w}(x + B_\gamma(x)). \tag{91}$$

The following statements are valid:

- $\varphi^{(n)}(x)$ in n, (92)
- $\varphi^{(n)}(x) \geq 2\,\overset{\circ}{w}(x + B_\gamma(x)), \quad n = 0, 1, 2, \ldots$ (93)
- $\varphi^{(n)}(x) \leq f(x), \quad n = 0, 1, 2, \ldots$ (94)

The last inequality follows from (88) and from the following obvious inequality

$$\overset{\circ}{w}(x + B_\gamma(x)) \leq \overset{\circ}{w}(x + \delta_0), \quad x > 0. \tag{95}$$

Therefore there exists the limit of sequences of function $\{\varphi^{(n)}(x)\}_0^\infty$

$$\lim_{n\to\infty} \varphi^{(n)} = \varphi(x) \leq f(x)$$

and $\varphi(x)$ satisfies the equation (90).

Step. VII. Notice that the function $\widetilde{E}(x)=2B_\gamma(x)-\varphi(x)$ satisfies the following homogeneous integral equation

$$E(x)=\lambda(x)\mu(x)\int_0^\infty[K(x-t)+\varepsilon\overset{\circ}{K}(x+pt)]E(t)dt,\quad x>0. \tag{96}$$

Really we have

$$\lambda(x)\mu(x)\int_0^\infty[K(x-t)+\varepsilon\overset{\circ}{K}(x+pt)](2B_\gamma(t)-\varphi(t))dt=$$

$$=2\lambda(x)\mu(x)\int_0^\infty[K(x-t)+\varepsilon\overset{\circ}{K}(x+pt)]B_\gamma(t)dt+$$

$$+2\overset{\circ}{w}(x+B_\gamma(x))-\varphi(x)=2\lambda(x)B_\gamma(x)+2\overset{\circ}{w}(x+B_\gamma(x))-\varphi(x)=2B_\gamma(x)-\varphi(x).$$

Now we consider the following iteration

$$\begin{gathered}E^{(n+1)}(x)=\lambda(x)\mu(x)\int_0^\infty[K(x-t)+\varepsilon\overset{\circ}{K}(x+pt)]E^{(n)}(t)dt,\\ E^{(0)}(x)=2B_\gamma(x),\quad n=0,1,2,\dots\end{gathered} \tag{97}$$

By induction we check that

- $E^{(n)}(x)\downarrow$ in n, (98)

- $E^{(n)}(x)\geq\widetilde{E}(x),\quad n=0,1,2,\dots$ (99)

Therefore there exists the solution $E(x)$ of equation (96), moreover the double inequalities hold

$$2\lambda(x)B_\gamma(x)\geq E(x)\geq\widetilde{E}(x). \tag{100}$$

As $B_\gamma(x)\geq\gamma\beta_0\geq\max(æ,\gamma)\geq æ\geq f(x)\geq\varphi(x)$, then

$$\widetilde{E}(x)\geq B_\gamma(x),\ \text{ therefore} \tag{101}$$

$$2\lambda(x)B_\gamma(x)\geq E(x)\geq\widetilde{E}(x),\quad x>0. \tag{102}$$

It is obvious that if $E(x)$ satisfies equation (86) and the chain of inequalities (102), then the function

$$Y(x)=\frac{E(x)}{\lambda(x)}$$

will satisfy the equation

$$Y(x) = \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)]\lambda(t)Y(t)dt, \quad x > 0, \tag{103}$$

and the inequalities

$$B_\gamma(x) \le \widetilde{E}(x) \le E(x) \le Y(x) \le 2B_\gamma(x), \quad x > 0. \tag{104}$$

The next steps the chain of inequalities (104) will be of essential use in future.

3 One parameter family of solutions

Step. VIII. At this stage we construct one parameter family of solutions for the following class Hammerstein type nonlinear integral equation:

$$N(x) = \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)](N(t) - w(t, N(t)))dt, \quad x > 0 \tag{105}$$

with respect to unknown functions $N(x)$.

We consider the following iteration

$$\begin{gathered} N^{(p+1)}(x) = \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)](N^p(t) - w(t, N^p(t)))dt, \\ N^{(0)}(x) = 2B_\gamma(x), \quad p = 0, 1, 2, \dots \end{gathered} \tag{106}$$

First we prove that

$$N^{(p)}(x) \ge Y(x), \quad p = 0, 1, 2, \dots \tag{107}$$

In the case when $p = 0$, the inequality (107) immediately follows from (104). We suppose that (107) is true for some $p \in \mathbb{N}$ and prove the assertion when $p+1$. Using (10), (12) and obvious inequalities

$$Y(x) \ge B_\gamma(x) \ge \delta_0, \quad x \in (0, +\infty)$$

from (106) we have

$$N^{(p+1)}(x) \ge \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)](Y(t) - w(t, Y(t)))dt \ge$$

$$\ge \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)](Y(t) - w_0(t + Y(t)))dt \ge$$

$$\geq \mu(x)\int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)](Y(t)-w_0(t+B_\gamma(t)))dt =$$

$$= \mu(x)\int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)](Y(t)-(1-\lambda(t))B_\gamma(t))dt \geq$$

$$\geq \mu(x)\int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)](Y(t)-(1-\lambda(t))Y(t))dt = Y(x).$$

Now we prove that

$$N^p(x)\downarrow \text{ in } p\,. \tag{108}$$

We have

$$N^{(1)}(x)\leq \mu(x)\int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)]N^{(0)}(t)dt = 2B_\gamma(x),$$

because $w(t,N^{(0)}(t)) = w(t,2B_\gamma(t)) \geq 0$, since $2B_\gamma(x) \geq 2\gamma\beta_0 \geq 2\delta_0 \geq \delta_0$. Assuming $N^{(p)}(x)\leq N^{(p-1)}(x)$, from (106), taking into account (10) we obtain

$$N^{(p+1)}(x)\leq N^{(p)}(x).$$

Thus the sequences of functions $\{N^{(p)}(x)\}_0^\infty$ have the pointwise limit

$$\lim_{p\to\infty} N^{(p)}(x) = N(x)\geq Y(x),$$

moreover the following chain inequalities are valid:

$$B_\gamma(x)\leq \widetilde{E}(x)\leq E(x)\leq Y(x)\leq N(x)\leq 2B_\gamma(x), \quad x>0. \tag{109}$$

Using B. Levi's theorem it is easy to check that $N(x) = N_\gamma(x)$ satisfies the equation (105).

Now we prove that to different parameters $\gamma\in\triangle$ different solution of equation (105) correspond. Really we take arbitrary numbers $\gamma_1>\gamma_2$ and consider the corresponding iterations

$$\begin{gathered} N_{\gamma_j}^{(p+1)}(x) = \mu(x)\int\limits_0^\infty [K(x-t)+\varepsilon \overset{\circ}{K}(x+pt)](N_{\gamma_j}^{(p)}(t)-w(t,N_{\gamma_j}^{(p)}(t)))dt, \\ j=1,2, \quad p=0,1,2,\ldots, \quad N_{\gamma_j}^{(0)}(t) = 2B_{\gamma_j}(x). \end{gathered} \tag{110}$$

By induction we prove that

$$N_{\gamma_1}^{(p)} - N_{\gamma_2}^{(p)} \geq 2(B_{\gamma_1}(x)-B_{\gamma_2}(x)), \quad p=0,1,2,\ldots \tag{111}$$

For $p = 0$ it is obvious. Let (111) take place for some $p \in \mathbb{N}$. Then from (111) taking into account (10) we have

$$N_{\gamma_1}^{(p+1)}(x) - N_{\gamma_2}^{(p+1)}(x) \geq \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)][2B_{\gamma_1}(t) - 2B_{\gamma_2}(t)]dt +$$

$$+\mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)](w(t, N_{\gamma_2}^{(p)}(t)) - w(t, N_{\gamma_1}^{(p)}(t)))dt =$$

$$= 2\mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)][B_{\gamma_1}(t) - B_{\gamma_2}(t)]dt =$$

$$= 2B_{\gamma_1}(x) - 2B_{\gamma_2}(x) + \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)])(w(t, N_{\gamma_2}^{(p)}(t)) - w(t, N_{\gamma_1}^{(p)}(t)))dt.$$

On the other hand

$$B_{\gamma_1}(x) - B_{\gamma_2}(x) = (\gamma_1 - \gamma_2)B(x) \geq (\gamma_1 - \gamma_2)\beta_0 > 0.$$

Therefore

$$N_{\gamma_1}^{(p+1)}(x) - N_{\gamma_2}^{(p+1)}(x) \geq 2(B_{\gamma_1}(x) - B_{\gamma_2}(x)) \geq 2(\gamma_1 - \gamma_2)\beta_0.$$

Passing to limit in (111) we obtain

$$N_{\gamma_1}(x) - N_{\gamma_2}(x) \geq 2(B_{\gamma_1}(x) - B_{\gamma_2}(x)) \geq 2(\gamma_1 - \gamma_2)\beta_0 > 0,$$

i.e.

$$N_{\gamma_1}(x) \geq N_{\gamma_2}(x).$$

Thus the following theorem is valid.

Theorem 1. *Let conditions (2)–(7), (9)–(11), (76) be fulfilled. Then equation* (105) *possesses one parameter family of positive solutions* $\{N_\gamma(x)\}_{\gamma \in \triangle}$*, moreover for each function from the family the asymptotic equality holds:* $N_\gamma(x) = O(x)$ *as* $x \to +\infty$.

4 Solution of basic equation (1)

Step. IX. In this step by means of previous results we proof the existence of solution of basic equation (1). We introduce special iterations:

$$F^{(n+1)}(x) = \mu(x) \int_0^\infty [K(x-t) + \varepsilon \overset{\circ}{K}(x+pt)]Q(t, F^{(n)}(t))dt, \qquad (112)$$
$$F^{(0)}(x) = 2B_\gamma(x), \quad n = 0, 1, 2, \ldots$$

Whence taking into consideration (8) and Theorem 1 we get

$$\bullet \quad F^{(n)}(x) \downarrow \text{ in } n, \tag{113}$$

$$\bullet \quad F^{(n)}(x) \geq N_\gamma(x), \quad n = 0, 1, 2. \tag{114}$$

Therefore there exists

$$\lim_{n\to\infty} F(x) = F(x) \tag{115}$$

which satisfies equation (1). From (114) and (112) it follows that

$$N_\gamma(x) \leq F(x) \leq 2B_\gamma(x). \tag{116}$$

Thus the following theorem holds.

Theorem 2. *Let conditions (9)–(11), (76) be fulfilled. Then equation (1) has positive solutions with asymptotic $F(x) = O(x)$, as $x \to \infty$.*

Examples of function $Q(t, z)$.

$$1) \quad Q(t, z) = (z^2 - w(t, z)z)^{\frac{1}{2}}, \tag{117}$$

$$2) \quad Q(t, z) = \frac{1}{2} w(t, z)u(z) + z - \frac{1}{2} w(t, z), \tag{118}$$

where $u(z)$ is a defined on $(-\infty, +\infty)$, measurable function and

$$0 \leq u(z) \leq 1, \quad u(z) \uparrow \text{ in } z \text{ on } [\delta_0, +\infty), \quad u \in C[\delta_0, +\infty), \tag{119}$$

$$3) \quad Q(t, z) = \frac{2(z^2 - zw(t, z))}{2z - w(t, z)}, \tag{120}$$

$$4) \quad Q(t, z) = z - w(t, z) + \ln \frac{1 + e^{w(t,z)}}{2}. \tag{121}$$

Step. X. Using Theorem 2 we get a more general result.

Theorem 3. *Let all conditions of Theorem 2 be fulfilled. Assume $R(x, \tau)$ is a measurable function on $(0, +\infty) \times (-\infty, +\infty)$ satisfying the following conditions:*

$$\bullet \quad R(x, \tau) \in Carat(\Omega_{\delta_0}), \tag{122}$$

$$\bullet \quad R(x, \tau) \uparrow \ \textit{in } \tau \ \textit{on } [\delta_0, +\infty) \ \textit{for each } x > 0, \tag{123}$$

$$\bullet \quad \mu(x) \leq R(x, \tau) \leq 1, \quad (x, \tau) \in \Omega_{\delta_0}. \tag{124}$$

Then the equation

$$\chi(x) = R(x, \chi(x)) \int_0^\infty [K(x - t) + \varepsilon \overset{\circ}{K}(x + pt)] Q(t, \chi(t)) dt, \quad x > 0, \tag{125}$$

possesses positive solution with asymptotic $\chi(x) = O(x)$, $x \to \infty$. Moreover the inequalities are valid

$$N_\gamma(x) \leq F(x) \leq \chi(x) \leq 2B_\gamma(x). \tag{126}$$

Examples of function $R(x,\tau)$. Below we give two examples of function R:

1) $R(x,\tau) = \dfrac{1-\mu(x)}{2}u(\tau) + \dfrac{1+\mu(x)}{2}$, where u satisfies conditions (119),

2) $R(x,\tau) = (1-\mu(x))P(x,\tau) + \mu(x)$, where

- $0 \leq P(x,\tau) \leq 1, \quad (x,\tau) \in \Omega_{\delta_0}$,
- $P \uparrow$ on τ on $[\delta_0, +\infty)$,
- $P(x,\tau) \in \mathrm{Carat}(\Omega_{\delta_0})$.

References

[1] Kufner A., Fuchik S.*Nonlinear differential equations*, Moscow, Nauka, 1988 (in Russian).

[2] Yengibaryan N. B. *On one nonlinear radiative transfer problem.* Astrophysica, 1966, **4**, 31–36 (in Russian).

[3] Khachatryan Kh. A. *One parameter family of solutions for one class of Hammerstein nonlinear equations on a half line.* Doklady Mathematics, 2009, **80**, 872–876.

[4] Yengibaryan N. B., Yengibaryan B. N. *Convolution type integral equation on a half line with completely monotonic kernel.* Math. sb., 1996, **187**, 53–72 (in Russian).

[5] Arabadjyan L. G. *On one integral equation of transfer theory in nonhomogeneous medium.* Differential equations, 1987, **23**, 1618–1622 (in Russian).

[6] Arabadjyan L. G., Yengibaryan N. B. *Convolution equations and nonlinear functional equations.* Itogi nauki i techniki, 1984, **22**, 175–242 (in Russian).

[7] Kolmogorov A. N., Fomin V. S. *Elements of functions theory and functional analysis*, Moscow, 1981 (in Russian).

A. Kh. Khachatryan
Institute of Mathematics
of National Academy of Sciences
Armenia
E-mail: *Aghavard@hotbox.ru*

Kh. A. Khachatryan
Institute of Mathematics
of National Academy of Sciences
Armenia
E-mail: *Khach82@rambler.ru*

Received October 27, 2009