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A criterion for parametrical completeness in the 8-valued algebraic model of modal logic $S5$

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Abstract. The problem of parametrical completeness in the logic of 8-element topological Boolean algebra with trivial open elements is considered. The conditions permitting to determine the parametrical completeness of an arbitrary system of formulas in the mentioned logic are established in terms of 25 parametrical pre-complete classes of formulas.

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In mathematical logic, in modern algebra and in its applications the problem to construct an algorithm that, for every list of formulas, could recognize those formulas (functions or operations) which can be obtained from the initial list by means of some given tools remains to be actual. The best known method of the expressibility of some Boolean functions by the other ones by means of superpositions has its background in the works of American mathematician E. Post [9, 10] who described all closed with respect to superposition classes of two-valued Boolean functions.

A. V. Kuznetsov [6] proposed the notion of parametrical expressibility (in short, p. expressibility), which is a generalization of the traditional expressibility of formulas (or functions). On the base of the p. expressibility the notion of parametrical completeness (p. completeness) of an arbitrary system of formulas appeared. A. V. Kuznetsov [6] described all parametrical closed classes of Boolean functions. The number of these classes was determined to be equal to 25. On the base of this description one not very complex criterion for p. completeness in the classical logic may be formulated. He also obtained an original criterion for p. expressibility in the k-valued logic ($k = 3, 4, \dots$).

In the present paper we establish the necessary and sufficient conditions for the p. completeness of an arbitrary system of formulas in the 8-valued extension of $S5$ modal logic, which is the logic of 8-element topological Boolean algebra with two trivial open elements.

1 Basis notions and preliminary formulations

We consider the (propositional) *modal logic* in the sense of works [5, 7, 8, 13], which is based on *modal formulas* built in a traditional way from small letters of the Latin alphabet p, q, $r, \ldots$ (possibly indexed) with the help of symbols for the logical operations & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation), $\Box$ (necessitation) and parentheses. Arbitrary formulas, as a rule, are denoted by capital letters of the Latin alphabet possibly indexed.

We come to note down by $F(p_1, \ldots, p_n)$ the formula F which does not contain other variables besides $p_1, \ldots, p_n$. Let $\pi_1, \pi_2, \ldots, \pi_n$ be some pairwise different variables. Then $F[\pi_1/D_1, \ldots, \pi_n/D_n]$ or, shorter $F[D_1, \ldots, D_n]$ denotes [8] the result of the substitution of formulas $D_1, \ldots, D_n$, respectively, for variables $\pi_1, \ldots, \pi_n$ into F. We use analogical symbolism also in the case of substitution of variable values.

The modal calculus $S4$ is defined by means of classical propositional calculus completed with the *modal axioms*

$$(\Box p \supset p), (\Box p \supset \Box\Box p), (\Box(p \supset q) \supset (\Box p \supset q)),$$

as well as by the necessity rule (*Gödel rule*), which permits the transition from formula A to formula $\Box A$. We identify conventionally the logic $S4$ with the set of all formulas deducible in the calculus $S4$.

Any set of modal formulas which includes the axioms of $S4$ modal calculus and is closed with respect to its 3 deduction rules is called a (normal) *modal logic* [7]. If one logic is included into the other, then the last of them is called the *extension* of the first logic [7].

The $S5$ *modal logic* is defined by $S5$ *calculus* which contains all axioms and deduction rules of $S4$ calculus plus the formula $(\Diamond p \supset \Box\Diamond p)$ as a new axiom. So the $S5$ logic is the extension of $S4$ logic generated by the mentioned formula.

For the interpretation of modal formulas we use the notion of *topological Boolean algebra* [11], which is an algebra $\theta = \langle A; \&, \vee, \supset, \neg, \Box \rangle$ of type (2, 2, 2, 1, 1) such that the system $\langle A; \&, \vee, \supset, \neg \rangle$ is a Boolean algebra and the $\Box$ operation, called the interior operation, for any two elements α and β of A, satisfies the conditions:

$$\Box(\alpha\&\beta) = (\Box\alpha\&\Box\beta), \quad \Box\Box\alpha = \Box\alpha, \quad \Box 1 = 1 = \neg 0, \quad \Box\alpha \leq \alpha.$$

We define the partially ordered relation $\alpha \leq \beta$ by means of the relation $\alpha = \alpha\&\beta$ (or $\beta = \alpha \vee \beta$). We denote the smallest element of algebra - zero by the symbol 0, and we denote the biggest element - the unity by 1. An element α is called *open* if $\alpha = \Box\alpha$.

A formula A is called *valid* in Boolean algebra X if for every evaluation of the formula A with elements from X the value of F is identically equal to 1. Remind that for every X algebra, the set of all valid formulas in X constitutes a some modal logic denoted below by the symbol LX [11]. A modal logic is called *tabular* if it coincides with the logic of some finite topological Boolean algebra. A modal logic L is called *locally tabular* if the logic of any finitely generated topological Boolean algebra including all valid in L formulas is finite.

The following series of finite topological Boolean algebras with trivial open elements

$$\theta_1, \theta_2, \ldots, \theta_k, \ldots \qquad (1)$$

called the *series of Scroggs* [13], played an significant role in the study of $S5$ modal logic.

The following inclusions of logics take place:

$$L\theta_1 \supseteq L\theta_2 \supseteq L\theta_3 \dots \ . \tag{2}$$

In the logic $L\theta_1$ the formulas p, $\Box p$ and $\Diamond p$ are pairwise equivalent, and so the logic $L\theta_1$ coincides whit the propositional classical logic. The following relation:

$$S5 = L\theta_1 \cap L\theta_2 \cap L\theta_3 \dots \tag{3}$$

was established by Scroggs [13]. From the last equality it follows that the modal logic $S5$ is not tabular.

Two formulas F and G are called *equivalent in a logic* L if their equivalence $F \sim G$ is valid in L ($F \sim G == (F \supset G)\&(G \supset F)$). They say that an F formula *is explicitly expressible* in a logic L via a system of formulas Σ if F can be obtained from variables and formulas which belong to Σ by a finite number of applications of the weak rule of substitution (that is the transition from two formulas B and C to $B[\pi/C]$, where π is a variable in B), and a finite number of applications of the replacement by equivalent rule in L. The relation of explicit expressibility is *transitive*. If all transitions from some formulas to the another consist only of applications of weak rule of substitution, then they say that F *is directly* expressible via Σ. The system Σ is called *explicitly complete in* L if all formulas of the language of L are explicitly expressible in L via Σ.

A formula F is called *parametrically expressible* [6] (in short, p. expressible) in L logic via the system of formulas Σ , if there exist such numbers l and m, variables π_1, ...,π_l without any appearances in F, formulas B_1, C_1, ..., B_m, C_m explicitly expressible in L via Σ and formulas D_1, ..., D_l which do not contain the variables π_1, ..., π_l such that the following relations take place:

$$(F \sim \pi) \supset (B_1 \sim C_1)\& \dots \&(B_m \sim C_m)[\pi_1/D_1] \dots [\pi_l/D_l], \tag{4}$$

$$(B_1 \sim C_1)\& \dots \&(B_m \sim C_m) \supset (F \sim \pi). \tag{5}$$

Remind that for the cases of classical logic and of k-valued logic, the pithy sense of these relations consists essentially in that the predicate $F \sim \pi$ is equivalent (in classical sense) with $\exists\pi_1 \cdots \exists\pi_l S$, where S is the following conjunction of equalities [6, p. 27]:

$$(B_1 = C_1)\& \dots \&(B_m = C_m),$$

or, in other words, S is the system of equations

$$B_i = C_i \ (i = 1, \dots m).$$

The implicit expressibility is a particular case of parametrical expressibility [6].

The system Σ of formulas is called *parametrically complete* (*p. complete*) in the L logic if all formulas of the language of L are p. expressible in L via Σ. A Σ system is called *parametrically pre-complete* (*p. pre-complete*) in L if Σ is not p. complete in L, but for every formula F which is not p. expressible in L via Σ the system $\Sigma \cup \{F\}$ is p. complete in L.

We'll say that a formula $F(p_1, \dots, p_n)$ preserves the predicate $R(x_1, \dots, x_m)$ on the algebra $\theta = \langle M; \&, \vee, \supset, \neg, \Box \rangle$ if for every elements

$$\alpha_{ij} \in M (i = 1, \dots, m; j = 1, \dots, n)$$

because the following affirmations are true

$$R(\alpha_{11}, \dots, \alpha_{m1}), \dots, R(\alpha_{1n}, \dots, \alpha_{mn})$$

it results [6]

$$R(F(\alpha_{11}, \dots, \alpha_{1n}), \dots, F(\alpha_{m1}, \dots, \alpha_{mn})).$$

In the case when the predicate is defined on a finite set M it is often convenient to talk about the conservation of a matrix

$$(\alpha_{ij}) \quad (i = 1, \dots, m; j = 1, \dots, l)$$

of elements of M, corresponding to R, such that R is true on those and only on those sets of elements of M which are met in columns of the given matrix.

They say that two functions $f(p_1, \dots, p_m)$ and $g(p_1, \dots, p_n)$ are permutable [4] if they are bound up with each other by the following identity

$$f(g(p_{11}, \dots, p_{1n}), \dots, g(p_{m1}, \dots, p_{mn})) = g(f(p_{11}, \dots, p_{m1}), \dots, f(p_{1n}, \dots, p_{mn})).$$

Remind that the situation when the functions f and g are permutable is equivalent to the fact that f preserves the predicate

$$g(x_1, \dots x_n) = x_{n+1},$$

where x_{n+1} differs from $x_1, \dots, x_n$, and at the same time it is equivalent to the fact that g preserves the predicate

$$f(x_1, \dots x_m) = x_{m+1},$$

where x_{m+1} is distinct from $x_1, \dots, x_m$ [6]. So, any class of formulas preserving the predicate of type $f(x_1, \dots, x_m) = x_{m+1}$ $(x_{m+1} \neq x_1, \dots, x_m)$ is closed with respect to p. expressibility [6, p. 28]. The set of all formulas which preserve the predicate of type

$$f(x_1, \dots, x_m) = x_{m+1} \ (x_{m+1} \neq x_1, \dots, x_m)$$

on the θ algebra may be defined also as the *centralizer* of the function $f(p_1, \dots, p_m)$ [4, p. 142-143].

2 Criterion for parametrical completeness in the 8-element extension of $S5$ logic

In 2001 the author [2] realized the first step from classical logic to the modal logic $S5$ in the approach of p. completeness problem for the $S5$ modal logic. Namely we obtained a criterion of p. completeness in the logic of 4-element algebra θ_2. The mentioned criterion is based on 22 p. pre-complete classes C_1, C_2, ..., C_{22}, the first six of them are known from the case of classical logic.

In the present article the author accomplishes the second step to the logic $S5$. And namely we present the criterion for p. completeness in the 8-element model of $S5$ modal logic.

Let specify that the base of θ_3 algebra is the set

$$E = \{0, \rho, \mu, \varepsilon, \omega, \nu, \sigma, 1\},$$

and the signature consists of the set

$$\Omega = \{\&, \vee, \supset, \neg, \Box\}.$$

The partial order of elements of base E is the following: $0 < \rho < \omega < 1$, $0 < \rho < \nu$, $0 < \mu < \omega$, $0 < \mu < \sigma$, $0 < \varepsilon < \nu < 1$, $0 < \varepsilon < \sigma < 1$, the elements of triplets $\langle \rho, \mu, \varepsilon \rangle$ and $\langle \omega, \nu, \sigma \rangle$ are pairwise incomparable. The base operations of algebra θ_3 are expressed by formulas of the list

$$(p\&q), (p \vee q), (p \supset q), \neg p, \Box p, \tag{6}$$

and may be represented through the 8-valued tables.

Consider the following subalgebras of θ_3 algebra

$$\langle \{0, \mu, \nu, 1\}; \Omega \rangle; \quad \langle \{0, \varepsilon, \omega, 1\}; \Omega \rangle. \tag{7}$$

Observe that these subalgebras are pairwise isomorphic with the subalgebra

$$\theta_2 = \langle \{0, \rho, \sigma, 1\}; \Omega \rangle$$

and any of them determines one and the same 4-valued modal logic $L\theta_2$. So if an operation of θ_3 algebra is expressed by mean of some 1-ary formula, then this operation is completely determined by the values of this formula on θ_2 algebra. Therefore it takes place

Lemma 1. *If 1-ary formula $F(p)$ is expressible through the Σ system of formulas in $L\theta_2$ logic, then $F(p)$ is expressible through Σ also in the $L\theta_3$ logic.*

Remind that there exist 16 unary pairwise non-equivalent in $L\theta_3$ formulas. Below we present these formulas by means of theirs tables together with some of its designations.

p	0	1	$\Box p$	$\Diamond p$	$\neg\Box p$	$\neg\Diamond p$	$\neg p$	Δp	$\neg\Delta p$
0	0	1	0	0	1	1	1	1	0
ρ	0	1	0	1	1	0	σ	0	1
μ	0	1	0	1	1	0	ν	0	1
ϵ	0	1	0	1	1	0	ω	0	1
ω	0	1	0	1	1	0	ϵ	0	1
ν	0	1	0	1	1	0	μ	0	1
σ	0	1	0	1	1	0	ρ	0	1
1	0	1	1	1	0	0	0	1	0

p	$N_\rho(p)$	$\neg N_\rho(p)$	$Z_\rho(p)$	$Z_\sigma(p)$	$U_\rho(p)$	$U_\sigma(p)$
0	1	0	0	0	1	1
ρ	ρ	σ	ρ	σ	ρ	σ
μ	μ	ν	μ	ν	μ	ν
ϵ	ϵ	ω	ϵ	ω	ϵ	ω
ω	ω	ϵ	ω	ϵ	ω	ϵ
ν	ν	μ	ν	μ	ν	μ
σ	σ	ρ	σ	ρ	σ	ρ
1	0	1	0	0	1	1

Consider the following matrix

$$\begin{pmatrix} 0 & \rho & \mu & \epsilon & \omega & \nu & \sigma & 1 \\ 0 & \rho & \epsilon & \mu & \nu & \omega & \sigma & 1 \\ 0 & \mu & \rho & \epsilon & \omega & \sigma & \nu & 1 \\ 0 & \mu & \epsilon & \rho & \sigma & \omega & \nu & 1 \\ 0 & \epsilon & \rho & \mu & \nu & \sigma & \omega & 1 \\ 0 & \epsilon & \mu & \rho & \sigma & \nu & \omega & 1 \end{pmatrix}. \tag{8}$$

We note that every its line contains without repetition all elements of algebra θ_3, and any two different lines of matrix determine some automorphism [4] of θ_3 algebra. Any formula of list [6] preserves on θ_3 matrix (8), and so any formula conserves the matrix (8).

In the present work the following matrices will play a special role:

$$M_1 = \begin{pmatrix} 0 & \rho & \mu & \epsilon & \omega & \nu & \sigma & 1 \\ 0 & \rho & \rho & \rho & \sigma & \sigma & \sigma & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & \rho & \mu & \epsilon & \omega & \nu & \sigma & 1 \\ 0 & \rho & \nu & \omega & \epsilon & \mu & \sigma & 1 \end{pmatrix}, \tag{9}$$

$$M_3 = \begin{pmatrix} 0 & \rho & \mu & \epsilon & \omega & \nu & \sigma & 1 \\ 0 & \sigma & \nu & \omega & \epsilon & \mu & \rho & 1 \end{pmatrix}.$$

Denote by symbols C_{23}, C_{24}, and C_{25} the classes of formulas preserving on θ_3 algebra, respectively, these matrices. Considering the definitions of the classes $C_1, \dots, C_{22}$ [2], it is not difficult to verify that the classes C_1, C_2, ..., C_{25} are pairwise incomparable by inclusion.

The following criterion for parametrical completeness in the logic $L\theta_3$ is the basic result of the present work.

Theorem 1. *In order that a system Σ of formulas to be parametrically complete in the 8-valued logic $L\theta_3$ it is necessary and sufficient that Σ be parametrically complete in the 4-valued logic $L\theta_2$, and for every of classes C_{23}, C_{24} and C_{25} there exist in Σ formulas F_{23}, F_{24} and F_{25} not belonging, respectively, to these classes.*

Necessity results from the fact that the logic $L\theta_3$ is included in the logic $L\theta_2$ (on the base of relations (2)), the classes C_{23}, C_{24} and C_{25} are closed with respect to p. expressibility in the logic $L\theta_3$, and these classes are not parametrically complete in $L\theta_3$ logic.

Sufficiency. Let Σ system be p. complete in the $L\theta_2$ logic, and the formulas

$$F_{23}(p_1, \dots, p_n), F_{24}(p_1, \dots, p_n), F_{25}(p_1, \dots, p_n) \tag{10}$$

be such that $F_i \in \Sigma \backslash C_i (i = 23, 24, 25)$. Then on base of Lemma 1 any 1-ary formula is p. expressible through Σ in the $L\theta_3$ logic. From the fact that Σ is p. complete in the $L\theta_2$ logic and the subalgebras (7) are isomorphic with θ_2 it results that all the 1-ary formulas, and also any formula containing no more than one variable which is not under one of operator $\Box$, $\Diamond$ or Δ are p. expressible via Σ in $L\theta_3$. For example, the following formulas

$$(p \& \Box q), (p \vee \Diamond q), (\Box p \supset q), (p \sim \Box F) \tag{11}$$

are p. expressible through Σ in $L\theta_3$ (it is sure that F is p. expressible in $L\theta_3$ through Σ).

It is sufficient to prove that conjunction $(p \& q)$ is p. expressible in $L\theta_3$ through 1-ary formulas, the formulas of type (11) and formulas F_{23}, F_{24}, F_{25}. But the last follows from the next three Theorems, which we prezent without demonstrations.

Theorem 2. *If system Σ of formulas is p. complete in the $L\theta_2$ logic, then the formula $\Box(p \sim q)$ is p. expressible in $L\theta_3$ through Σ and the formulas F_{23} and F_{25}.*

Theorem 3. *The formula $\Box(p \vee q) \vee \Box(p \supset q) \vee \Box(q \supset p)$ is p. expressible in the $L\theta_3$ logic through 1-ary formulas, the formulas $\Box(p \sim q)$, F_{24}, F_{25} and the formulas $K(p, q)$ and $D(p, q)$ which express the conjunction and, respectively, disjunction in the logic $L\theta_2$.*

Theorem 4. *The conjunction $(p \& q)$ is p. expressible in the $L\theta_3$ logic through 1-ary formulas, the formulas $\Box(p \sim q)$, $\Box(p \vee q) \vee \Box(p \supset q) \vee \Box(q \supset p)$, and the formulas $K(p, q)$ and $D(p, q)$ which express the conjunction and, respectively, disjunction in the logic $L\theta_2$.*

Theorem 5. *There is an algorithm, practically not complex, which for every finite system of formulas can recognize whether this system is p. complete in the $L\theta_3$ logic.*

Theorem 6. *There are exactly 25 classes of formulas which are p. pre-complete in the $L\theta_3$ logic, and namely, the following classes and only they: $C_1, C_2, \ldots, C_{25}$.*

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