BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(63), 2010, Pages 31–58
ISSN 1024–7696

Abstract complexes, their homologies and applications

Cataranciuc Sergiu, Soltan Petru

Abstract. The complex of multi-ary relations $\mathcal{K}^n$ is defined in a more natural way than it was defined in [18, 58, 59]. The groups of homologies and co-homologies of this complex over the group of integer numbers are constructed. The methods used for these constructions are for the most part analogous with classical methods [2, 32, 52], but sometimes they are based on methods from [18, 44, 58]. The importance and originality consist in application of the multi-ary relations of a set of objects in construction of homologies. This allows to extend areas of theoretical researches and non-trivial practical applications in a lot of directions. Other abstract structures, which are developed in a natural way from generalized complex of multi-ary relations are also examined. New notions such as the notions of abstract quasi-simplex and its homologies, the complex of abstract simplexes and the complex of the n-dimensional abstract cubes are introduced.

Mathematics subject classification: 18F15, 32Q60, 55-99.
Keywords and phrases: Complex, manifold, abstract cube, quasi-simplex, multi-dimensional Euler tour.

1 Generalized complex of multi-ary relations

The topology of multi-ary relations is certainly a part of algebraic topology [32, 56, 61, 62], but it is that branch of the topology which represent high modern abstract examinations and purely theoretical researches. The results of this article deal with investigations made during ten years. They were made in order to obtain not only practical aspects but also non-trivial applications. So only after we obtained those results we decided to present them in this article in a brief way. At the beginning there was the complex of multi-ary relations for which the sequence of elements of any relation does not admit repetition of elements. This complex is an abstract cell complex (W), but the complex presented in this article is more operable.

Defining multi-ary relations over the Cartesian product of a set of elements X allows to generalize some previous results [58], and it is also more natural to do in theoretical and applicative terms. The notion of generalized complex of multi-ary relations over an arbitrary set of elements was presented first in [17]. Then the groups of homologies over the group of integer numbers were constructed. This complex generalizes several classical notions like the notion of finite and directed graph. The directed graph is defined in terms of binary relations [5], but it is without loops. This fact cannot be ignored as can be seen from applications [7]. The complex defined in [58] is a finite, discrete structure, without loops. It needs some additional effort when it is used in studying and solving some theoretical and practical problems. This mathematical object has a lot of properties which are related to the subject of discrete combinatorial geometry [37]. From these considerations, on the one hand, it would be natural to study the impact of the topology of multi-ary relations over the research objects where the mentioned situation can be treated, for example, by using following families of objects: a) the system Q of quasi-groups with n algebraic operations or the system Q of n groups with n independent elements and one algebraic operation [64], b) the system of fuzzy sets [47], etc. To be more precise, it would be interesting to study an abstract, multidimensional, oriented and without boundaries manifold [15] that is determined from mentioned objects. On the other hand, by using multi-ary relations there could be modelated a lot of processes of applicative domains. For example, the water formula H_2O could be treated as an element of a family of three-ary relations $\{(H,\ H,\ O)\}$ which does not belong to the complex of multi-ary relations, because in the triplet $\{(H,\ H,\ O\}$ the element H appears two times (see [58]). From the above results the necessity it results to expand the notion of complex of multi-ary relations by introducing a notion of complex of multi-ary relations that would be more general. In that case any chemical formula [25] could be treated as a generalized complex of multi-ary relations. There are also other facts which deal with the generalized complex. Anticipating the following results, we repeat that this object, in its abstract form, leads to the non-trivial applications. The applications appear for example in the process of transmission and reception of information, in the formation of a finite database of any dimension, in the cryptography and in graph theory when a class of graphs is indicated, elements of which could be involved in the 1-dimensional skeleton of the multidimensional cube from R_1^n (here R_1^n is the vectorial n-dimensional space with norm the $||x|| = |x_1| + |x_2| + \ldots + |x_n|$). The last mentioned allows to present an algorithm of the median calculation (the Torricelli point) of labeled and weighted graphs, and the algorithm is metric-free (without using any metric) [59, 60].

Now, define this object.

Let $X = \{x_1,\ x_2,\ \ldots,\ x_r\}$ be a finite set of elements, that is a subset of a set M, $cardM \leq \infty$. Let $X = X^1,\ X^2,\ \ldots,\ X^{n+1},\ \ldots,\ (n \geq 1)$ be the succession of Cartesian products [40–42] of the set $X:\ X^{m+1} = X^m \cdot X,\ 1 \leq m \leq n$. Any nonempty subset $R^m \subset X^m$ is said to be an *m-ary relation* of elements from X (the set $R^1 \subset X^1$ is a subset of elements from X). According to the mentioned above, an *m-ary relation* R^m is a family of ordered successions named sequences. Each sequence consists of m elements of X. Generally speaking, the sequence $(x_{i_1},\ x_{i_2},\ \ldots,\ x_{i_m}) \in R^m$ could contain some elements from X several times. For this kind of sequence any subsequence $(x_{j_1},\ x_{j_2},\ \ldots,\ x_{j_l}),\ 1 \leq l \leq m$, which preserves the order of elements of $(x_{i_1},\ x_{i_2},\ \ldots,\ x_{i_m})$ is called a *hereditary subsequence.*

Now, let us consider a finite family of relations $\{R^1,\ R^2,\ \ldots,\ R^{n+1}\}$.

Definition 1.1. *A family of relations* $\{R^1,\ R^2,\ \ldots,\ R^{n+1}\}$ *which satisfies the conditions:*

1. *$R^1 = X^1 = X$;*

2. *$R^{n+1} \neq \emptyset$;*

3. *any hereditary subsequence $(x_{j_1},\ x_{j_2},\ \ldots,\ x_{j_l})$, $1 \leq l \leq m \leq n+1$, of the sequence $(x_{i_1},\ x_{i_2},\ \ldots,\ x_{i_m}) \in R^m$ belongs to the l-ary relation R^l,*

is called a ***generalized complex of multi-ary relations ($\mathcal{G}$-complex)*** *and is denoted by*

$$\mathcal{R}^{n+1} = (R^1,\ R^2,\ \ldots,\ R^{n+1}).$$

From Definition 1.1 we obtain that the set R^m of a generalized complex $\mathcal{R}^{n+1}$ is not empty for each $1 \leq m \leq n+1$.

The study of generalized complex of multi-ary relations is interesting because this notion covers a lot of classical notions like graphs [5, 7, 12], hypergraphs [5, 6, 24, 71], matroids [9, 10, 66], simplicial complexes, etc. Certainly, this complex could be interpreted as a particular case of the **abstract cellular complex** [32] (CW), but these new mathematical structures serve as effective models for solving a lot of theoretical and applicative problems [28, 29, 43, 46, 50, 53]. Remark that the object $\mathcal{R}^{n+1}$ has advantage over the structures mentioned above. Thus, if it is compared with cellular complex (CW) then it is seen that the $\mathcal{R}^{n+1}$ is formed from the elementary "bricks", maybe with non-isomorphic deformations, like there are the abstract quasi-simplexes (the so-named finite-dimensional **loops**). Consider the generalized complex of relations $\mathcal{R}^2 = (R^1,\ R^2)$. It is obvious that this complex (Definition 1.1) represents a directed graph [7,11,12]. This allows us to consider the generalized complex of relations $\mathcal{R}^{n+1}$ as an **oriented** and **hereditary hypergraph** (according to C. Berge [6]). The last notion could be rarely found in the bibliography of speciality, and it represents a structure different from the notion of hypergraph [5,6,45,71]. Next, we will describe a procedure that allows to obtain the notion of hypergraph in the form of generalized complex and the so-called cycles of hypergraph in a natural form. This will be different from the known one, and it will be starting from the notion of oriented hypergraph [65] transformed into a complex of abstract simplexes.

Definition 1.2. *If there are two $\mathcal{G}$-complexes of multi-ary relations $\mathcal{R}_1^{m+1} = (R_1^1,\ R_1^2,\ \ldots,\ R_1^{m+1})$ and $\mathcal{R}^{n+1} = (R^1,\ R^2,\ \ldots,\ R^{n+1})$, $1 \leq m \leq n$, such that $R_1^l \subset R^l$, for all l, $1 \leq l \leq m+1$, then $\mathcal{R}_1^{m+1}$ is called a* ***$\mathcal{G}$-subcomplex*** *of $\mathcal{R}^{n+1}$. In case $\mathcal{R}_1^{m+1} = (R^1,\ R^2,\ \ldots,\ R^{m+1})$, then the subcomplex $\mathcal{R}_1^{m+1}$ is called the* ***skeleton*** *of $\mathcal{R}^{n+1}$ and is denoted* ***sk(m+1)***$\mathcal{R}^{n+1}$.

It is easy to see that **sk(2)**$\mathcal{R}^{n+1}$ of any $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^{n+1}, n \geq 1$, is a directed graph [6]. This complex could be represented by the pair $\mathcal{R}^2 = (X,\ R^2)$, where X is the set of vertices and R^2 is a binary relation, which can also have elements $(x,\ x)$, called **loops** [5,7,45]. Relations of degree more than two with repetition of elements will be studied later.

Definition 1.3. *The $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$ is said to be* ***connected*** *if for any two elements $x_i,\ x_j \in R^1$, there is a sequence $x_i = x_{t_1},\ x_{t_2},\ \ldots,\ x_{t_s} = x_j$ of elements from R^1 such that at least one of the pairs $(x_{t_r},\ x_{t_{r+1}})$ and $(x_{t_{r+1}},\ x_{t_r})$ belongs to relation R^2, for any $r = 1,\ 2,\ \ldots,\ s-1$. The sequence of pairs $(x_{t_1},\ x_{t_2}),\ (x_{t_2},\ x_{t_3}),\ \ldots,\ (x_{t_{s-1}},\ x_{t_s})$, is called a* ***linear chain*** *of dimension one that joins the elements x_i and x_j. The $x_i,\ x_j$ are called* ***extremities*** *of this chain.*

We will denote the chain of dimension one that joins the elements x_i and x_j by $L^1(x_i,\ x_j)$. Later we will pass to the algebraic representation of the chain $L^1(x_i,\ x_j)$.

Definition 1.4. *If $\mathcal{R}^{n_1+1} = (R_1^1,\ R_1^2,\ \ldots,\ R_1^{n_1+1})$ and $\mathcal{R}^{n_2+1} = (R_2^1,\ R_2^2,\ \ldots,\ R_2^{n_2+1})$, $n_1 \leq n_2$, are two $\mathcal{G}$-complexes of multi-ary relations, then*

$$\begin{aligned}\mathcal{R}^{n+1} \quad = \quad & \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} = (R_1^1 \cup R_2^1,\ R_1^2 \cup R_2^2,\ \ldots, \\ & R_1^{n_1+1} \cup R_2^{n_2+1},\ R_2^{n_1+2},\ \ldots R_2^{n_2+1})\end{aligned}$$

is called ***the union****, and*

$$\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1} = (R_1^1 \cap R_2^1,\ R_1^2 \cap R_2^2,\ \ldots,\ R_1^{n_1+1} \cap R_2^{n_2+1}).$$

is called ***the intersection*** *of these two $\mathcal{G}$-complexes.*

According to [40, 41] it is easy to verify that both the union and the intersection of two $\mathcal{G}$-complexes of multi-ary relations is also a $\mathcal{G}$-complex of multi-ary relations.

If in a $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^{n+1} = (R^1,\ R^2,\ \ldots,\ R^{n+1})$ we have $R^1 = \emptyset$, and therefore $R^2 = R^3 = \ldots = R^{n+1} = \emptyset$, then $\mathcal{R}^{n+1}$ is called an *empty complex*. Two $\mathcal{G}$-complexes of multi-ary relations the intersection of which is an empty $\mathcal{G}$-complex are called *disjoint*.

Theorem 1.1. *A $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^{n+1}$ is connected if and only if it does not contain two nonempty $\mathcal{G}$-complexes of multi-ary relations $\mathcal{R}^{n_1+1}$ and $\mathcal{R}^{n_2+1}$ which are disjoint and satisfy the equality:*

$$\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1}.$$

Proof. Necessity. Let $\mathcal{R}^{n+1} = (R^1,\ R^2,\ \ldots,\ R^{n+1})$ be a connected $\mathcal{G}$-complex of multi-ary relations and $x \in R^1 = X$ be an arbitrary element. We will denote by R'_x the set of all elements x' from X for which there exists at least one chain $L^1(x,\ x')$. We assert that if an element, for example x_{i_j}, of an ordered sequence $(x_{i_1},\ x_{i_2},\ \ldots,\ x_{i_j},\ \ldots,\ x_{i_m}) \in R^m$, $1 \leq m \leq n+1$, is from the set R'_x, then any other element of this sequence is from R'_x. By this way, we construct the $\mathcal{G}$-subcomplex of multi-ary relations $\mathcal{R}^{n_1+1} = (R_1^1,\ R_1^2,\ \ldots,\ R_1^{n_1+1})$ such that any sequence of elements $(x'_{i_1},\ x'_{i_2},\ \ldots,\ x'_{i_{m_1}})$ from $R_1^{m_1}$, $1 \leq m_1 \leq n_1+1$, contains an element x' for which there exists the chain $L^1(x,\ x')$.

Now we construct the second $\mathcal{G}$-complex of relations $\mathcal{R}^{n_2+1} = (R_2^1,\ R_2^2,\ \ldots,\ R_2^{n_2+1})$ from $\mathcal{R}^{n+1}$ so that $R_2^{m_2} = R^{m_2} \backslash R_1^{m_2}$, $m_2 = 1,\ 2,\ \ldots,\ n+1$. The next equalities:

$$R_2^1 = \emptyset,\ R_2^2 = \emptyset,\ \ldots,\ R_2^{m_2+1} = \emptyset,$$

are true, because we obtain a contradiction with the fact that the complex $\mathcal{R}^{n+1}$ is connected otherwise. It follows that $\mathcal{R}^{n_2+1}$ is empty.

Because the element $x \in X$ was arbitrarily chosen, we obtain that in $\mathcal{R}^{n+1}$ two nonempty, disjoint $\mathcal{G}$-complexes it does not exist as it is mentioned in the theorem.

Sufficiency. Let us suppose that $\mathcal{R}^{n+1}$ does not contain two nonempty, disjoint $\mathcal{G}$-subcomplexes of multi-ary relations $\mathcal{R}^{n_1+1}$ and $\mathcal{R}^{n_2+1}$ which satisfy the condition from theorem, and $\mathcal{R}^{n+1}$ is not connected. In the same way as it is described in first part of the proof of this theorem, we construct two nonempty, disjoint $\mathcal{G}$-complexes of multi-ary relations $\mathcal{R}^{n_1+1}$ and $\mathcal{R}^{n_2+1}$ so that $\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1}$. This contradicts theorem's condition. □

Now let us fix an arbitrary element $\overline{x} \in X$. We will denote by $\overline{X}$ the subset of all elements $x' \in X$, including $\overline{x}$, for which there exists at least one chain $L^1(\overline{x},\ x')$ in $\mathcal{R}^{n+1}$. Next, we construct the sequence of Cartesian products of the set $\overline{X}$: $\overline{X}^1 = \overline{X},\ \overline{X}^2,\ \ldots,\ \overline{X}^m,\ \ldots$, where $\overline{X}^{m+1} = \overline{X}^m \cdot \overline{X}$, and form the sets

$$\overline{R}^m = \overline{X}^m \cap R^m,\ m = 1,\ 2,\ \ldots,\ n+1.$$

Let $n_1 \leq n$ be the maximal index value which satisfies the relation $\overline{R}^{n_1+1} \neq \emptyset$.

Remark 1.1. *The sets of relations* $\overline{R}^1,\ \overline{R}^2,\ \ldots,\ \overline{R}^{n_1+1}$ *satisfy conditions I-III from Definition 1.1 and, thus, this family represents a generalized complex of multi-ary relations.*

Definition 1.5. *The $\mathcal{G}$-complex of multi-ary relations* $\mathcal{R}_{\overline{x}}^{n_1+1} = (\overline{R}^1,\ \overline{R}^2,\ \ldots,\ \overline{R}^{n_1+1})$ *is called a* ***connected component*** *of* $\mathcal{R}^{n+1}$ *complex.*

Each element $\overline{x} \in \overline{R}^1$ determines in $\mathcal{R}^{n+1}$ the same connected component. So the index $\overline{x}$ can be omitted from the notation $\mathcal{R}_{\overline{x}}^{n_1+1}$, and then we will denote the connected component just $\mathcal{R}^{n_1+1}$. It is obvious that $\mathcal{R}^{n_1+1} \subset \mathcal{R}^{n+1}$, i. e. $\mathcal{R}^{n_1+1}$ is a $\mathcal{G}$-subcomplex of $\mathcal{R}^{n+1}$.

Theorem 1.2. *If* $\{\mathcal{R}^{n_1+1},\ \mathcal{R}^{n_2+1},\ \ldots,\ \mathcal{R}^{n_q+1}\}$ *represent the family of all connected and pairwise distinct components of $\mathcal{G}$-complex of* $\mathcal{R}^{n+1}$ *relations, then the relations:*

$$\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} \cup\ \ldots \cup\ \mathcal{R}^{n_q+1} \tag{1.1}$$

are true, where $\mathcal{R}^{n_i+1} \cap \mathcal{R}^{n_j+1} = \emptyset$ *for all* $i \neq j,\ i,\ j = 1,\ 2,\ \ldots,\ q$.

Proof. Considering the construction of a connected component, that contains a given element $\overline{x} \in X$, we obtain that is true the inclusion

$$\mathcal{R}^{n+1} \subset \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} \cup\ \ldots \cup\ \mathcal{R}^{n_q+1}.$$

At the same time, because of the fact that connected components are some $\mathcal{G}$-subcomplexes of multi-ary relations of the complex $\mathcal{R}^{n+1}$, we obtain

$$\mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} \cup \ldots \cup \mathcal{R}^{n_q+1} \subset \mathcal{R}^{n+1}.$$

From these two relations we have the equality (1.1).

Now, let us suppose that there exist two connected components $\mathcal{R}^{n_i+1}$ and $\mathcal{R}^{n_j+1}$, $i \neq j$, so that $\mathcal{R}^{n_i+1} \cap \mathcal{R}^{n_j+1} \neq \emptyset$.

In this case we obtain that for each element $x \in \overline{R}_i^1$ and for each $y \in \overline{R}_j^1$ there is a chain $L^1(x, y)$ in $\mathcal{R}^{n+1}$. Therefore we have $\mathcal{R}^{n_i+1} \subset \mathcal{R}^{n_j+1}$, and the reverse $\mathcal{R}^{n_j+1} \subset \mathcal{R}^{n_i+1}$. So we obtain that $\mathcal{R}^{n_i+1} = \mathcal{R}^{n_j+1}$.This contradicts theorem's conditions. It follows that the assumption $\mathcal{R}^{n_i+1} \cap \mathcal{R}^{n_j+1} \neq \emptyset$ is false. □

Definition 1.6. *The $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$ is called **locally complete** if for any $m = 1, 2, \ldots, n$ and for any sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in R^m$, the relation R^m also contains all sequences that correspond to the $m!$ permutations of elements $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$.*

Any directed and symmetric graph can be an example of locally complete $\mathcal{G}$-complex of multi-ary relations [4,45]. We will denote such a graph by $\mathcal{R}^2 = (R^1, R^2)$, where R^2 is a binary and symmetric relation defined on the set of elements from R^1. A locally complete $\mathcal{G}$-complex of multi-ary relations is, also, a $\mathcal{G}$-complex constructed on the family of Cartesian products of the set X, i. e.

$$\mathcal{R}^{n+1} = (X = X^1, X^2, \ldots, X^{n+1}).$$

Following the goal announced in the title of this article, by analogy to the known classical bibliography in the combinatorial topology and topological algebra fields [1, 2, 8, 13, 16, 27, 39, 51, 52, 56, 63, 70], further we will also use other notations and notions, that are equivalent to those mentioned. These notations and notions will be used to study the properties of the complex of multi-ary relations, which are needed to solve practical problems.

Definition 1.7. *The sequence $(x_{i_0}, x_{i_1}, \ldots, x_{i_m}) \in R^{m+1}$, which has pairwise distinct elements, is said to be an **abstract simplex of dimension m** and is denoted by $S_i^m = (x_{i_0}, x_{i_1}, \ldots, x_{i_m}) \in R^{m+1}$, $m = dim S_i^m$. Any sequence of elements $(x_{j_0}, x_{j_1}, \ldots, x_{j_l}) \in R^{l+1}$, which is a hereditary subsequence from S_i^m, is called a **face** of dimension l of a simplex S_i^m, and it will be denoted by $S_j^l = (x_{j_0}, x_{j_1}, \ldots, x_{j_l})$, $S_j^l \subset S_i^m$. Sometimes we will use to call the faces of dimension zero - **vertices**, and those of dimension one are called **edges** of simplex S_i^m, $0 \leq m \leq n$.*

If $\mathcal{R}^{n+1}$ is finite and any sequence from R^m, $1 \leq m \leq n+1$, satisfies Definition 1.7, then $\mathcal{R}^{n+1}$ is a complex of multi-ary relations as in [17, 18, 44]. Having in mind the examinations of this particular case of $\mathcal{R}^{n+1}$, we will remind some additional aspects, that will be useful in the future.

A subset formed by $m+1$ pairwise distinct elements from the set X can generate several abstract simplexes of dimension m. The maximal number of it coincides with the number of different permutations of the $m+1$ elements. This means that there are $(m+1)!$ simplexes. It follows that distinct abstract simplexes of dimension m that are stretched on $m+1$ vertices from X could be imagined as membranes that strain these vertices.

Further we will denote by S^m the set of all simplexes with dimension m that are determined by sequences from R^{m+1}.

By this way the complex of relations $\mathcal{R}^{n+1} = (R^1,\ R^2,\ \dots,\ R^{n+1})$ can be represented as follows:

$$\mathbb{S}^0 = R^1,\ \mathbb{S}^1 = R^2,\ \dots,\ \mathbb{S}^n = R^{n+1}$$

and

$$(\mathbb{S}^0,\ \mathbb{S}^1,\ \dots,\ \mathbb{S}^n) = \mathcal{K}^n. \tag{1.2}$$

We will also keep the name of *complex* for $\mathcal{K}^n$, where $n = dim\mathcal{K}^n$ is called the *dimension* of $\mathcal{K}^n$. We have $\mathbf{sk(m+1)}\mathcal{R}^{n+1}\mathbf{=sk(m)}\mathcal{K}^n$.

Let $S_i^m \in \mathcal{S}^m$ be an abstract simplex of dimension m.

Definition 1.8. *The set of all simplexes of the dimension greater or equal to m from $\mathcal{K}^n$ with a common face $S_i^m \in \mathbb{S}^m$ is called the **star** of the simplex S_i^m, $m = 0,\ 1,\ 2,\ \dots,\ n$, and it is denoted by $\mathbf{st}S_i^m$ [16, 33].*

Remark 1.2. *The complex of relations $\mathcal{K}^n$ is not a simplicial abstract complex [32], because the same set of vertices can determine more that one abstract simplex.*

Depending on what is needed, the complex of multi-ary relations will be represented in one of the two equivalent forms [57]:

$$\mathcal{K}^n = (R^1,\ R^2,\ \dots,\ R^{n+1}),$$
$$\mathcal{K}^n = (\mathbb{S}^0,\ \mathbb{S}^1,\ \dots,\ \mathbb{S}^n).$$

The family of simplexes $\mathbb{S}^m$ from $\mathcal{K}^n$ will be represented in the following way

$$\mathbb{S}^m = \{S_1^m,\ S_2^m,\ \dots,\ S_{\alpha_m}^m\}, \tag{1.3}$$

where $0 \le m \le n$, and α_m is the cardinal of this family, $\alpha_m = card\ \mathbb{S}^m$.

Definition 1.9 (see [2, 4, 32])**.** *For any complex of relations $\mathcal{K}^n = (\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^n)$, we define the function of integer values*

$$\chi(\mathcal{K}^n) = \sum_{i=0}^{n} (-1)^i \alpha_i, \tag{1.4}$$

*that is called the **Euler characteristic** of the complex $\mathcal{K}^n$, where $\alpha_i = card\mathbb{S}^i$, $0 \le i \le n$.*

Further we will operate as in Definition 1.7.

Definition 1.10. *Any sequence* $(x_{i_0},\ x_{i_1},\ \dots,\ x_{i_m}) \in R^{m+1}$ *of a* $\mathcal{G}$*-complex of multi-ary relations* $\mathcal{K}^n = R^{n+1} = (R^1,\ R^2,\ \dots,\ R^{n+1})$ *is said to be an* ***abstract quasi-simplex of dimension m****, and it is denoted by* $Q^m = (x_{i_0},\ x_{i_1},\ \dots,\ x_{i_m})$*. The family of all quasi-simplexes of dimension* m *is denoted by* $\mathbb{Q}^m$*,* $0 \le m \le n$*.*

By Definition 1.1, we have that a sequence $(x_{i_0},\ x_{i_1},\ \dots,\ x_{i_m})$ from the relation R^{m+1} could have some repetitions of elements. For example, the sequence $(x_{i_j},\ x_{i_j},\ \dots,\ x_{i_j})$, with only one element $x_{i_j} \in X$. it is possible that the complex $\mathcal{R}^{n+1}$. Here the element x_{i_j} is $m+1$ times repeated, and this sequence represents a generalization of loop from graph theory [5]. We will keep the notion of *loop of more dimension*. But, at the same time the generalized loops could have more complicated forms. The problem of loops classification for a sequence $(x_{i_0},\ x_{i_1},\ \dots,\ x_{i_m}) \in R^{m+1}$, $1 \le m \le n$, is solved through the notion of isomorphism, and it is connected to the subject of another article. This leads us to the following remark.

Remark 1.3. *Any abstract simplex is a quasi-simplex, but not any quasi-simplex is an abstract simplex. The sequence* $(x_{i_1},\ x_{i_1},\ \dots,\ x_{i_m})$ *is not an abstract simplex because the element* x_{i_1} *appears in this sequence at least two times.*

Further we will denote by $\mathbb{Q}^m$, $0 \le m \le n$, the set of all quasi-simplexes with m dimension that are determined by the elements from R^{m+1}.

Definition 1.11. *A quasi-simplex* $Q^l = (x_{j_0},\ x_{j_1},\ \dots,\ x_{j_l}) \in \mathbb{Q}^l$ *which is a hereditary subsequence of the quasi-simplex* $Q^m = (x_{i_0},\ x_{i_1},\ \dots,\ x_{i_m}) \in \mathbb{Q}^m$ *is called a* ***quasi-face*** *of dimension* l *of the quasi-simplex* Q^m*.*

It is obvious that the face of $Q^m,\ 1 \le m \le n$, represents a quasi-simplex too (see Definition 1.10).

It is easy to verify next two assertions, that are mentioned in [58].

Assertion 1.1. *If* $\mathcal{K}^{n_1}$ *and* $\mathcal{K}^{n_2}$ *are two subcomplexes of multi-ary relations of the complex* $\mathcal{K}^n$ *then*

$$\chi(\mathcal{K}^{n_1} \cup \mathcal{K}^{n_2}) = \chi(\mathcal{K}^{n_1}) + \chi(\mathcal{K}^{n_2}) - \chi(\mathcal{K}^{n_1} \cap \mathcal{K}^{n_2}).$$

Theorem 1.3. *If the sequence* $\mathcal{K}^{n_1},\ \mathcal{K}^{n_2},\ \dots,\ \mathcal{K}^{n_q}$ *represent all connected components of the* $\mathcal{K}^n$ *then*

$$\chi(\mathcal{K}^n) = \sum_{i=1}^{q} \chi(\mathcal{K}^{n_i}). \tag{1.5}$$

We have to mention that multi-ary relations $R^1,\ R^2,\ \dots,\ R^{n+1}$ that represent the $\mathcal{G}$-complex of multi-ary relations R^{n+1} (see Definition 1.1) are families of abstract quasi-simplexes is of dimension $0, 1, ..., n$. We denote them by $\mathbb{Q}^0,\ \mathbb{Q}^1,\ \dots,\ \mathbb{Q}^n$. Thus the $\mathcal{G}$-complex of R^{n+1} relations can be regarded as a complex of abstract quasi-simplexes $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \dots,\ \mathbb{Q}^n)$.

2 The orientation of quasi-simplexes and the incidence matrices

Let $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$ be a $\mathcal{G}$-complex of relations, and $Q_j^m \in \mathbb{Q}^m$, $j \in \Lambda_m$, be an arbitrary quasi-simplex, represented by the sequence $Q_j^m = (x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_m})$, $0 \leq m \leq n$. We consider $\Lambda_m = card\ \mathbb{Q}^m$, $m = 0, 1, ..., n$.

For the sequence of indexes $(j_0,\ j_1,\ \ldots,\ j_m)$ we will denote by $t(j_0,\ j_1,\ \ldots,\ j_m)$ the number of all transpositions [40] of this sequence. We have to mentioned that if $j_r > j_s$, where $s > r$, and if after j_r there are some indexes equal to j_s then the number of transpositions grows with this number.

Definition 2.1. *If the number $t(j_0,\ j_1,\ \ldots,\ j_m)$ related to the quasi-simplex $Q_j^m \in \mathbb{Q}^m$ is* ***even*** *then the Q_j^m is said to be a* ***positively oriented*** *quasi-simplex and it is denoted by $+Q_j^m$. Otherwise, if $t(j_0,\ j_1,\ \ldots,\ j_m)$ is* ***odd*** *then the Q_j^m is said to be a* ***negatively oriented*** *quasi-simplex and it is denoted by $-Q_j^m$. The oriented quasi-simplex $Q_j^m \in \mathbb{Q}^m$ will be denoted by:*

$$Q_j^m = [x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_m}].$$

Now let $Q_{j_k}^{m-1} \in \mathbb{Q}^{m-1}$ be a quasi-simplex of dimension $m-1$ which is obtained from the quasi-simplex $Q_j^m \in \mathbb{Q}^m$ by the elimination of the vertex x_{j_k}. Thus $Q_{j_k}^{m-1}$ is a quasi-face of the simplex Q_j^m that is opposite to the vertex x_{j_k}. From these considerations the quasi-simplex $Q_{j_k}^{m-1}$ can be denoted as follows:

$$Q_{j_k}^{m-1} = [x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_{k-1}},\ x_{j_{k+1}},\ \ldots,\ x_{j_m}].$$

Definition 2.2. *Two quasi-simplexes $Q_j^m \in \mathbb{Q}^m$ and $Q_i^{m-1} \in \mathbb{Q}^{m-1}$ are called coherent if:*

1. *Q_i^{m-1} is a quasi-face of the quasi-simplex Q_j^m that is opposite to the vertex $x_{j_k} \in Q_j^m$, i.e. $Q_i^{m-1} = Q_{j_k}^{m-1}$;*

2. *the quasi-simplexes Q_j^m and $Q_i^{m-1} = Q_{j_k}^{m-1}$ are equally oriented, i.e. both are positively oriented or both are negatively oriented.*

If the second condition from Definition 2.2 does not hold then quasi-simplexes Q_j^m and $Q_i^{m-1} = Q_{j_k}^{m-1}$ are called non-coherent. In this case the quasi-simplexes Q_i^m and $(-1)Q_{j_k}^{m-1}$ are equally oriented.

We have to mention that according to the rule of sign determination of a quasi-simplex all quasi-simplexes with zero dimension are considered to be positively oriented [1, 13, 52, 57].

We will not extend the notions of coherence and non-coherence to the quasi-simplexes whose difference of dimensions is bigger than 1.

Definition 2.3. *The coefficient of* Δ*-**incidence** of the quasi-simplex* $Q_j^m \in \mathbb{Q}^m$, $0 \le m \le n$, *with respect to quasi-simplex* $Q_i^{m-1} \in \mathbb{Q}^{m-1}$ *is the following number*

$$\varepsilon_j^i(m,\ \Delta) = \begin{cases} +1, & \text{if } Q_j^m \text{ and } Q_i^{m-1} \text{ are coherent quasi-simplexes,} \\ -1, & \text{if } Q_j^m \text{ and } Q_i^{m-1} \text{ are non-coherent quasi-simplexes,} \\ 0, & \text{in the rest,} \end{cases}$$

where $i \in \Lambda_{m-1} = card\ \mathbb{Q}^{m-1}$, $j \in \Lambda_m = card\ \mathbb{Q}^m$.

The coefficient of Δ-incidence of quasi-simplexes Q_j^m and Q_i^{m-1} in the given order will be denoted by $\left[Q_j^m : Q_i^{m-1}\right] = \varepsilon_j^i(m,\ \Delta)$.

Definition 2.4. *The coefficient of* ∇*-**incidence** of quasi-simplex* $Q_j^m \in \mathbb{Q}^m$, $0 \le m \le n$, *with respect to quasi-simplex* $Q_l^{m+1} \in \mathbb{Q}^{m+1}$ *is the following number*

$$\varepsilon_j^l(m,\ \nabla) = \begin{cases} +1, & \text{if } Q_j^m \text{ and } Q_i^{m+1} \text{ are coherent quasi-simplexes,} \\ -1, & \text{if } Q_j^m \text{ and } Q_i^{m+1} \text{ are non-coherent quasi-simplexes,} \\ 0, & \text{in the rest,} \end{cases}$$

where $i \in \Lambda_m = card\ \mathbb{Q}^m$, $l \in \Lambda_{m+1} = card\ \mathbb{Q}^{m+1}$.

The coefficient of ∇-incidence of quasi-simplexes Q_j^m and Q_l^{m+1} in the given order, will be denoted by $\left[Q_j^m : Q_l^{m+1}\right] = \varepsilon_j^l(m,\ \nabla)$.

For the Δ-incidence and the ∇-incidence coefficients it is easy to verify the following relations:

$$\begin{aligned} \varepsilon_j^i(m,\ \Delta) &= \varepsilon_i^j(m-1,\ \nabla), \quad \text{where } 1 \le m \le n, \\ \varepsilon_j^l(m,\ \nabla) &= \varepsilon_l^j(m+1,\ \Delta), \quad \text{where } 0 \le m \le n-1, \end{aligned}$$

where $i \in \Lambda_{m-1} = card\ \mathbb{Q}^{m-1}$, $j \in \Lambda_m = card\ \mathbb{Q}^m$, $l \in \Lambda_l = card\ \mathbb{Q}^l$.

The concepts of Δ-incidence and ∇-incidence are met in bibliography [1,13,35,56] and they are dealing with different mathematical objects.

Remark 2.1. *The symbols* Δ *and* ∇ *are borrowed from works [14, 36], and in our opinion they are more convenient for further explanations.*

Remark 2.2. *Coefficient of incidence* $\left[Q_j^m : Q_i^{m-1}\right]$ *of the quasi-simplexes* $Q_j^m = [x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_m}]$ *and* $Q_i^{m-1} = [x_{i_0},\ x_{i_1},\ \ldots,\ x_{i_{m-1}}]$ *for which the sequence* $x_{i_0},\ x_{i_1},\ \ldots,\ x_{i_{m-1}}$ *if formed from the elements of* Q_j^m, *but it is not an hereditary subsequence of succession* $[x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_m}]$, *is equal to zero. In this case the quasi-simplex* Q_i^{m-1} *is not a quasi-face of the quasi-simplex* $Q_j^m = [x_{j_0},\ x_{j_1},\ \ldots,\ x_{j_m}]$.

Definition 2.5. *We consider the next two matrices for the* $\mathcal{G}$*-complex of relations* $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$:

1. *$I^m(\Delta) = (\varepsilon_j^i(m,\ \Delta))$, where i and j are the number of the lines and respectively of the columns of matrix $I^m(\Delta)$, and $i \in \Lambda_{m-1}$, $j \in \Lambda_m$, $1 \leq m \leq n$. This matrix is called the **matrix of Δ-incidence of dimension** m;*

2. *$I^m(\nabla) = (\varepsilon_j^l(m,\ \nabla))$, where l and j are the number of the lines and respectively of the columns of matrix $I^m(\nabla)$, and $j \in \Lambda_m$, $l \in \Lambda_{m+1}$, $0 \leq m \leq n-1$. This matrix is called the **matrix of ∇-incidence of dimension** m.*

The fact that the set X from which the $\mathcal{G}$-complex of multi-ary relations is constructed is a finite set implies that the matrices $I^m(\Delta)$, $I^m(\nabla)$ are also finite (see Definition 1.1). Without making any significant efforts the $\mathcal{G}$-complex could be defined over an infinite set X. In these conditions, obviously, the matrices will be infinite.

Assertion 2.1. *For the generalized complex of $\mathcal{K}^n$ relations we have that the pairs of matrices $I^m(\Delta)$, $I^{m-1}(\nabla)$, $1 \leq m \leq n$, and $I^m(\nabla)$, $I^{m+1}(\Delta)$, $0 \leq m \leq n-1$, are conjugated, i.e.:*

$$(I^m(\Delta))^* = I^{m-1}(\nabla),$$
$$(I^m(\nabla))^* = I^{m+1}(\Delta).$$

3 The homologies of $\mathcal{G}$-complex of multi-ary relations

Suppose a $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$ be given and $\mathbf{Z}$ is the additive group of integer numbers. By analogy with Definition 1.8, for the quasi-simplex $Q_j^m \in \mathcal{K}^n$ we will introduce the notion of **quasi-star** of Q_j^m. It will be the set of all quasi-simplexes from $\mathcal{K}^n$ for which Q_j^m is their quasi-face and it will be denoted by **qst** (Q_j^m), $0 \leq m \leq n$. For the quasi-simplex $Q_j^n \in \mathbb{Q}^n$ the set **qst** (Q_j^n) is empty.

The quasi-simplex $Q_j^m \in \mathbb{Q}^m$ contains $m+1$ faces of $m-1$ dimension, which we will denote by $Q_{\beta_0}^{m-1}, Q_{\beta_1}^{m-1}, ..., Q_{\beta_m}^{m-1}$. Let us consider that in the quasi-star **qst** (Q_j^m) there are some t quasi-simplexes of $m+1$ dimension for which Q_j^m is a hereditary face. These quasi-simplexes will be denoted by $Q_{\gamma_1}^{m+1}, Q_{\gamma_2}^{m+1}, ..., Q_{\gamma_t}^{m+1}$.

Definition 3.1. *The following sums:*

$$\Delta Q_j^m = \varepsilon_j^{\beta_0}(m,\ \Delta)Q_{\beta_0}^{m-1} + \varepsilon_j^{\beta_1}(m,\ \Delta)Q_{\beta_1}^{m-1} + \ldots + \varepsilon_j^{\beta_m}(m,\ \Delta)Q_{\beta_m}^{m-1}, \qquad (3.1)$$

where $1 \leq m \leq n$, $j \in \Lambda_m$, and

$$\nabla Q_j^m = \varepsilon_j^{\gamma_1}(m,\ \nabla)Q_{\gamma_1}^{m+1} + \varepsilon_j^{\gamma_2}(m,\ \nabla)Q_{\gamma_2}^{m+1} + \ldots + \varepsilon_j^{\gamma_t}(m,\ \nabla)Q_{\gamma_t}^{m+1}, \qquad (3.2)$$

*where $0 \leq m \leq n-1$, $j \in \Lambda_m$, will be called, respectively, the **Δ-border (algebraic border)** and **∇-border (co-border)** of the quasi-simplex $Q_j^m \in \mathbb{Q}^m$. Δ-border of*

Q_j^m will be denoted by ΔQ_j^m, and ∇-border of this quasi-simplex will be denoted by ∇Q_j^m, $0 \le m \le n$.

We consider $\Delta Q_i^0 = 0$ and $\nabla Q_j^n = 0$ for each quasi-simplex $Q_i^0 \in \mathbb{Q}^0$ and for each quasi-simplex $Q_j^n \in \mathbb{Q}^n$, where $i \in \Lambda_0 = card\ X$, $j \in \Lambda_n = card\mathbb{Q}^n$.

The formulas (3.1) and (3.2) can be simplified. For example, let the quasi-simplex $Q_j^m = (x_{j_0}, x_{j_1}, \ldots, x_{j_k}, \ldots, x_{j_m})$ be represented by the corresponding indexes: $Q_j^m = (j_0, j_1, \ldots, j_k, \ldots, j_m)$, and let its quasi-faces of $(m-1)$ dimension $Q_{j_k}^{m-1}$ of Q_j^m be the opposite face to the vertex j_k, $0 \le k \le m$. Then, according to the definition of coherence of quasi-simplexes Q_j^m and $(-1)^k Q_{j_k}^{m-1}$, where $(-1)^k = \varepsilon_j^k(m, \Delta)$, the sum (3.1) can be written as follows:

$$\Delta Q_j^m = (-1)^0 Q_{\beta_0}^{m-1} + (-1)^1 Q_{\beta_1}^{m-1} + \ldots + (-1)^k Q_{\beta_k}^{m-1} + \ldots + (-1)^m Q_{\beta_m}^{m-1}. \quad (3.1')$$

As in [32, 52] it could be proved that $\Delta\Delta Q_j^m = 0$, $\nabla\nabla Q_j^m = 0$, for any quasi-simplex $Q_j^m \in \mathbb{Q}^m$, $0 \le m \le n$.

Other coefficients of Δ-incidence, which do not occur in the sum (3.1′), are equal to zero by Definition 2.2.

The advantages of the formulas (3.1) and (3.2) will be applied below. Now let $f : \mathcal{K}^n \to \mathbf{Z}$ be an unequivocal application of the complex of relations $\mathcal{K}^n$ to the group of integer numbers. For a negatively oriented (see Definition 2.1) simplex $Q^m \in \mathbb{Q}^m$ we agree to write $f_m(-Q^m) = -f_m(Q^m)$. Let us denote $f(Q_i^m) = p_i$, where $p_i \in \mathbf{Z}$, for any $Q_i^m \in \mathbb{Q}^m$. For simplicity and for keeping in mind the pro-image of p_i, instead of $f(Q_i^m) = p_i$ we will write $p_i Q_i^m$, $0 \le m \le n$.

Definition 3.2. *For the family of quasi-simplexes $\mathbb{Q}^m = \{Q_1^m, Q_2^m, ..., Q_{\alpha_m}^m\}$, $0 \le m \le n$, the finite sum*

$$p_1 Q_1^m + p_2 Q_2^m + ... + p_{\alpha_m} Q_{\alpha_m}^m \quad (3.3)$$

*is called the **m-dimensional chain** of the $\mathcal{G}$-complex of relations $\mathcal{K}^n$ and it is denoted by L^m. The set of all chains L^m will be denoted by $\mathcal{L}^m$, $0 \le m \le n$.*

Let $L_1^m = \sum_{Q_i^m \in \mathbb{Q}^m} p_i^1 Q_i^m$ and $L_2^m = \sum_{Q_i^m \in \mathbb{Q}^m} p_i^2 Q_i^m$ be two m-dimensional chains of the $\mathcal{G}$-complex $\mathcal{K}^n$.

Definition 3.3. *The relation*

$$L_1^m + L_2^m = \sum_{Q_i^m \in \mathbb{Q}^m} (p_i^1 + p_i^2) Q_i^m \quad (3.4)$$

*is called the **sum of the chains** L_1^m and L_2^m.*

Next theorem is obvious.

Theorem 3.1. *The set $\mathcal{L}^m$ of all m-dimensional chains of a $\mathcal{G}$-complex of multiary relations $\mathcal{K}^n$ with the addition operation defined by the relation* (3.4) *forms a commutative group.*

The group of m-dimensional Δ-chains will be denoted by $\Delta\mathcal{L}^m$.

Definition 3.4. *For a finite m-dimensional chain $L^m \in \mathcal{L}^m$, $0 \leq m \leq n$, the equality*

$$\Delta L^m = \sum_{Q_i^m \in \mathbb{Q}^m} p_i \Delta q_i^m \tag{3.5}$$

is called the $\Delta-$**algebraic border** *of the chain L^m.*

In case $m = 0$, according to Definition 3.1, it follows $\Delta L^0 = 0$.

From all mentioned above, each chain $L^m \in \mathcal{L}^m$ will be also called Δ-*chain*.

In classical literature the terminology mentioned in [32] is also applied.

Remark 3.1. *The operation of defining a Δ-algebraic border of the Δ-chain $L^m \in \mathcal{L}^m$ is a homomorphism [40]:*

$$\Delta(m) : \mathcal{L}^m \to \mathcal{L}^{m-1}, \ \ 1 \leq m \leq n.$$

It is natural to call this homomorphism a Δ-*homomorphism*. This is even necessary, since we will use also other homomorphism, which signification is different. The mentioned Δ-homomorphism can be obtained by applying the respective operations for creating the Δ-border, for Q_i^m (see $(3.1')$), $Q_i^m \in \mathbb{Q}^m$, and if we consider $\Delta Q_i^m = L^{m-1} \in \mathcal{L}^{m-1}$, $0 \leq m \leq n$. We will denote by $Im\Delta(m)$ the *image*, and by $Kern\Delta(m)$ the *kernel* of the homomorphism $\Delta(m)$ [40].

Theorem 3.2. *For each $L^m \in \mathcal{L}^m$ the following equality holds*

$$\Delta\Delta L^m = 0, \ \ m = 0,\ 1,\ \ldots,\ n.$$

The proof of this theorem can be done exactly as in [32] applying the known relation $\Delta\Delta Q_i^m = 0$ (see Definition 3.1).

From the proof of Theorem 3.2 we obtain

Assertion 3.1. *For a $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ the next equality holds*

$$I^{m-1}(\Delta) \cdot I^m(\Delta) = 0, \tag{3.6}$$

$1 \leq m \leq n$.

Definition 3.5. *The chain $L^m \in \mathcal{L}^m$ with the property $\Delta L^m = 0$ is said to be the* Δ-***cycle*** *of dimension m of the $\mathcal{G}$-complex $\mathcal{K}^n$, and it is denoted by $Z^m(\Delta) = L^m$, $0 \leq m \leq n$.*

Let

$$L_1^m = p_1^1 Q_1^m + p_2^1 Q_2^m + \ldots + p_{\alpha_m^1}^1 Q_{\alpha_m^1}^m$$

and

$$L_2^m = p_1^2 Q_1^m + p_2^2 Q_2^m + \ldots + p_{\alpha_m^2}^2 Q_{\alpha_m^2}^m$$

be two chains of dimension m, $\alpha_m = card\mathbb{Q}^m$.

Is obvious

Theorem 3.3. *With respect to the addition of Δ-chains, the set of all Δ-cycles of dimension m forms a commutative subgroup of the group $\mathcal{L}^m$.*

We will denote the subgroup of Δ-cycles of $\Delta\mathcal{L}^m$ by $\mathbb{Z}^m(\Delta)$, $0 \leq m \leq n$.

Definition 3.6. *If there exist two Δ-chains $L^m \in \mathcal{L}^m$ and $L^{m+1} \in \mathcal{L}^{m+1}$ with properties:*

a) $L^m = \Delta L^{m+1}$;

b) $\Delta L^m = 0$,

then L^m is called Δ-cycle of dimension m ***Δ-homological with 0****. In this case we will use the notation $L^m = Z^m(\Delta) \sim 0$. Two Δ-cycles $Z_1^m(\Delta)$ and $Z_2^m(\Delta)$ that belong to $\mathcal{Z}^m(\Delta)$ are said to be* ***Δ-homologous*** *if $Z_1^m(\Delta) - Z_2^m(\Delta) \sim 0$, $0 \leq m \leq n$ [1, 13, 59, 63, 70].*

The fact that ΔL^m represents a Δ-cycle homological with 0 shows the situation that in $\mathcal{K}^n$ the chain ΔL^m bounds in the m-dimensional skeleton of $\mathcal{K}^n$ a $\mathcal{G}$-subcomplex of the $\mathbf{sk(m)}\mathcal{K}^n$.

The following theorem is obvious.

Theorem 3.4. *The set of all m-dimensional Δ-cycles that are Δ-homologous with 0, with respect to the additive operation defined in $\Delta\mathcal{L}^m$, forms a subgroup of the group $\mathbb{Z}^m(\Delta)$. We denote this group by $\mathbb{Z}_0^m(\Delta)$.*

We obtain the existence of the group $\mathbb{Z}_0^m(\Delta)$ from Theorem 3.2. It is obvious that $\mathbb{Z}_0^n(\Delta) \sim 0$, i.e. in a $\mathcal{G}$-complex of relations $\mathcal{K}^n$ Δ-chains of dimensions $n+1$ do not exist.

Definition 3.7. *The factor-group $\mathbb{Z}^m(\Delta)/\mathbb{Z}_0^m(\Delta)$ of the $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ is called* ***the group of Δ-homologies (the group of direct homologies)*** *of dimension m over the group $\mathbf{Z}$, and it is denoted by $\Delta^m(\mathcal{K}^n)$, $0 \leq m \leq n$. The ranks of these groups are called* ***Betti numbers****. In works [1, 32, 52, 56] these groups are denoted by $\mathcal{H}_m(\mathcal{K}^n)$, $0 \leq m \leq n$.*

Obviously we can write it as follows [32]:

$$\Delta^m(\mathcal{K}^n) = Kern\Delta(m)/Im\Delta(m), \;\; 1 \leq m \leq n.$$

Our goal is to narrow the concept of complex of relations like in [17], but not so much as it is done in [58, 59]. It is worth to mention that all results could be also generalized in the case of a more general complex of relations, which is without any restrictions on dimension (by analogy see [54, 55]).

Further applying the group of integer numbers $\mathbf{Z}$, we form the so-called groups of co-homologies [13, 32] of $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$, with restriction that the chain $L^m \subset \mathcal{L}^m$ represents, obviously, a finite sum. From now this chain will be called ∇-chain (co-chain).

Simplifying the notations, we introduce the notion of ∇-chain (**co-chain**), which coincides with the notion of the chain with the respective dimension.

Definition 3.8. *For a ∇-chain $L^m \in \mathcal{L}^m$, $0 \leq m \leq n$, the equality*

$$\nabla L^m = \sum_{Q_i^m \in \mathbb{Q}^m} p_i \nabla Q_i^m$$

*is called the $\nabla-$**algebraic border (co-border)** of the chain ∇L^m. In case $m = n$ we consider $\nabla L^n = 0$.*

Remark 3.2. *The operation of creating a ∇-border of the ∇-chain $L^m \in \mathcal{L}^m$ is a homomorphism:*

$$\nabla(m) : \mathcal{L}^m \to \mathcal{L}^{m+1}, \;\; 0 \leq m \leq n-1.$$

We will denote by $Im\nabla(m)$ the *image*, and by $Kern\nabla(m)$ the *kernel* of ∇-homomorphism $\nabla(m)$, $0 \leq m \leq n-1$.

Definition 3.9. *The ∇-chain $L^m \in \mathcal{L}^m$ with the property $\nabla L^m = 0$ is said to be the ∇-**cycle** of dimension m of the $\mathcal{G}$-complex of $\mathcal{K}^n$ relations, and it is denoted by $Z^m(\nabla) = L^m$, $0 \leq m \leq n$.*

$\nabla L^n = Z^n(\nabla)$ is a ∇-cycle of dimension n, according to Definition 3.8. If we introduce the notion of sum of ∇-chains, we obtain

Theorem 3.5. *The set of all ∇-cycles of dimension m with respect to the addition of ∇-chains forms a commutative subgroup of the group $\nabla\mathcal{L}^m$.*

This subgroup will be denoted by $\mathbb{Z}^m(\nabla)$, $0 \leq m \leq n$.

Definition 3.10. *If there exist two Δ-chains $L^m \in \mathcal{L}^m$ and $L^{m-1} \in \mathcal{L}^{m-1}$ with the properties:*

a) $L^m = \nabla L^{m-1}$;

b) $\nabla L^m = 0$,

*then L^m is said ∇-cycle of dimension m being ∇-**homological with 0**. In this case we will use the (usual) notation [32]:*

$$L^m = Z^m(\nabla) \sim 0.$$

*Two cycles $Z_1^m(\nabla)$ and $Z_2^m(\nabla)$ from $\mathbb{Z}^m(\nabla)$ are said to be ∇-**homologous** if*

$$Z_1^m(\nabla) - Z_2^m(\nabla) \sim 0, 1 \leq m \leq n.$$

Is obvious

Theorem 3.6. *With respect to the addition defined in $\nabla\mathcal{L}^m$, the set of all m-dimensional ∇-cycles ∇-homological with zero, forms a subgroup of the group $\mathbb{Z}^m(\nabla)$ and will be denoted by $\mathbb{Z}_0^m(\nabla)$.*

Definition 3.11. *The factor-group $\mathbb{Z}^m(\nabla)/\mathbb{Z}_0^m(\nabla)$ of the $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ is called **the group of** ∇-**homologies** (simply **co-homologies**) of dimension m over the group $\mathbf{Z}$ of integers, and it is denoted by $\nabla_m(\mathcal{K}^n)$, $0 \leq m \leq n$.*

Remark 3.3. *We remind that $\nabla_m(\mathcal{K}^n)$, $0 \leq m \leq n$, is defined considering the finite ∇-chains of $\mathcal{K}^n$.*

Obviously we can, also, write it as follows:

$$\nabla_m(\mathcal{K}^n) = Kern\nabla(m)/Im\nabla(m), \;\; 0 \leq m \leq n.$$

Let us remember that the existence of the group $\mathbb{Z}_0^m(\Delta)$, $0 \leq m \leq n$, of a $\mathcal{G}$-complex of multi-ary relations results from the equality $\Delta\Delta L^m = 0$, $0 \leq m \leq n$. This situation generates the necessity to formulate some additional results.

Theorem 3.7. *For each ∇-chain $L^m \in \nabla\mathcal{L}^m$ of the $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ the next equality holds*

$$\nabla\nabla L^m = 0, \;\; 0 \leq m \leq n.$$

The proof of Theorem 3.7 is done exactly as in [32], by applying the known relation $\nabla\nabla Q_i^m = 0$ (see Definition 3.1).

Remark 3.4. *For the groups of direct homologies and co-homologies of a $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$, the procedure of orientation of its quasi-simplexes is an auxiliary problem and it does not depend on the structure of these groups [13, 32, 52].*

Definition 3.12. *A $\mathcal{G}$-complex of multi-ary relations $n \geq 1$ is called **acyclic** if*

$$\Delta^1(\mathcal{K}^n) = \Delta^2(\mathcal{K}^n) = \ldots = \Delta^n(\mathcal{K}^n) \cong 0.$$

In order to formulate the next theorem we need some classical notions (see [13, 52]), and we will prove an auxiliary lemma (we repeat it for our abstract case).

Let $\mathbb{Q}^0 = \{Q_1^0, \; Q_2^0, \ldots, \; Q_{\alpha_0}^0\}$ be a family of 0-dimensional simplexes (vertices) of a $\mathcal{G}$-complex of multi-ary relations, and let $L^0 = p_1Q_1^0 + p_2Q_2^0 + \ldots + p_{\alpha_0}Q_{\alpha_0}^0$ be an arbitrary Δ-chain from $\Delta\mathcal{L}^0$.

Definition 3.13. *The operator*

$$I : L^0 \to \mathbb{Z},$$

*with the property $I(L^0) = p_1 + p_2 + \ldots + p_{\alpha_0}$, is called the **index** of Δ-chain L^0.*

It is obvious that for any two arbitrary Δ-chains L_1^0, $L_2^0 \in \mathcal{L}^0$ is true the relation

$$I(L_1^0 + L_2^0) = I(L_1^0) + I(L_2^0). \tag{3.7}$$

Lemma 3.1. *If $\Delta\mathcal{L}^0$ is the group of 0-dimensional Δ-chains of a $\mathcal{G}$-complex of relations $\mathcal{K}^n = (\mathbb{Q}^0, \; \mathbb{Q}^1, \; \ldots, \; \mathbb{Q}^n)$, then each chain $L^0 \in \Delta\mathcal{L}^0$ is homological with 0 if and only if $I(L^0) = 0$.*

Proof. Let $Q^1 \in \mathbb{Q}^1$ be an arbitrary quasi-simplex, positively oriented and represented by pair $Q^1 = (Q_i^0,\ Q_j^0)$, where $Q_i^0,\ Q_j^0 \in \mathbb{Q}^0$, $i \neq j$. In this case $\Delta(p \cdot Q^1) = p\Delta Q_j^0 - p\Delta Q_i^0$, it follows we have $I(\Delta(p \cdot Q^1)) = 0$. According to relation (3.7), for each $Z^0(\Delta) \in \mathbb{Z}^0(\Delta)$ we obtain $I(Z^0(\Delta)) = 0$, so the necessary condition of lemma is true.

Now prove the reverse assertion. From connectedness of the $\mathcal{G}$-complex $\mathcal{K}^n$ we obtain that for each two elements $Q_i^0,\ Q_j^0 \in \mathbb{Q}^0$, there exists a sequence of 1-dimensional quasi-simplexes $Q_{i_1}^1,\ Q_{i_2}^1,\ \ldots, Q_{i_t}^i$, for which $Q_{i_k}^1$ and $Q_{i_{k+1}}^1$, $1 \leq k \leq t-1$, are adjacent and for which the origin of $Q_{i_1}^1$ coincides with Q_i^0, and extremity of $Q_{i_1}^1$ coincides with Q_j^0. More over, the elements of this sequence can be oriented in such a way that all of them be positively oriented in the Δ-chain $L^1 = pQ_{i_1}^1 + pQ_{i_2}^1 + \ldots + pQ_{i_t}^1$, where $p \in \mathbf{Z}$ is also positive. We observe that $\Delta L^1 = pQ_j^0 - pQ_i^0$. Thus, according to (3.7), $pQ_j^0 \sim pQ_i^0$. This leads to the relation $L^1 \in \Delta\mathcal{L}^1$, which means that Δ-chain is homological with pQ_i^0. Then, because L^1 is homological with pQ_i^0, it results $I(L^1) = p$. By this way we have the relation $L^1 \sim I(L^1)Q_i^0$, from where results that if $I(L^1) = 0$, then $L^1 \sim 0$. □

We will just remind the next statement because it is classic and it is abstractly proved.

Theorem 3.8. *If $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$ is connected, then $\Delta^0(\mathcal{K}^n)$ is isomorphic with the group of integer numbers* $\mathbf{Z}$.

Remark 3.5. *If a $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ satisfies equality* (1.1)*, then*

$$\Delta^m(\mathcal{K}^n) \cong \Delta^m(\mathcal{K}_1^n) \oplus \Delta^m(\mathcal{K}_2^n) \oplus \ldots \oplus \Delta^m(\mathcal{K}_q^n),$$

where $0 \leq m \leq n$. If $m = 0$ then

$$\Delta^0(\mathcal{K}^n) \cong \underbrace{\mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}.}_{q\ times} \tag{3.8}$$

Definition 3.14. *A connected and acyclic $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n$ is called an **oriented n-tree** of multi-ary relations.*

The importance of these notions will be showed in further researches, when the results mentioned in the beginning of first section will be elaborated. For $n = 1$ the respective construction represents an oriented, connected graph without cycles. So, if $\mathcal{K}^1$ is connected then this represents an oriented tree [7].

Definition 3.15. *If $\mathcal{K}^n$ is a locally complete $\mathcal{G}$-complex of multi-ary relations and it is transformed into an abstract quasi-simplicial complex, according to those mentioned in §1, then $\mathcal{K}^n$ will be called a **symmetric** $\mathcal{G}$-complex of multi-ary relations.*

If it will be necessary, each element of $\mathbb{Q}^m$, $0 \leq m \leq n$, can be oriented, according to § 2.

In § 1, the notion of Euler characteristic of complex of multi-ary relations $\mathcal{K}^n$ was introduced (see the relation 1.3).

The importance of this characteristic is well known [52,60]. Let $\rho_m(\Delta^m(\mathcal{K}^n))$ be the rank of the group $\Delta^m(\mathcal{K}^n)$ of $\mathcal{K}^n$.

$$\chi(\mathcal{K}^n) = \sum_{m=0}^{n} (-1)^m \alpha_m \tag{3.9}$$

is the Euler characteristic (see (1.4)), where α_m means now the number of quasi-simplexes of dimension m of $\mathcal{G}$-complex $\mathcal{K}^n$, $0 \leq m \leq n$.

Then, according to (3.9), we have an analogical result to that obtained by Poincare and Kolmogorov [13,63].

Theorem 3.9 (**Euler-Kolmogorov**). *For any $\mathcal{G}$-complex of multi-ary relations $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$ the equality holds*

$$\chi(\mathcal{K}^n) = \sum_{m=0}^{n} (-1)^m \rho_m(\Delta^m(\mathcal{K}^n)).$$

The proof is done exactly as in works [52,58].

At the end of this section we propose a definition and an important assertion.

Let $\mathcal{K}^n = (\mathbb{Q}^0,\ \mathbb{Q}^1,\ \ldots,\ \mathbb{Q}^n)$ be a complex. Analyze the complex $\mathcal{K}^n_d = (\mathbb{Q}^0_d,\ \mathbb{Q}^1_d,\ \ldots,\ \mathbb{Q}^n_d)$, where every m-dimensional abstract quasi-simplex Q^m_i is considered to be a cell complex [32] with the dimension $n-m$, denoted by $Q^{n-m}_{d_i}, 0 \leq m \leq n$, $i \in \Lambda_m$. All these complexes with the dimension $n-m$ are denoted by $\mathbb{Q}^{n-m}_d, 0 \leq m \leq n$, and they of course respect the incidences.

Definition 3.16. *The abstract complex $\mathcal{K}^n_d$ constructed above is called the **dual complex of** K^n.*

For example, a zero-dimensional simplex, which is incident with n simplexes of $(n-1)$ dimension of a n-dimensional simplex, represents a CW with dimension n. It has the form of a n-dimensional simplex, and it has as cells the set of all faces, including one improper face (the interior of n-dimensional simplex).

Assertion 3.2. *The complex $\mathcal{K}^n_d$ is a connected cell-complex (CW).*

Let

$$\mathcal{H}^0_d(\mathcal{K}^n_d, \mathbf{Z}), \mathcal{H}^1_d(\mathcal{K}^n_d, \mathbf{Z}), ..., \mathcal{H}^n_d(\mathcal{K}^n_d, \mathbf{Z}) \tag{3.10}$$

be the group succession of the direct homologies of $\mathcal{K}^n_d$. If we examine more carefully the sequence (3.10) then we have:

Theorem 3.10. *For $\mathcal{K}^n_d$ complex the following relations are true:*

$$\begin{aligned} \nabla_0(\mathcal{K}^n, \mathbf{Z}) &\cong \mathcal{H}^0_d(\mathcal{K}^n_d, \mathbf{Z}), \\ \nabla_1(\mathcal{K}^n, \mathbf{Z}) &\cong \mathcal{H}^1_d(\mathcal{K}^d_n, \mathbf{Z}), \\ &\cdots \\ \nabla_n(\mathcal{K}^n, \mathbf{Z}) &\cong \mathcal{H}^n_d, \mathbf{Z}). \end{aligned} \tag{3.11}$$

According to this theorem we obtain that the Klomogorov-Alexander theorem [13, 32], which is about the duality of groups of homologies and co-homologies for topological spaces, is also valid in the case of a generalized complex of multi-ary relations $\mathcal{K}^n$. Thus, we obtain:

Theorem 3.11. *For the complex $\mathcal{K}^n$ the following equalities are true:*

$$\begin{aligned} \Delta^0(\mathcal{K}^n, \mathbf{Z}) &\cong \nabla_n(\mathcal{K}^n, \mathbf{Z}), \\ \Delta^1(\mathcal{K}^n, \mathbf{Z}) &\cong \nabla_{n-1}(\mathcal{K}^n, \mathbf{Z}), \\ &\dots \\ \Delta^n(\mathcal{K}^n, \mathbf{Z}) &\cong \nabla_0(\mathcal{K}^n, \mathbf{Z}). \end{aligned} \tag{3.12}$$

Remark 3.6. *It was possible not to use CW. But in this case it was necessary to define what represents an abstract n-dimensional polyhedron (the border of which is an abstract $(n-1)$-dimensional sphere). This thing is a niggling question, and we leave it for the moment without any attention. Further we will be interested only in the definition of the abstract n-dimensional cube.*

4 The complex of abstract cubes

As it was mentioned above any sequence $\left(x_{i_1}, x_{i_2}, ..., x_{i_{m+1}}\right) \in R^{m+1}$ that does not contain repetition of elements could be considered an abstract m-dimensional simplex, $0 \leq m \leq n$. Therefore, each complex of multi-ary relations $\mathcal{K}^n$ can be also regarded as a complex of abstract simplexes

$$\mathcal{K}^n = \left\{\mathbb{S}^0, \mathbb{S}^1, ..., \mathbb{S}^n\right\},$$

where $\mathbb{S}^m$ represents the set of all m-dimensional simplexes that are determined by the elements from R^{m+1} and any two simplexes $S^l \in \mathbb{S}^l$ and $S^k \in \mathbb{S}^k$, $0 \leq l,\ k \leq n$, have either an empty intersection or their intersection is a p-dimensional simplex $S^p \in \mathbb{S}^p$, $p \leq \min\{l, k\}$. Using abstract simplexes the abstract n-dimensional cube [15, 19] and the abstract complex of cubes are defined.

We will use the inductive definition of the abstract n-dimensional cube and its vacuum.

Definition 4.1.

1. *The abstract* 0*-dimensional cube and the abstract* 1*-dimensional cube coincide with the abstract simplexes of the same dimension. The vacuums of these cubes coincide with the vacuums of the respective simplexes.*

2. *Consider two pairs of* 0*-dimensional cubes S_1^0, S_2^0 and S_3^0, S_4^0. The 2-ary relations of these two pairs of cubes S_1^0, S_2^0 and S_3^0, S_4^0 determine the existence of* 1*-dimensional cubes $S_1^1 = (S_1^0,\ S_2^0)$, $S_2^1 = (S_3^0,\ S_4^0)$, $S_3^1 = (S_1^0,\ S_3^0)$ and*

$S_4^1 = (S_2^0,\ S_4^0)$. Form the 2- and 3-ary relations between these pairs of cubes. These determine a simplicial complex that is formed from the new abstract simplexes $S_5^1 = (S_1^0,\ S_4^0)$, $S_1^2 = (S_1^0,\ S_3^0,\ S_4^0)$, $S_2^2 = (S_1^0,\ S_2^0,\ S_4^0)$ and the four 1-dimensional simplexes mentioned above S_1^1, S_2^1, S_3^1, S_4^1. The union of vacuums $\overset{0}{S}{}_1^2$, $\overset{0}{S}{}_2^2$ and $\overset{0}{S}{}_5^1$ of new obtained simplexes is called the vacuum of the 2-dimensional abstract cube and it will be denoted:

$$\overset{0}{I}{}^2 = \overset{0}{S}{}_1^2 \cup \overset{0}{S}{}_2^2 \cup \overset{0}{S}{}_5^1.$$

The abstract 2-dimensional cube is denoted by I^2, and it is defined by its vacuum as follows:

$$I^2 = \bigcup_{i=1}^{4} S_i^1 \cup \overset{0}{I}{}^2.$$

The complex determined by the mentioned family of simplexes is said to be a 2-dimensional pro-cube. The 2-dimensional pro-cube will be denoted by $I^2(\Delta)$.

3. *Suppose that the notion of* ***abstract*** i***-dimensional cube*** *and* ***pro-cube*** *is known I^i and $I^i(\Delta)$, $1 \le i \le n-1$, as well as the notion of cube's vacuum with the same dimension is known.*

4. *We will define the notion of abstract n-dimensional cube by using the notion of $(n-1)$-dimensional cube. Consider $2n$ copies of $(n-1)$-dimensional cube I_1^{n-1}, I_2^{n-1}, …, I_{2n}^{n-1} with the corresponding pro-cubes $I_1^{n-1}(\Delta)$, $I_2^{n-1}(\Delta)$, …, $I_{2n}^{n-1}(\Delta)$. Let us have i-ary relations, $2 \le i \le n$, among the 0-dimensional cubes that determine some simplexes which form an abstract simplicial complex.* ***The vacuum of the*** n***-dimensional cube****, denoted by $\overset{0}{I}{}^n$, represents the union of all vacuums of the simplexes that do not intersect the pro-cubes $I_j^{n-1}(\Delta)$, $1 \le j \le 2n$.* ***The abstract*** n***-dimensional cube*** *is defined by using its vacuum in the following way: $I^n = \bigcup_{i=1}^{2n} I_i^{n-1} \cup \overset{0}{I}{}^n$. The simplicial complex $I^n(\Delta)$ will be called* ***pro-cube*** *of I^n.*

Definition 4.2. *[19] An abstract cube I^m, which according to Definition 4.1 is used by the I^n cube formation, $0 \le m \le n-1$, is called an* ***own face*** *of the n-dimensional cube I^n.*

Definition 4.3. *The family of non-empty and finite abstract cubes $\mathcal{I}^n = \{I^m,\ 0 \le m \le n\}$ is said to be an* ***abstract cubic complex*** *with dimension n if the following properties are true:*

1. *for $\forall I^s,\ I^t \in \mathcal{I}^n$, $0 \le s,\ t \le n$, the relation $I^s \cap I^t \in \mathcal{I}^n$ or $I^s \cap I^t = \emptyset$ is true;*
2. *each face I^k of any $I^n \in \mathcal{I}^n$, $0 \le k < n$, is an element from $\mathcal{I}^n$;*
3. *$\exists I^n \in \mathcal{I}^n$.*

By analogy with the orientation of quasi-simplexes the orientation of a n-dimensional cube I^n, that is constructed on a set of vertices $X = \{x_1, x_2, \ldots, x_{2^m}\}$ is defned. The positively oriented cube is denoted by $+I^n$, and the negatively oriented cube is denoted by $-I^n$.

Let us consider an unequivocal application $h : \mathcal{I}^n \to \mathbf{Z}$, with the property: if $I_i^m \in \mathcal{I}^n$, $0 \leq m \leq n$ is an abstract negative oriented cube, then $h(-I_i^m) = -h(I_i^m)$. We consider the image $h(I_i^m) = g_i \in \mathbf{Z}$. For convenience, instead of $h(I_i^m)$ we'll use the notation $g_i I_i^m$, and instead of $h(-I_i^m)$ we'll write $-g_i I_i^m$.

The following notions allow, by analogy with the elements of the complex of multi-ary relations, to define the coherence and non-coherence of the abstract cube $I^m \in \mathcal{I}^n$ and cubic's variety.

Definition 4.4. *The sum of all m-dimensional cubes of the cubic complex $\mathcal{I}^n$*

$$L^m = g_1 I_1^m + g_2 I_2^m + ... + g_{\beta_m} I_{\beta_m}^m, 2 \leq m \leq n, \tag{4.1}$$

*where β_m represents the cardinal of the set of all abstract cubes of dimension m from I^n, is called a m-**dimensional** $\square$-**chain** of cubes of the $\mathcal{I}^n$ complex.*

Definition 4.5. *The sum of two $\square$-**chains** of abstract cubes $L_1^m = \sum\limits_{i=1}^{\beta_m} g_i^1 I_i^m$ and $L_2^m = \sum\limits_{i=1}^{\beta_m} g_i^2 I_i^m$, is the expression:*

$$L_1^m + L_2^m = \sum_{i=1}^{\beta_m} \left(g_i^1 + g_i^2\right) I_i^m. \tag{4.2}$$

It is easy to verify

Theorem 4.1. *With respect to the operation 4.2, the set of all m-dimensional $\square$-chains, denoted by $\mathcal{L}_\square^m$, $0 \leq m \leq n$, of cubic complex $\mathcal{I}^n$ forms an abelian group.*

We denote by $\mathbb{I}^m$ the family of all m-dimensional cubes of cubic complex $\mathcal{I}^n$, $0 \leq m \leq n$. We have to mention that each cube $I^m \in \mathbb{I}^m$ has m pairs of opposite faces with $m-1$ dimension. Let $I_{i_j 0}^{m-1}$ and $I_{i_j 1}^{m-1}$ be opposite faces of an m-dimensional cube $I_i^m \in \mathcal{I}^n$, $j \in \{0, 1, ..., m-1\}$.

Definition 4.6. *For the m-dimensional cube $I_i^m \in \mathcal{I}^n$, $1 \leq m \leq n$ the expression*

$$\square I_i^m = \sum_{j=0}^{m-1} (-1)^j \left(I_{i_j 0}^{m-1} - I_{i_j 1}^{m-1}\right)$$

*is called $\square$-**border** of I_i^m cube.*

As has been done in the case of generalized complex of multi-ary relations, the notion of coherent and non-coherent abstract cubes is defined, as well as the notion of coefficients of incidence.

By analogy with the classic situation we define cubic homologies.

Definition 4.7. *$\square$-**border** of the chain $L^m \in \mathcal{L}^m_\square$ is the following sum*

$$\square L^m = \sum_{i=1}^{\beta_m} g_i \square I_i^m, \ g_i \in Z, \ 1 \le i \le \beta_m.$$

*The $\square$-chain L^m, for which $\square L^m = 0$ is called $\square$-**cycle** of dimension m.*

Theorem 4.2. *Let $L^m \in \mathcal{L}^m_\square$ be a $\square$-chain. The following equality is true:*

$$\square\square L^m = 0.$$

There are two types of $\square$-cycles of cubic complex $\mathcal{I}^n$:

1. m-dimensional *square*-cycles which represent a $\square$-border of a $\square$-chain L^{m+1};
2. m-dimensional *square*-cycles which do not represent a $\square$-border of a $\square$-chain L^{m+1}.

With respect to the addition for chains, the set of all $\square$-chains of dimension m of $\mathcal{I}^n$, $0 \le m \le n$, which verify the B property from above, form an abelian group. This is denoted by $\mathcal{Z}^m(\square) \subset \mathcal{L}^m_\square$, $m \in \{0, 1, ..., n\}$.

With respect to the same addition, the set of all $\square$-chains of dimension m of the complex $\mathcal{I}^n$, $0 \le m \le n$, which verify the A property from above, form an abelian group. It is denoted $\mathcal{Z}_0^m(\square) \subset \mathcal{L}^m_\square$, $m \in \{0, 1, ..., n\}$.

Definition 4.8. *The factor-group $\mathcal{Z}^m(\square)/\mathcal{Z}_0^m(\square)$ is said to be **the group of $\square$-homologies of dimension** m of the cubic complex $\mathcal{I}^n$, and it is denoted by $\square^m(\mathcal{I}^n, \mathbf{Z})$, $0 \le m \le n$.*

The following conditions are true

$$\square^0 \left(\mathcal{I}^n, \ \mathbb{Z}\right) \cong \mathbb{Z}$$

$$\square^1 \left(\mathcal{I}^n, \ \mathbb{Z}\right) \cong \square^2 \left(\mathcal{I}^n, \ \mathbb{Z}\right) \cong \ldots \cong \square^n \left(\mathcal{I}^n, \ \mathbb{Z}\right) \cong 0,$$

means that the complex of abstract cubes $\Im^n$ is connected and acyclic. For such a complex an analogue result to the Helly theorem [14] can be formulated.

Let $\Im^n$ be a family of n-dimensional cubic complexes that are connected and acyclic.

Hypothesis. *If for any two complexes $\mathcal{I}^1, \mathcal{I}^2 \in \Im^n$ the condition*

$$\mathcal{I}^1 \cap \mathcal{I}^2 \in \Im^n$$

is true, then the intersection of all cubic complexes from $\Im^n$ is not empty and it represents an abstract n-dimensional cubic complex.

5 Applicative aspects

As it was mentioned in the beginning, the results presented in this article were obtained long ago but they weren't presented for the publication because the authors were aware of the possible applicative ideas. And only, when those ideas took shape, and the results were obtained (the articles are being prepared for the publications), we decided to present the main theoretical "trunk", giving in a brief form three nontrivial applicative aspects.

5.1 Application in the hypergraphs theory

Hypergraphs represent some discrete mathematical structures that, according to those mentioned above, can be regarded as particular cases of $\mathcal{G}$-complex of multi-ary relations and which are used in solving many theoretical and applicative problems.

The notion of hypergraph in Berge sense [6] is known: a hypergraph is the pair $H = (X;\ E)$, where X is a labeled set of vertices, and E is a set of edges that contains at least two vertices. For this hypergraph H, a succession of edges $e_1, e_2, ..., e_p$, with the property $e_i \cap e_{i+1} \neq \varnothing$, $1 \leq i \leq p$, $p \oplus 1 = 1(\text{mod:p})$, is called a cycle. This notion of cycle is not operable. Having the mentioned "trunk" we asked ourselves: would it be possible to define in a more natural way the notion of cycle for a hypergraph? The time proved that it is possible.

According to the definition, both the set of vertices X and the set of edges E, of a hypergraph H can be finite or infinite; more over - the number of vertices that forms an edge $e \in E$ can be infinite [3]. For simplicity, let consider now only the case where X and E are finite sets.

Let $E = \{e_1, e_2, ..., e_m\}$ be a set of edges of a hypergraph. For every edge $e_i = (x_{i_1}, x_{i_2}, ..., x_{i_{q_i}})$, $1 \leq i \leq m$, we construct in accordance with those mentioned in §1, a generalized complex of multi-ary relations $\mathcal{R}_1^{q_i}$ on the set of vertices $X_i = \left\{\ x_{i_1}, x_{i_2}, ..., x_{i_{q_i}}\right\}$. The union $\mathcal{R}^{q_1} \cup \mathcal{R}^{q_2} \cup ... \cup \mathcal{R}^{q_m}$ is a $\mathcal{G}$-complex of multi-ary relations $\mathcal{R}^q$, with dimension $q = \max\limits_{1 \leq i \leq m} \{q_i\}$, constructed on the set of vertices $X = \{x_1, x_2, ..., x_n\}$.

From $\mathcal{G}$-complex $\mathcal{R}^q$, in accordance with those described in §2, we obtain the simplicial-abstract complex $\mathcal{K}^{q-1}$ and the group succession of direct homologies.

Each element of such group, i.e. a r-dimensional Δ-cycle, $0 \leq r \leq q-1$ is a cycle in Berge sense. But, if in hypergraph a cycle in Berge sense exists, then the hypergraph contains only one 1-dimensional Δ-cycle. We believe that Δ-cycles are more natural: they offer more detailed information about H. But if we introduce also the co-homologies, then we obtain much more information about the hypergraph $\mathcal{H}$, which possibly will lead to other practical aspects.

We have to mention that those described above could be also extended to infinite hypergraphs, where the edges are determined by an arbitrary number of vertices.

5.2 Median calculation

The median applications are well-known in solving some practical problems. If $(X,\ d)$ is a finite metric space and $f(x) = \sum_{i=1}^{n} d(x,\ x_i)p(x_i)$ is a definite function on the set $X = \{x_1,\ x_2,\ ...,\ x_n\}$ with the elements weight $p(x_i) > 0$, $1 \leq i \leq n$, then the point $x^* \in X$ which minimizes the function $f(x)$ is called a **median**.

Median calculation of a metric space by examining the corresponding function could become quite difficult, what will lead to an inefficient solving of this problem. In the works [20–22, 60] for some special complexes efficient algorithms of median calculation are exposed, without using the metric. The ideas described in the mentioned works can be used at the median calculation of an abstract complex of n-dimensional cubes, which is a particular case of the complex of multi-ary relations.

From the complex of abstract cubes

$$\mathcal{I}^n = \left\{Q_\lambda^p : 0 \leq p \leq n, \quad \lambda \in \Lambda, \quad \dim \Lambda < \infty\right\},$$

defined above, we will require that for homology groups of this complex over the group of **Z** integers, the relations hold

$$\Box^0(\mathcal{I}^n,\ \mathbf{Z}) \cong \mathbf{Z}, \Box^1(\mathcal{I}^n,\ \mathbf{Z}) \cong ... \cong \Box^n(\mathcal{I}^n,\ \mathbf{Z}) \cong 0.$$

The median problem formulated on the 1-dimensional skeleton of the $\mathcal{I}^n$ is solving efficiently, without using metric. 1-Dimensional skeleton $\mathbf{sk}(1)\mathcal{I}^n$ is a graph which we will denote by $H = (X, : U)$. The median of this graph H is calculated on $\mathbf{sk}(1)I^m$, where I^m is an unitary cube of the normed space R_1^m, and m is determined by $\mathcal{I}^n$. For $x \in \mathbf{sk}(1)I^n$ the norm is $\| x \| = | x_1 | + | x_2 | + ... + | x_m |$. The median of $H = (X, : U)$ coincides with the median calculated on $\mathbf{sk}(1)I^m$. Our hypothesis is: a metric graph, wider than the one mentioned, for which the median would be calculated without using metric, does not exist.

5.3 Posthumous problem and the generalization of Euler border

Let a sequence be formed from k elements 0 and 1. We formulate the problem: if we have a disk with a big enough border, how many sequences of k length is possible to place on this border so that none of them would be repeated (Posthumous [7]). The problem is solved with a strongly oriented graph, which represents an Euler tour, obtaining 2^k sequences which represent a generalized complex of 4-ary relations. This situation is applied in the communication systems which are operating with two elements: 0 and 1.

Next we expose our generalizations about an Euler tour, formed from p-dimensional and concordant cubes, $1 \leq p \leq n$ [13].

In the works [19, 23] it is mentioned that an n-dimensional cubic tour $V_1^n(\Box)$, which is concordantly oriented, has an Euler tour of any dimension m, $1 \leq m \leq n$. We will remark that such property isomorphicly possesses each n-dimensional manyfold that is oriented and without borders [13, 15, 68]. In this situation the

Posthumous problem is positively solved, no matter how big the length k of the sequence will be. The exception is only the abstract sphere.

Let $V_1^n(\Box)_1$ and $V_1^n(\Box)_2$ be two abstract n-dimensional isomorphic cubic tours, that are non-concordant oriented. We eliminate the interior $\mathbf{int}I_1^n$ and $\mathbf{int}I_2^n$ of the cubes I_1^n, I_2^n and stick together their isomorphic and coherent borders. Thus a manifold of the second level $V_2^n(\Box)$ is obtained. This manifold is concordantly oriented, and it has an n-dimensional Euler tour $E_1 \cup E_2 \setminus (\mathbf{int}I_1^n \cup \mathbf{int}I_2^n)$. If k_1 is the length of the tour E_1 (the number of cubes), and k_2 is the length of E_2, then the Euler tour in $V_2^n(\Box)$ will have the length $k_1 + k_2 - 2$ [15].

Using the same idea of sticking together the manifold of second level $V_2^n(\Box)$ with a new manifold of first level $V_1^n(\Box)$, we obtain a manyfold of third level $V_2^n(\Box)$, for which also an Euler tour exists. Thus, in an inductive way we can obtain a manifold $V_p^n(\Box)$ of p level which has an Euler tour. The Posthumous problem is solved with the same classic algorithm [7], by obtaining an n-dimensional circular tour (Euler tour) which covers the whole manifold $V_p^n(\Box)$. From here two applicative aspects result.

1. It is possible to construct a virtual device of a compact shape of a manifold $V_p^n(\Box)$ which would be covered by an Euler tour and which can be made mobile E_p^n so that each n-dimensional cube would contain a package of information that does not repeat. For this tour E_p^n a **timer** would also exist to stop the tour at the necessary cube. This situation will give the possibility to store as more information as possible by the increase of k and p.

2. The idea exposed in 1), in $V_p^2(\Box)$ case allows us to generalize the constructed device according to the classical problem of Posthumous for the transmission and reception of information with Euler bidimensional tour with the binary elements 0 and 1, however big the length is, depending on the p increase, having in tangency other manifold $V_p^n(\Box)$. The 2-dimensional cube of $V_p^2(\Box)$ can be done in three-dimensional package. So on such a manifold any volume of knowledge can be stored.

The practical importance of this problem is well known and can be applied in different problems of transmission and reception of information. The application from cryptography, for example, would have the following issue: to determine a word where every arrangement from k letters of the alphabet appears just once.

The device from 2 represents an invention with possible practical applications and would be accomplished by engineers.

References

[1] Alexandroff P. (Aleksandrov P. S.), Hopf H. *Topologie, I,* Springer, Berlin, 1935.

[2] Aleksandrov P. S. *Introduction to Homological dimension theory and general combinatorial topology*, Moscow, 1975 (in Russian).

[3] Baranyai Z. *On the factorization of the complete uniform hypergraph,* Infinite and finite sets, vol. 1, Proceeding of a Colloquium held at Keszthely, June 25 – July 1, 1973, Dedicated to Paul Erdos on his 60th Birthday. (Ed. A. Hajnal, R. Rado, V. T. Sos), North Holland, Amsterdam, 1975, 91–108.

[4] Bauese H. J. *Combinatorial foundation of homology and homotopy*, Springer, Berlin, 1999.

[5] Berge C. *Graphs and hypergraphs*, North-Holland, Amsterdam, 1973.

[6] Berge G. *Hypergraphs: combinatorics of finite sets*, North Holland, Amsterdam, 1989.

[7] Berge C. *Theorie des graphes et ses applications*, DUNOD, Paris, 1958.

[8] Björner A. *Topological methods* (in Handbook of Combinatorics, eds. R. Grahfm, M. Grotschel, L. Lovasz), North Holland, Amsterdam, 1995, 1819–1872.

[9] Björner A., Ziegler G. M. *Introduction to Greedoids in Matroid applications* (ed. N. White), Cambridge Univ. Press, London, 1992, 284–357.

[10] Björner A., Las Vergnas M., Sturmfels B., White N., Ziegler G. M. *Oriented Matroids*, Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge Univ. Press, London, 1993.

[11] Bollobas B. *On generalized graphs*, Acta Math. Acad. Sci. Hung., 1965, **16**, 447–452.

[12] Bollobas B. *Modern Graph Theory*, Graduate text in Mathematics, vol. 184, Springer, New-York, USA, 1998.

[13] Boltyanski V. *The homotopic theory of continuous images and vector fields*, Trudy Matematicheskogo instituta "V. A. Steklov" AN SSSR, **XLVII**, Moscow, 1955, p. 199 (in Russian).

[14] Boltyanski V., Martini H., Soltan P. S. *Excursions into Combinatorial Geometry*, Springer, Berlin, 1997.

[15] Bujac M. *Classification of abstract multidimensional orientable manifolds without borders*, Analele Stiintifice ale USM. Seria "Stiinte fizico-matematice", Chişinău, 2003, 247–250 (in Romanian).

[16] Cartan H., Eilenberg S. *Homological Algebra*, Princeton, University Press, 1956.

[17] Cataranciuc S. *$\mathcal{G}$-complex of multi-ary relations*, Analele ştiinţifice ale USM, Seria "Ştiinţe fizico-matematice", Chişinău, 2006, 119–122 (in Romanian).

[18] Cataranciuc S., Soltan P. *Hypergraphs and their homology*, Trends in the Development of the Information and Communication Technology in Education and Management. International Conference, March 20–21, 2003, Chişinău, 294–300 (in Romanian).

[19] Cataranciuc S., Bujac M., Soltan P. *On the Division of Abstract Manifolds in Cubes*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2006, No. 2(51), 29–34.

[20] Cataranciuc S., Scripnic M., Soltan P. *About the median does not depend on the space metric*, The 30-th Annual Congres of the American Romanian Academy of Arts and Sciences (ARA), Proceedings, Chişinău, July 5–10, 2005, 58–61.

[21] Cataranciuc S., Scripnic M., Soltan P. *A metric space with independent median*, Annals of Tiberiu Popoviciu, Seminar of Functional Equations, Approximation and Convexity, vol. 3, Cluj-Napoca, 2005, 3–12.

[22] Cataranciuc S. *The complex of parallelepipeds and the 3-dimensional tree*, The 33-rd Annual Congres of the American Romanian Academy of Arts and Sciences (ARA), Proceedings, Sibiu, Romania, July 2–7, 2009, vol. II, 293–296.

[23] Bujac M., Cataranciuc S., Soltan P. *On the properties of Multidimensional Torus*, The XIV Conference on Applied and Industrial Mathematics, dedicated to the 60^{th} anniversary of the foundation of the Faculty of Mathematics and Computer Science of Moldova State University, Chişinău, 2006, 66–70.

[24] Dewdney A. K. *Degree sequences in complexes and hypergraphs*, Proc. Amer. Math. Soc., 1975, **53**, No. 2, 535–540.

[25] *Chemical applications of topology and graph theory,* Edited by R. B. King. ELSEVIER, 1983.

[26] Dodson C. T. J., Lok R. *Hypergraphs, homotopy and neighbourhood homology*, Ars. Com., 1983, **16-A**, 107–130.

[27] Dowker C. H. *Homology groups of relations*, Annals of mathematics, 1952, **56**, No. 2, 84–95.

[28] Erdos P. *On extremal problems on graphs and generalized graphs*, Israel J. Math., 1964, **2**, 183–190.

[29] Fritsch R., Fritsch G. *The four-color theorem*, New-York, Springer-Verlag, 1998.

[30] Gantmacher F. R. *Matrix theory*, Moscow, 1967 (in Russian).

[31] Ghouila-Houri A. *Caracterisation des matrices totalement unimodulaires*, C. R. Acad. Sci. (Paris), 1962, **254**, 1192–1194.

[32] Hilton P. J., Wylie Sh. *Homology Theory: an introduction to Algebraic topology*, Cambridge University Press, 1960.

[33] Hoffman A. J., Kruskal J. B. *Integral boundary points of convex polyhedra*, in Kuhn H. W., Tucker A. W. *Linear Inequalities and Related Systems*, Annals of Mah. Studies, 1956, **28**, Princeton Univ. Press, 223–246.

[34] Holl M. *Combinatorics*, Moscow, Mir, 1970 (in Russian).

[35] Hurewicz W., Wallman H. *Dimension Theory*, Princeton Univ. Press, 1941.

[36] James R. Munkres. *Topological Results in Combinatorics*, Michigan math. J., 1984, **31**, issue 1, 113–128.

[37] Kenmochi Yukiko, Atsushi Imiya, Ichikawa Akira, *Discrete combinatorial geometry,* Pattern Recognition, 1997, **30**, No. 10, 1719–1728.

[38] Kleene St. C. *Mathematical Logic*, New York: Wiley, 1967 (Russian edition: Moscow, 1973).

[39] Knaster B., Kuratowski C., Mazurkiewicz S. *Ein Beweis des Fixpunktsatzes fur n-dimensionale Simplexe*, Fundam. Math., 1929, **14**, 132–137.

[40] Kurosh A. G. *Course of Higher Algebra*, Moscow, 1968 (in Russian).

[41] Kurosh A. G. *Lectures on general algebra*, Moscow, 1962 (in Russian).

[42] Lang S. *Algebra*, Reading, Mass., 1965 (Russian edition: Moscow, 1968).

[43] Lovasz L. *On chromatic number of finite sets-system*, Acta Math. Acad. Sci. Hung., 1968, **19**, 59–67.

[44] Martini H., Soltan P. *On the Homologies of Multi-ary Relations*, IC MCC, 2003, **45**, 219–243.

[45] Melnicov O., Tyshkevich R., Yemelichev V., Sarvanov V. *Lectures on graph theory*, Mannheim, BT-Wiss-Verl, 1994. (Russian edition: Moscow, 1990).

[46] Meyer J. C. *Quelques problemes concernant les cliques des hypergraphes h-complets et q-parti h-complets*, Hypergraph seminar (Columbus, 1972, Berge and Ray-Chaudhuri, eds.), Lecture Notes in Math., **411**, Springer-Verlag, Berlin, 1974, 127–139.

[47] Moisil Grigore. *Lectii despre logica rationamentului nuantat*, Bucuresti, 1975 (in Romanian).

[48] Naik R. N., Rao S. B., Shrikhande S. S., Singhi N. M. *Intersection graphs on k-uniform linear hypergraphs*, Europ. J. Combinatorics, 1982, **3**, 159–172.

[49] Oliver R. *Fixed-point sets of group actions on finite aciclic complexes*, Comment. Math. Helvetici, 1975, **50**, 155–177.

[50] Padberg M. W. *Total unimodularity and the Euler subgraph problem*, Operations Research Letters, 1988, **7**, 173–179.

[51] Poincare H. *Complement a l'analysis situs*, Rend. Circ. Mat., Palermo, 1989, **13**, 285–343.

[52] Pontryagin A. S. *Basis of Combinatorial Topology*, Moscow, 1976 (in Russian).

[53] Ringel Gerhard. *Map Color Theorem*, Springer-Verlag, Berlin–Heidelberg–New-York, 1974.

[54] Rishkov S. *About the combinatorial topology in Hilbert space*, Uspekhi Matematichesckikh Nauk, 1958, **XIII**, No. 4(82), 216–217 (in Russian).

[55] Rishkov S. *Invariant of homologies groups by infinity dimension*, All-Union Mathematics congress, vol. IV, Moscow, 1958 (in Russian).

[56] Rotman J. J. *An introduction to algebraic topology*, Springer-Verlag, New-York, 1998.

[57] Saunders MacLane. *Homology*, Springer, Berlin, 1963.

[58] Soltan P. *On the Homologies of Multy-ary relations and Oriented Hipergraphs*, Studii in metode de analiza numerica si optimizare, 2000, **2**, No. 1(3), 60–81.

[59] Soltan P. *Selected lectures in graph theory*, Chişinău, USM, 2001 (in Romanian).

[60] Soltan P. S, Zambitki D. K., Prisacaru K. F. *Extremal problemes on graphs and algorithms of their solving*, Kishinev, 1974 (in Russian).

[61] Spanier E. H. *Algebraic Topology*, Springer, New-York, 1966 (Russian edition: Moscow, 1971).

[62] Teleman S. *Elements of topology and differential varieties*, Bucharest, 1964 (in Romanian).

[63] The Mathematical Encyclopedia, 1977, **1** (in Russian).

[64] The Mathematical Encyclopedia, 1982, **3** (in Russian).

[65] Tomescu I. *Sur le probleme du coloriage des graphes generalises*, C. R. Acad. Sci. (Paris), 1968, **267**, 250–252.

[66] Tutte W. T. *Lectures on matroids*, J. Research Nat. Bureau of Standards (USA), Sect., 1965, **B69**, 1–47.

[67] Tuza Z. *Critical hypergraphs and intersecting set-pair system*, J. Comb. Theory, 1985, **B39**, 134–145.

[68] Veblen O., Alexander J. W. *Manifolds of n-dimensions*, Ann. of Math., 1913, **14**, No. 2, 163–178.

[69] Voloshin V. L. *On the upper chromatic number of a hypergraph*, Australian Journal of Combinatorics, 1995, **11**, 25–45.

[70] Wylie S. *Intercept-finite cell complexes,* Algebraic Geometry and Topology, Symposium in honor of S. Lefschetz, Princeton Math. Series, 1957, No. 12, 389–399.

[71] Zykov A. A. *Hypergraphs,* Uspekhi Mathematicheskikh Nauk, 1974, **29**, No. 6(180), 89–154 (in Russian).

Cataranciuc Sergiu, Soltan Petru
State University of Moldova
60 A. Mateevici str., MD-2009, Chisinau
Moldova

E-mail: *caseg@usm.md*, *psoltan@usm.md*

Received February 8, 2010