BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(63), 2010, Pages 3–19
ISSN 1024–7696

Properties of final unrefinable chains of groups topologies

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Abstract. Let G be a nilpotent group and $(\mathfrak{M}, <)$ be the lattice of all group topologies or the lattice of all group topologies in each of which the group G possesses a basis of neighborhood of unit consisting of subgroups. If τ and τ' are group topologies from $\mathfrak{M}$ such that $\tau = \tau_0 \prec_{\mathfrak{M}} \tau_1 \prec_{\mathfrak{M}} \ldots \prec_{\mathfrak{M}} \tau_n = \tau'$, then $k \leq n$ for any chain $\tau = \tau'_0 < \tau'_1 < \ldots < \tau'_k = \tau'$ of topologies from $\mathfrak{M}$.

Mathematics subject classification: 22A05.
Keywords and phrases: Topological groups, lattice of group topologies, modular lattice, chain of topologies, unrefinable chain.

1 Introduction

After the problem of the existence of non discrete Hausdorff topologies in infinite Abelian groups and some infinite rings was solved (see for example, [1, p. 351–390]), and the existence of a lange number of group topologies in infinite Abelian groups, was proved it was an interesting to study the lattice of all group topologies and lattices of all ring topologies and their sublattices.

So in work [7] it was proved that the lattice of all group topologies of an Abelian group is modular.

As properties of finite unrefinable chains in any modular lattice were investigated well (see, for example, Theorem 3.10), then for any Abelian groups in any sublattice of the lattice of all group topologies the properties of unrefinable chains are investigated well enough.

As the lattice of all group topologies is non modular even for nilpotent groups (see [2]) it is natural to study properties of finite unrefinable chains of group topologies.

The present work is devoted to the study of properties of finite unrefinable chains of group topologies for nilpotent groups.

The basic results of work are Theorem 4.6 and Corollary 4.7 in which some properties of a unrefenable chains of group topologies are proved. These results give the positive answer to the question 14.5 (б) from [8].

2 Notations

In this work if another will be not stipulated we shall use the following notations:

2.1. $\mathbb{N}$ is the set of all natural numbers.

2.2. $G(\cdot)$ or simply G is a group.

2.3. If G is a group, A and $B \subseteq G$ then we shall put:

$< A >$ is the subgroup of the group G which is generated by the set A and $[A, B] = \{a \cdot b \cdot a^{-1} \cdot b^{-1} | a \in A, b \in B\}$.

2.4. By induction, for any natural number k we shall define a normal subgroup $G_{[k]}$ of the group G as follows:

Put $G_{[0]} = G$ and take as $G_{[k+1]}$ a subgroup generated by the set $[G_{[k]}, G]$, i.e. $G_{[k+1]} =< [G_{[k]}, G] >$.

By induction on number i it is easily checked that $G_{[i]}$ is a normal subgroup of the group G.

2.5. If τ_1 and τ_2 are topologies on a set X, then we shall consider, that $\tau_1 \leq \tau_2$, if $\tau_1 \subseteq \tau_2$.

2.6. If I is some normal subgroup of a group G it is easy to notice then the set $\{I\}$ satisfies conditions 3.6.1 - 3.6.5 (see below Remark 3.6) and hence it sets on the group G some group topology for which this set is a basis of neighborhoods of unit.

We shall denote this topology by $\tau(I)$.

2.7. Let (G, τ) be a topological group and I be some normal subgroup of the group G. If Ω is some basis of neighborhoods of unit in the topological group (G, τ), then it is easy to notice that the set $\{V \bigcap I | V \in \Omega\}$ satisfies conditions $3.6.1 - 3.6.5$ (see below Remark 3.6) and hence it sets on the group G some group topology for which this set is a basis of neighborhoods of unit.

We shall denote this topology by τ_I.

2.8. Let (G, τ) be a topological group and I is some normal subgroup of the group G. If Ω is some basis of neighborhoods of unit in the topological group (G, τ) then it is easy to notice that the set $\{V \cdot I | V \in \Omega\}$ satisfies conditions $3.6.1 - 3.6.5$ (see below Remark 3.6) and hence it sets on the group G some group topology for which this set is a basis of neighborhoods of unit.

This topology we shall designate by $\tau \cdot I$.

2.9. If $(X, <)$ is a partially ordered set, $S \subseteq X$ and $a, b \in X$, then:

- We consider that $a = inf_X S$ if $a \leq x$ for any element $x \in S$ and if $d \in X$ is an element such that $d \leq x$ for all $x \in S$, then $d \leq a$;

- We consider that $b = sup_X S$ if $b \geq x$ for any element $x \in S$ and if $d \in X$ is an element such that $d \geq x$ for all $x \in S$, then $d \geq b$.

3 Definitions and auxiliary results

Results of this section have been received by the author of present article together with I. V. Vdovichenko. As by the moment of preparation of present article they are not published, then for completeness of the statement we adduce them.

3.1. Definition (see [3, 5, 6]). A partially ordered set (X, ¡) is called:
- A lattice if for any two elements $a, b \in X$ there exist $inf_X\{a, b\}$ and $sup_X\{a, b\}$;
- A full lattice if for any nonempty subset $S \subseteq X$ there exist $inf_X S$ and $sup_X S$.

3.2. Remark. If (G, τ) is a topological group, then from one definitions of right and left uniform structures in (G, τ) (see [4, p. 224, Definition 1]) the following statement easily follows:

In a topological group (G, τ) right and left uniform structures coincide if and only if for any neighborhood V_0 of unit there exists a neighborhood V_1 of unit such that $g \cdot V_1 \cdot g^{-1} \subseteq V_0$ for any element $g \in G$.

3.3. Definition (see [1, 4]). A topological group (G, τ) is called precompact if for any neighborhood V of unit in (G, τ) there exists a finite subset $S \subseteq G$ such that $G = S \cdot V$.

3.4. Remark. As $G = G^{-1}$ and $(S \cdot V)^{-1} = V^{-1} \cdot S^{-1}$ then a topological group (G, τ) is precompact if and only if for any neighborhood V of unit in (G, τ) there exist a finite subset $S \subseteq G$ such that $G = V \cdot S$.

3.5. Proposition. If the topological group (G, τ) is precompact then in (G, τ) the right and left uniform structures coincide.

Proof. Let V_0 be a neighborhood of unit in the topological group (G, τ) and V_1 be a neighborhood of unit in topological group (G, τ) such that $V_1 \cdot V_1 \cdot V_1^{-1} \subseteq V_0$. There is a finite subset S in G such that $V_1 \cdot S = G$ and there exists a neighborhood V_2 of unit in (G, τ) such that $V_2 \subseteq V_1$ and $g \cdot V_2 \cdot g^{-1} \subseteq V_1$ for any $g \in S$.
If $g \in G$ then $g = v \cdot h$ for some $v \in V_1$ and $h \in S$. Then

$$g \cdot V_2 \cdot g^{-1} = (v \cdot h) \cdot V_2 \cdot (v \cdot h)^{-1} = v \cdot (h \cdot V_2 \cdot h^{-1}) \cdot v^{-1} \subseteq V_1 \cdot V_1 \cdot V_1^{-1} \subseteq V_0.$$

The proposition is completely proved.

3.6. Remark (see [4, p. 203, Proposition 1]). Let G be a group and Ω be a set of subsets of the group G such that the following conditions are true:

3.6.1. $e \in V$ for any $V \in \Omega$;

3.6.2. For any V and U from Ω there exists such $W \in \Omega$ that $W \subseteq V \bigcap U$;

3.6.3. For any $V \in \Omega$ there exists such $U \in \Omega$ that $U^{-1} \subseteq V$;

3.6.4. For any $V \in \Omega$ there exists such $U \in \Omega$ that $U \cdot U \subseteq V$;

3.6.5. For any $V \in \Omega$ and any element $g \in G$ there exists such $U \in \Omega$ that $g \cdot U \cdot g^{-1} \subseteq V$.

Then on the group G there exists the unique group topology τ such that Ω is a basis of neighborhoods of unit in the topological group (G,τ).

3.7. Proposition. For any group G the following statements are true:

3.7.1. The set $\mathfrak{M}$ of all group topologies on the group G with order which was defined in 2.5, is a full lattice;

3.7.2. The set $\mathfrak{G}$ of all group topologies on the group G in each of which the topological group possesses a basis of neighborhoods of unit consisting of subgroups with order which has been defined in 2.5, is a full lattice;

3.7.3. The set $\mathfrak{N}$ of all group topologies on the group G in each of which the topological group possesses a basis of neighborhoods of unit consisting of normal subgroups with order which has been defined in 2.5, is a full lattice;

3.7.4. The set $\mathfrak{S}$ of all group topologies on the group G in each of which right and left uniform structures coincide with order which is defined in 2.5, is a full lattice;

3.7.5. The set $\mathfrak{C}$ of all group topologies on the group G in each of which the topological group is a precompact group with order which is defined in 2.5, is a full lattice.

Proof.

3.7.1. In the beginning we shall show that there exists $sup_{\mathfrak{M}}\mathcal{S}$ for any nonempty subset $\mathcal{S} \subseteq \mathfrak{M}$.

For each group topology $\tau \in \mathcal{S}$ we shall choose some basis Ω_τ of neighborhoods of unit in the topological group (G,τ) and also we shall consider the set $\Omega = \bigcup\limits_{\tau\in\mathcal{S}} \Omega_\tau$. If $\widetilde{\Omega}$ is the set of all finite subsets of the set Ω, then for every $\Delta \in \widetilde{\Omega}$ take $\widetilde{W}_\Delta = \bigcap\limits_{V\in\Delta} V$. Show that the set $\Theta = \{\widetilde{W}_\Delta | \Delta \in \widetilde{\Omega}\}$ satisfies conditions 3.6.1 – 3.6.5 of Remark 3.6.

As $e \in V$ for any $V \in \Omega$ then $e \in W_\Delta$ for any $\Delta \in \widetilde{\Omega}$, i.e. the condition 3.6.1 is executed.

Let $\Delta_1 \in \widetilde{\Omega}$ and $\Delta_2 \in \widetilde{\Omega}$. If $\Delta = \Delta_1 \bigcup \Delta_2$, then $\Delta \in \widetilde{\Omega}$, and $\widetilde{W}_\Delta = \widetilde{W}_{\Delta_1} \bigcap \widetilde{W}_{\Delta_2}$, i.e. the condition 3.6.2 is executed.

Let $\Delta = \{V_1, \dots, V_s\} \in \widetilde{\Omega}$. As (G,τ) is a topological group for any $\tau \in \mathfrak{M}$ then for any $1 \le i \le s$ there exists $U_i \in \Omega$ such that $U_i^{-1} \subseteq V_i$. Then $\Delta' = \{U_1, \dots, U_s\} \in \widetilde{\Omega}$, and

$$\widetilde{W}_{\Delta'}^{-1} = (\bigcap_{i=1}^{s} U_i)^{-1} \subseteq \bigcap_{i=1}^{s} U_i^{-1} \subseteq \bigcap_{i=1}^{s} V_i = \widetilde{W}_\Delta,$$

i.e. the condition 3.6.3 is executed.

Let $\Delta = \{V_1, \dots, V_s\} \in \widetilde{\Omega}$. As (G,τ) is a topological group for any $\tau \in \mathfrak{M}$ then for any $1 \le i \le s$ there exists $U_i \in \Omega$ such that $U_i \cdot U_i \subseteq V_i$. Then $\Delta' = \{U_1, \dots, U_s\} \in \widetilde{\Omega}$, and

$$\widetilde{W}_{\Delta'} \cdot \widetilde{W}_{\Delta'} = (\bigcap_{i=1}^{s} U_i) \cdot (\bigcap_{i=1}^{s} U_i) \subseteq \bigcap_{i=1}^{s} (U_i \cdot U_i) \subseteq \bigcap_{i=1}^{s} V_i = \widetilde{W}_{\Delta},$$

i.e. the condition 3.6.4 is executed.

Let $\Delta = \{V_1, \ldots, V_s\} \in \widetilde{\Omega}$ and $g \in G$. As (G, τ) is a topological group then for any $\tau \in \mathfrak{M}$ for any $1 \leq i \leq s$ there exists $U_i \in \Omega$ such that $g \cdot U_i \cdot g^{-1} \subseteq V_i$. Then $\Delta' = \{U_1, \ldots, U_s\} \in \widetilde{\Omega}$, and

$$g \cdot \widetilde{W}_{\Delta'} \cdot g^{-1} = g \cdot (\bigcap_{i=1}^{s} U_i) \cdot g^{-1} \subseteq \bigcap_{i=1}^{s} (g \cdot U_i \cdot g^{-1}) \subseteq \bigcap_{i=1}^{s} V_i = \widetilde{W}_{\Delta},$$

i.e. the condition 3.6.5 is executed.

According to Remark 3.6, on the group G there exists a group topology $\tau* \in \mathcal{M}$ in which the set $\Theta = \{\widetilde{W}_{\Delta} | \Delta \in \widetilde{\Omega}\}$ is a basis of neighborhoods of unit.

As $\Omega_\tau \subseteq \Theta$ for any topology $\tau \in \mathcal{S}$ then $\tau \leq \tau*$ for any topology $\tau \in \mathcal{S}$.

Let now $\tau' \in \mathfrak{M}$ be a group topology on group G such that $\tau \leq \tau'$ for any topology $\tau \in \mathcal{S}$.

Then any subset $V \in \Omega$ is a neighborhood of unit in the topological group (G, τ'). If $\widetilde{W}_{\Delta} \in \Theta$, then $\widetilde{W}_{\Delta}$ is the intersection of finite number of sets from Ω, and hence, it is a neighborhood of unit in the topological group (G, τ').

Hence $\tau* \leq \tau'$.

So, we have proved that $\tau* = sup_{\mathfrak{M}}\mathcal{S}$.

Now show that in $\mathfrak{M}$ there exists $inf_{\mathfrak{M}}\mathcal{S}$ for any nonempty subsets $\mathcal{S} \subseteq \mathfrak{M}$.

Consider the set $\mathcal{S}' = \{\tau' \in \mathfrak{M} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$. As the set $\mathcal{S}'$ contains the anti-discrete topology then $\mathcal{S}' \neq \emptyset$. Then, as it was proved above, in $\mathfrak{M}$ there exists $\widetilde{\tau} = sup_{\mathfrak{M}}\mathcal{S}'$.

Show that $\widetilde{\tau} = inf_{\mathfrak{M}}\mathcal{S}$.

If $\tau \in \mathcal{S}$, then $\tau' \leq \tau$ for all $\tau' \in \mathcal{S}'$. Then (see 2.9) $\widetilde{\tau} = sup_{\mathfrak{M}}S' \leq \tau$ for all $\tau \in \mathcal{S}$.

Moreover, if $\tau'' \leq \tau$ for all $\tau \in \mathcal{S}$, then $\tau'' \in \mathcal{S}'$, and hence, $\tau'' \leq sup_{\mathfrak{M}}\mathcal{S}' = \widetilde{\tau}$. Then $\widetilde{\tau} = inf_{\mathfrak{M}}\mathcal{S}$.

The statement 3.7.1 is proved.

3.7.2. Let $\emptyset \neq \mathcal{S} \subseteq \mathfrak{G}$ and $\tau* = sup_{\mathfrak{M}}\mathcal{S}$ (see 3.7.1).

In the proof of the statement 3.7.1 it has been shown that the set Θ is a basis of neighborhoods of unit in the topological group $(G, \tau*)$. As the intersection of any number of subgroups of the group G is a subgroup, then any of subsets $\widetilde{W}_{\Delta}$ is a subgroup, and hence, $\tau* \in \mathfrak{G}$.

As $\mathfrak{G} \subseteq \mathfrak{M}$, then $\tau* = sup_{\mathfrak{G}}\mathcal{S}$.

So, we have proved that there exists $sup_{\mathfrak{G}}\mathcal{S}$

Now show that in $\mathfrak{G}$ there exists $inf_{\mathfrak{G}}\mathcal{S}$ for any nonempty subset $\mathcal{S} \subseteq \mathfrak{G}$.

If $\mathcal{S}' = \{\tau' \in \mathfrak{G} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$ then as in the proof of the statement 3.7.1 is proved that $sup_{\mathfrak{G}}\mathcal{S}' = inf_{\mathfrak{G}}\mathcal{S}$.

The statement 3.7.2 is proved.

The proof of the statement 3.7.3 is analogues to proofs of the statement 3.7.2.

3.7.4. Let $\emptyset \neq \mathcal{S} \subseteq \mathfrak{S}$ and $\tau* = sup_{\mathfrak{M}}\mathcal{S}$ (see 3.7.1). We shall show that $\tau* \in \mathfrak{S}$. In the proof of the statement 3.7.1 it has been proved that the set $\Theta = \{\widetilde{W_\Delta} | \Delta \in \widetilde{\Omega}\}$ is a basis of neighborhoods of unit in the topological group $(G, \tau*)$.

Let $\Delta = \{V_1, \ldots, V_s\} \in \widetilde{\Omega}$. As for any topology $\tau \in \mathfrak{S}$ in topological group (G, τ) left and right uniform structures coincide, then (see Remark 3.2) for any number $1 \leq i \leq s$ there exists $U_i \in \Omega$ such that $g \cdot U_i \cdot g^{-1} \subseteq V_i$ for all $g \in G$. Then $\Delta' = \{U_1, \ldots, U_s\} \in \widetilde{\Omega}$, and

$$g \cdot W_{\Delta'} \cdot g^{-1} = g \cdot (\bigcap_{i=1}^{s} U_i) \cdot g^{-1} \subseteq \bigcap_{i=1}^{s} (g \cdot U_i \cdot g^{-1}) \subseteq \bigcap_{i=1}^{s} V_i = W_\Delta$$

for any $g \in G$, i.e. $\tau* \in \mathfrak{S}$.

As $\mathfrak{S} \subseteq \mathfrak{M}$, then $\tau* = sup_{\mathfrak{S}}\mathcal{S}$.

So, we have proved that there exists $sup_{\mathfrak{S}}\mathcal{S}$.

Now show that in $\mathfrak{S}$ there exists $inf_{\mathfrak{S}}\mathcal{S}$ for any nonempty subsets $\mathcal{S} \subseteq \mathfrak{S}$.

If $\mathcal{S}' = \{\tau' \in \mathfrak{S} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$ then, as in the proof of the statement 3.7.1, is proved that $sup_{\mathfrak{S}}\mathcal{S}' = inf_{\mathfrak{S}}\mathcal{S}$.

The statement 3.7.4 is proved.

3.7.5. Let $\emptyset \neq \mathcal{S} \subseteq \mathfrak{C}$ and $\tau* = sup_{\mathfrak{M}}\mathcal{S}$ (see 3.7.1).

It is easy to notice that in the proof of Proposition 4.4.11 in [1] the requirement of commutative of the group is not essential, and hence, this proof with little modification can be applied for proofs of that the topological group $(G, \tau*)$ is a pre-compact, i.e. that $\tau* \in \mathfrak{C}$.

As $\mathfrak{C} \subseteq \mathfrak{M}$, then $\tau* = sup_{\mathfrak{C}}\mathcal{S}$.

Now show that in $\mathfrak{C}$ there exists $inf_{\mathfrak{C}}\mathcal{S}$ for any nonempty subsets $\mathcal{S} \subseteq \mathfrak{C}$.

Consider the set $\mathcal{S}' = \{\tau' \in \mathfrak{C} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$. As in the proof of the statement 3.7.1, it is proved that $sup_{\mathfrak{C}}\mathcal{S}' = inf_{\mathfrak{C}}\mathcal{S}$.

The statement 3.7.5 is proved, and hence, the theorem is completely proved.

3.8. Definition. Let $\mathfrak{A}$ be any lattice and $a, b \in \mathfrak{A}$. If $a < b$ and between elements a and b there exist no other elements in the lattice $\mathfrak{A}$ then we shall say that the element b covers the element a in the lattice $\mathfrak{A}$ (see [3], p. 15), also we shall write $a \prec_{\mathfrak{A}} b$.

Notice, that if $\mathfrak{A}$ is a sublattice of a lattice $(\mathfrak{B}, <)$ and $a, b \in \mathfrak{A}$, then $a \prec_{\mathfrak{A}} b$ does not follows that $a \prec_{\mathfrak{B}} b$, but from that $a \prec_{\mathfrak{B}} b$ follows that $a \prec_{\mathfrak{A}} b$.

3.9. Definition. As it is usual (see [3], [6]), a lattice $\mathfrak{A}$ l is called a modular lattice[1] if in it the following condition is true:

[1]Such lattices sometimes are called Dedekind.

If $a, b, c \in \mathfrak{A}$ and $a \leq c$, then $\sup_{\mathfrak{A}}\{a, \inf_{\mathfrak{A}}\{b, c\}\} = \inf_{\mathfrak{A}}\{\sup_{\mathfrak{A}}\{a, b\}, c\}$.

It is easy to notice that any sublattice of a modular lattice is a modular lattice.

3.10. Theorem. Let $\mathfrak{A}$ be a modular lattice and $a, b \in \mathfrak{A}$. Then the following statements are true:

3.10.1. If $a = a_1 \prec_{\mathfrak{A}} a_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} a_n = b$ (i.e. this chain is a unrefinable chain of the lattice $\mathfrak{A}$) and $a = b_1 < b_2 < \ldots < a_k = b$, then $k \leq n$, and $k = n$ if and only if $a = b_1 \prec_{\mathfrak{A}} b_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} a_k = b$ (see [6], pp. 191 and 192);

3.10.2. If $a, b, c \in \mathfrak{A}$ and $a \prec_{\mathfrak{A}} b$, then $sup_{\mathfrak{A}}\{a,\ c\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{b,\ c\}$ and $inf_{\mathfrak{A}}\{a,\ c\} \preceq_{\mathfrak{A}} inf_{\mathfrak{A}}\{b,\ c\}$ (see [5], p. 213, theorem 4).

3.11. Proposition. Let G be a group, τ_1 and τ_2 be group topologies on the group G, Ω_1 and Ω_2 be some basis of neighborhoods of unit in topological groups (G, τ_1) and (G, τ_2), accordingly. Then the following statements are equivalent:

3.11.1. For any neighborhoods of unit $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $V_2 \cdot U_2 \subseteq U_1 \cdot V_1$;

3.11.2. For any neighborhoods of unit $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $U_2 \cdot V_2 \subseteq V_1 \cdot U_1$;

3.11.3. The set $\Omega_3 = \{U \cdot V | V \in \Omega_1,\ U \in \Omega_2\}$ is a basis of neighborhoods of unit in the topological group (G, τ_3), where $\tau_3 = inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\}$ in the lattice $\mathfrak{M}$ of all group topologies on the group G.

Proof. In the beginning, we shall prove , that 2.11.1 ⇒ 3.11.2

Let $V_0 \in \Omega_1$ and $U_0 \in \Omega_2$. There exist $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ such that $V_1^{-1} \subseteq V_0$ and $U_1^{-1} \subseteq U_0$. As the statement 3.11.1 is executed then there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $V_2 \cdot U_2 \subseteq U_1 \cdot V_1$ and also there exist $V_3 \in \Omega_1$ and $U_3 \in \Omega_2$ such that $V_3^{-1} \subseteq V_2$ and $U_3^{-1} \subseteq U_2$.

As $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$ for any $a, b \in G$ then

$$U_3 \cdot V_3 = ((U_3)^{-1})^{-1} \cdot (V_3^{-1})^{-1} = ((V_3^{-1}) \cdot (U_3^{-1}))^{-1} \subseteq$$

$$\subseteq (V_2 \cdot U_2)^{-1} \subseteq (U_1 \cdot V_1)^{-1} = V_1^{-1} \cdot U_1^{-1} \subseteq V_0 \cdot U_0,$$

i.e. the statement 3.11.2 is executed.

The further proof of the theorem can be found in ([7], the proof of the Theorem 3.2).

3.12. Proposition. Let G be a group, τ_1 and τ_2 be group topologies on G, Ω_1 and Ω_2 be some basis of neighborhoods of unit in topological groups (G, τ_1) and (G, τ_2), accordingly. If for any neighborhood of unit $V_1 \in \Omega_1$ there exist such $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$, for any $g \in U_2$, then for group topologies τ_1 and τ_2 the statement 3.11.2 is true, and hence, each of statements 3.11.1 – 3.11.3 is true.

Proof. Let $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$. There exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $V_2 \subseteq V_1$, $U_2 \subseteq U_1$ and $g \cdot V_2 \cdot g^{-1} \subseteq V_1$ for any $g \in U_2$. Then $g \cdot V_2 = g \cdot V_2 \cdot g^{-1} \cdot g \subseteq V_1 \cdot g$ for any $g \in U_2$, and hence, $U_2 \cdot V_2 \subseteq V_1 \cdot U_2 \subseteq V_1 \cdot U_1$, i.e. the statement 3.11.2 is true, and hence each of statements 3.11.1 – 3.11.3 is true.

The proposition is completely proved.

3.13. Corollary. Let G be a group and τ_1 be a group topology such that in the topological group (G, τ_1) the right and left uniform structures coincide. Then for any group topology τ_2 the pair of topologies τ_1 and τ_2 satisfies each of statements 3.11.1 – 3.11.3.

Really, for any topology τ_2 the set $U = G$ is a neighborhood of unit in the topological group (G, τ_2), and according to Remark 3.2, for any neighborhood V_1 of unit in the topological group (G, τ_1) there exists a neighborhood V_2 in the topological group (G, τ_1) such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$ for any $g \in G$. Then, from the previous proposition, the truths the corollary follows.

3.14. Proposition. For any group G the following statements are true:

3.14.1. The lattice $\mathfrak{N}$ of all group topologies on the group G in each of which the topological group possesses a basis of neighborhood of unit consisting of normal subgroups is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on the group G;

3.14.2. The lattice $\mathfrak{S}$ of all group topologies on the group G in each of which right and left uniform structures coincide is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on the group G

3.14.3. The lattice $\mathfrak{C}$ of all group topologies on the group G in each of which topological group is a precompact group is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on the group G.

Proof.

3.14.1. Let τ_1 and $\tau_2 \in \mathfrak{N}$, Ω_1 and Ω_2 be some basis of neighborhoods of unit in topological groups (G, τ_1) and (G, τ_2), accordingly, consisting from normal subgroups.

As the intersection of any number of normal subgroups is a normal subgroup then from the proof of the statement 3.7.1 follows, that topological group $(G, sup_{\mathfrak{M}}\{\tau_1, \tau_2\})$ possesses of basis of neighborhoods of unit, which will consist from normal subgroups, i.e. $sup_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{N}$.

Moreover, according to the statement 3.11.3, the set $\{U \cdot V \mid U \in \Omega_1,\ V \in \Omega_2\}$ is a basis of neighborhoods of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$. As product of two normal subgroups is a normal subgroup, then $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{N}$.

The statement 3.14.1 is proved.

3.14.2. Let τ_1, $\tau_2 \in \mathfrak{S}$ and Ω_1 and Ω_2 be some basis of neighborhoods of unit in topological groups (G, τ_1) and (G, τ_2), accordingly. Then (see 3.7.1) the set $\Omega_3 = \{U \bigcap V \mid U \in \Omega_1,\ V \in \Omega_2\}$ is a basis of neighborhoods of unit in the topological group $(G, sup_{\mathfrak{M}}\{\tau_1, \tau_2\})$.

If $U\bigcap V \in \Omega_3$, then there exist such $U_1 \in \Omega_1$ and $V_1 \in \Omega_2$ that $g \cdot U_1 \cdot g^{-1} \subseteq U$ and $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any $g \in G$. Then $U_1\bigcap V_1 \in \Omega_3$ and

$$g \cdot (U_1\bigcap V_1) \cdot g^{-1} \subseteq (g \cdot U_1 \cdot g^{-1})\bigcap(g \cdot V_1 \cdot g^{-1}) \subseteq U\bigcap V$$

for any $g \in G$, i.e. $sup_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{S}$.

Moreover, from Corollary 3.13 and the statement 3.11.3, it follows that the set $\Omega_4 = \{U \cdot V \ |U \in \Omega_1,\ V \in \Omega_2\}$ is a basis of neighborhoods of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$.

If $U \cdot V \in \Omega_4$ then there exist such $U_1 \in \Omega_1$ and $V_1 \in \Omega_2$ that $g \cdot U_1 \cdot g^{-1} \subseteq U$ and $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any $g \in G$. Then $U_1 \cdot V_1 \in \Omega_4$ and

$$g \cdot (U_1 \cdot V_1) \cdot g^{-1} = (g \cdot U_1 \cdot g^{-1}) \cdot (g \cdot V_1 \cdot g^{-1}) \subseteq U \cdot V$$

for any $g \in G$, i.e. $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{S}$.

The statement 3.14.2 is proved.

3.14.3. If τ_1 and $\tau_2 \in \mathfrak{C}$ then in the proof of the statement 3.7.5, it has been proved that $sup_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{C}$.

Moreover, from Definition 3.3 it follows that every group topology which is weaker than some precompact topology itself is a precompact topology then $inf_{\mathfrak{M}}\{\tau_1, \tau_2\}$ is a precompact topology, i.e. $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{C}$.

The proposition 3.14 is completely proved.

4 The basic results

4.1. Proposition. Let:

– G be a group;

– $\mathfrak{M}$ be the lattice of all group topologies on the group G;

– τ_1 and τ_2 be such group topologies that topological groups (G, τ_1) and (G, τ_2) possess basis of neighborhoods of unit consisting of subgroups.

If for any neighborhood V_0 of unit in the topological group (G, τ_1) there exist neighborhoods V_1 and U_1 of units in topological groups (G, τ_1) and (G, τ_2), accordingly, such that $g \cdot V_1 \cdot g^{-1} \subseteq V_0$ for any $g \in U_1$, then the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$ possesses a basis of neighborhoods of unit, consisting of subgroups.

Proof. Let Ω_1 and Ω_2 be basis of neighborhoods of units in topological groups (G, τ_1) and (G, τ_2), accordingly, consisting of subgroups.

From Proposition 3.12 and the statement 3.11.3 it follows that the set $\Omega_3 = \{V \cdot U \mid V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$.

For any $V \in \Omega_1$ and $U \in \Omega_2$ we shall consider the subgroup $W(V, U)$ which is generated by the set $V \cdot U$, i.e. $W(V, U) =< V \cdot U >$, and let

$\widetilde{\Omega} = \{W(V,U) \mid V \in \Omega_1, U \in \Omega_2\}$. As $V \cdot U \subseteq W(V,U)$, then any set $W(V,U)$ is a neighborhood of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$, i.e. $\widetilde{\Omega}$ consists of neighborhoods of unit of the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$.

If now $V_0 \cdot U_0 \in \Omega_3$ then there exist such $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$, that $g \cdot V_1 \cdot g^{-1} \subseteq V_0$ for any $g \in U_1$.

As Ω_3 is a basis of neighborhoods of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$ then there exist such $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ that $V_2 \subseteq V_1$, $U_2 \subseteq U_1$ and $(V_2 \cdot U_2)^{-1} \subseteq V_1 \cdot U_1$.

Show that $< V_2 \cdot U_2 > \subseteq V_0 \cdot V_0$.

Assume the contrary, i.e. that $< V_2 \cdot U_2 > \not\subseteq V_0 \cdot V_0$. Then there exist such integers $k_1, \ldots, k_n$ from the set $\{1, -1\}$ and elements $b_1, b_2, \ldots, b_n$ from $V_2 \cdot U_2$ that $b = b_1^{k_1} \cdot b_2^{k_2} \cdot \ldots \cdot b_n^{k_n} \notin V_0 \cdot U_0$.

As $V_2 \cdot U_2 \subseteq V_1 \cdot U_1$ and $(V_2 \cdot U_2)^{-1} \subseteq V_1 \cdot U_1$, then $b_i^{k_i} \in V_1 \cdot U_1$ for any $1 \leq i \leq n$, and hence, $b_i^{k_i} = v_i \cdot u_i$, where $v_i \in V_1$ and $u_i \in U_1$ for all $1 \leq i \leq n$. Then $b = (v_1 \cdot u_1) \cdot \ldots \cdot (v_n \cdot u_n)$.

For every $1 \leq s \leq n-1$ by induction we shall define element g_s, we put $g_1 = u_1$ and $g_{s+1} = g_k \cdot u_{k+1}$. As U_1 is a subgroup, then $g_s \in U_1$ for any $1 \leq s \leq n-1$. Then $g_s \cdot v_{s+1} \cdot g_s^{-1} \in V_1 \subseteq V_0$, and as V_0 is a subgroup, then $b = (v_1 \cdot u_1) \cdot \ldots \cdot (v_n \cdot u_n) =$

$$v_1(\cdot u_1 \cdot v_2 \cdot g_1^{-1}) \cdot (g_1 \cdot u_2 \cdot v_3 \cdot g_2^{-1}) \cdot \ldots \cdot (g_{n-2} \cdot u_{n-1} \cdot v_{n-1} \cdot g_{n-1}^{-1}) \cdot g_{n-1} \cdot u_n =$$

$$v_1(\cdot g_1 \cdot v_2 \cdot g_1^{-1}) \cdot (g_2 \cdot v_3 \cdot g_2^{-1}) \cdot \ldots \cdot (g_{n-1} \cdot v_{n-1} \cdot g_{n-1}^{-1}) \cdot g_n \in V_0 \cdot U_0.$$

We have obtained a contradiction with the choice of element b. Hence, $< V_2 \cdot U_2 > \subseteq V_0 \cdot U_0$ and hence, in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$ the set $\{< V \cdot U > \mid V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of unit consisting from subgroups.

The proposition is completely proved.

4.2. Proposition. If G is a group and $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on the group G such that for any two group topologies from $\mathfrak{A}$ one of statements 3.11.1 – 3.11.4 is true, then the lattice $\mathfrak{A}$ is modular.

Proof. Let $\tau_1,\ \tau_2,\ \tau_3 \in \mathfrak{A}$ be such group topologies that $\tau_1 \leq \tau_3$ and $\Omega_1,\ \Omega_2, \Omega_3$ be basis of neighborhoods of unit in topological groups (G, τ_1), (G, τ_2) and (G, τ_3), accordingly. Then, from the proof of the statement 3.11.3 it follows that sets:

$\Omega_4 = \{W \cdot V | V \in \Omega_2,\ W \in \Omega_3\}$;
$\Omega_5 = \{U \bigcap V | U \in \Omega_1,\ V \in \Omega_2\}$;
$\Omega_6 = \{U \bigcap (W \cdot V) | U \in \Omega_1,\ V \in \Omega_2,\ W \in \Omega_3\}$;
$\Omega_7 = \{W \cdot (V \bigcap U) | U \in \Omega_1,\ V \in \Omega_2,\ W \in \Omega_3\}$
are basis of neighborhoods of unit, accordingly, in topological groups:
$(G, inf_{\mathfrak{M}}\{\tau_3, \tau_2\})$;
$(G, sup_{\mathfrak{M}}\{\tau_1, \tau_2\})$;
$(G, sup_{\mathfrak{M}}\{\tau_1, inf_{\mathfrak{M}}\{\tau_3, \tau_2\}\})$;
$(G, inf_{\mathfrak{M}}\{\tau_3, sup_{\mathfrak{M}}\{\tau_1, \tau_2\}\})$.

If $U\bigcap(W\cdot V)\in\Omega_6$, then there exists such $U_1\in\Omega_1$ that $U_1\cdot U_1\subseteq U$, and as $\tau_1\le\tau_3$ then there exists such $W_1\in\Omega_3$ that $W_1\subseteq U_1\bigcap W$. Then $W_1\cdot(V\bigcap U_1)\in\Omega_7$ and

$$W_1\cdot(V\bigcap U_1)\subseteq(W_1\cdot U_1)\bigcap(W_1\cdot V)\subseteq(U_1\cdot U_1)\bigcap(W\cdot V)\subseteq U\bigcap(W\cdot V),$$

and hence, $sup_{\mathfrak{M}}\{\tau_1,inf_{\mathfrak{M}}\{\tau_3,\tau_2\}\}\le inf_{\mathfrak{M}}\{\tau_3,sup_{\mathfrak{M}}\{\tau_1,\tau_2\}\}$.

Let now $W\cdot(V\bigcap U)\in\Omega_7$. There exists such $U_1\in\Omega_1$ that $U_1\cdot U_1\subseteq U$, and as $\tau_1\le\tau_3$ then there exists such $W_1\in\Omega_3$ that $W_1\subseteq W$ and $W_1^{-1}\subseteq U_1$. Then $U_1\bigcap(W_1\cdot V)\in\Omega_6$. If $u\in U_1\bigcap(W_1\cdot V)$ then $u=w\cdot v$, where $w\in W_1$ and $v\in V$. Then $v=w^{-1}\cdot u\in W^{-1}\cdot U_1\subseteq U_1\cdot U_1\subseteq U$, and hence, $v\in V\bigcap U$, i.e. $u=w\cdot v\in W\cdot(V\bigcap U)$.

From the arbitrariness of the element u it follows that $U_1\bigcap(W_1\cdot V)\subseteq W\cdot(V\bigcap U)$, and hence, $inf_{\mathfrak{M}}\{\tau_3,sup_{\mathfrak{M}}\{\tau_1,\tau_2\}\}\le sup_{\mathfrak{M}}\{\tau_1,inf_{\mathfrak{M}}\{\tau_3,\tau_2\}\}$.

Then $inf_{\mathfrak{M}}\{\tau_3,sup_{\mathfrak{M}}\{\tau_1,\tau_2\}=sup_{\mathfrak{M}}\{\tau_1,inf_{\mathfrak{M}}\{\tau_3,\tau_2\}\}$.

As $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{M}$, then

$$inf_{\mathfrak{A}}\{\tau_3,sup_{\mathfrak{A}}\{\tau_1,\tau_2\}\}=inf_{\mathfrak{M}}\{\tau_3,sup_{\mathfrak{M}}\{\tau_1,\tau_2\}\}=$$

$$=sup_{\mathfrak{M}}\{\tau_1,inf_{\mathfrak{M}}\{\tau_3,\tau_2\}\}=sup_{\mathfrak{A}}\{\tau_1,inf_{\mathfrak{A}}\{\tau_3,\tau_2\}\},$$

i.e. the lattice $\mathfrak{A}$ is a modular lattice.

The theorem is completely proved.

4.3. Corollary. For any group G the following lattices are modular:

- The lattice $\mathfrak{N}$ of all group topologies on the group G in which the topological group possesses a basis of neighborhoods of unit consisting of normal subgroups;

- The lattice $\mathfrak{S}$ of all group topologies on the group G in which right and left uniform structures coincide;

- The lattice $\mathfrak{C}$ of all group topologies on the group G in which the topological group is precompact.

4.4. Theorem. Let G be a group and $\mathfrak{A}$ be a sublattice of the lattice $\mathfrak{M}$ of all group topologies on G or it be a sublattice of the lattices $\mathfrak{G}$ of all group topologies on G in which G possesses a basis of neighborhoods of unit consisting of subgroups. If $\tau(G_{[i]})\in\mathfrak{A}$ (them definition of $\tau(G_{[i]})$ see 2.4 and 2.6) for any $i\in\mathbb{N}$ and τ_0 and τ_1 are such group topologies from $\mathfrak{A}$ that $\tau_0\prec_{\mathfrak{A}}\tau_1$ (the definition of $\prec$ see in 3.8) and $(\tau_0)_{G_{[k]}}=(\tau_1)_{G_{[k]}}$ (see 2.7) for some natural number k, then the following statements are true:

4.4.1. If $n=min\{k|(\tau_0)_{G_{[k]}}=(\tau_1)_{G_{[k]}}\}$ then $\tau_0=inf_{\mathfrak{A}}\{\tau_1,(\tau_0)_{G_{[n-1]}}\}$;

4.4.2. For any neighborhood V of unit in the topological group (G,τ_1) there exist such neighborhoods V_1 and W_1 in topological groups (G,τ_1) and $(G,(\tau_0)_{G[n-1]})$

(definition of number n see in the formulation of the statement 4.4.1), accordingly, that $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any element $g \in W_1$;

4.4.3. If τ is a group topology from $\mathfrak{A}$ such that $\tau_0 \leq \tau$, then for any neighborhood V of unit in the topological group (G, τ_1) there exist such neighborhoods V_1 and U_1 of unit in topological groups (G, τ_1) and (G, τ), accordingly, that $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any element $g \in U_1$;

4.4.4. If τ is a group topology from $\mathfrak{A}$ such that $\tau_0 \leq \tau$, then $\tau \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{\tau, \tau_1\}$;

4.4.5. If τ is a group topology from $\mathfrak{A}$ then $sup_{\mathfrak{A}}\{\tau, \tau_0\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{\tau, \tau_1\}$.

Proof.

4.4.1. In the beginning we shall show that $(\tau)_{G_{[i]}} \in \mathfrak{A}$.

From 2.6 and proofs of statements 3.7.1 and 3.7.2 it follows that

$$\tau_{G_{[i]}} = sup_{\mathfrak{M}}\{\tau, \tau(G_{[i]})\} = sup_{\mathfrak{A}}\{\tau, \tau(G_{[i]})\} \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on G, then

$$\tau_{G_{[i]}} = sup_{\mathfrak{M}}\tau, \tau(G_{[i]})\} = sup_{\mathfrak{A}}\tau, \tau(G_{[i]})\} \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{G}$ of all group topologies on G in which group G possesses a basis of neighborhoods of unit consisting of subgroups, then

$$\tau_{G_{[i]}} = sup_{\mathfrak{G}}\{\tau, \tau(G_{[i]})\} = sup_{\mathfrak{A}}\{\tau, \tau(G_{[i]})\} \in \mathfrak{A}.$$

So, we have proved that in both cases $\tau_{G_{[i]}} \in \mathfrak{A}$ for any $\tau \in \mathfrak{A}$ and $i \in \mathbb{N}$.

As $V \bigcap G_{[n-1]} \subseteq V$ for any neighborhood V of unit of the topological group (G, τ_0) then $\tau_0 \leq (\tau_0)_{G_{[n-1]}}$, and hence, $\tau_0 \leq inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\}$.

From definition of the number n it follows that $(\tau_0)_{G_{[n-1]}} < (\tau_1)_{G_{[n-1]}}$.

Then

$$(inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})_{G_{[n-1]}} \leq (\tau_0)_{G_{[n-1]}} < (\tau_1)_{G_{[n-1]}},$$

and hence, $inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} < \tau_1$.

So, we have received that $\tau_0 \leq inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} < \tau_1$.

As $\tau_0 \prec_{\mathfrak{A}} \tau_1$, then $\tau_0 = inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\}$.

The statement 4.4.1 is proved.

4.4.2. Let V be a neighborhood of unit in the topological group (G, τ_1) and V_0 be such neighborhood of unit in the topological group (G, τ_1) that $V_0 \cdot V_0 \subseteq V$. From definition of the numbers n it follows, that there exists such neighborhood U of unit in topological group (G, τ_0) that $U \bigcap G_{[n]} \subseteq V_0$.

There exist such neighborhoods U_1 of unit in the topological group (G, τ_0) and V_0 in the topological group (G, τ_1) that $U_1^{-1} \cdot U_1 \cdot U_1 \cdot U_1^{-1} \subseteq U$ and $V_1 \subseteq V_0 \bigcap U_1$. Then $W_1 = U_1 \bigcap G_{[n-1]}$ will be a neighborhood of unit in the topological group $(G, (\tau_0)_{G_{[n-1]}})$.

As $g^{-1} \cdot a \cdot g \cdot a^{-1} \in [G_{n-1}, G] \subseteq G_{[n]}$ and $g^{-1} \cdot a \cdot g \cdot a^{-1} \in U_1^{-1} \cdot U_1 \cdot U_1 \cdot U_1^{-1} \subseteq U$ for any elements $g \in V_1$ and $a \in W_1$, then

$$a \cdot g \cdot a^{-1} = g \cdot (g^{-1} \cdot a \cdot g \cdot a^{-1}) \in V_1 \cdot (U \bigcap G_{[n]}) \subseteq V_0 \cdot V_0 \subseteq V$$

for any elements $g \in V_1$ and $a \in W_1$, i.e. $a \cdot V_1 \cdot a^{-1} \subseteq V$ for any element $a \in W_1$.

The statement 4.4.2 is proved.

4.4.3. Let V be a neighborhood of unit in the topological group (G, τ_1) and V_0 be such neighborhood of unit in the topological group (G, τ_1) that $V_0 \cdot V_0 \cdot V_0^{-1} \subseteq V$. According to the statement 4.4.2 there exist such neighborhoods V_1 and W_1 of unit in topological groups (G, τ_1) and $(G, (\tau_0)_{G_{[n-1]}})$, accordingly, that $V_1 \subseteq V_0$ and $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any element $g \in W_1$.

If $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{M}$ of all group topologies on G, then from Proposition 3.12 and statement 3.11.3 it follows that the set $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})$, and as $inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} = inf_{\mathfrak{A}}|\{\tau_1, (\tau_0)_{G_{[n-1]}}\}$ then $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group $(G, inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})$ in this case.

If $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{G}$ of all group topologies on G in which G possesses a basis of neighborhoods of unit consisting of subgroups, then from Corollary 4.3 and Proposition 4.1 it follows that the topological group $(G, inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})$ possesses a basis of neighborhoods of unit consisting of subgroups, and hence, $inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} \in \mathfrak{G}$. As $\mathfrak{G} \subseteq \mathfrak{M}$, then $inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} = inf_{\mathfrak{G}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\}$, and as $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{G}$, then

$$inf_{\mathfrak{M}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} = inf_{\mathfrak{G}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} = inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\}.$$

and hence also in this case $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group $(G, inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})$.

So, we have proved that in both cases the set $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group $(G, inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})$.

As (see definition of number n) $(\tau_1)_{G_{[n-1]}} \neq (\tau_0)_{G_{[n-1]}}$ then

$$(\tau_1)_{G_{[n-1]}} > (\tau_0)_{G_{[n-1]}} = ((\tau_0)_{G_{[n-1]}})_{G_{[n-1]}} \geq (inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\})_{G_{[n-1]}})_{G_{[n-1]}}.$$

Then $\tau_1 > inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} \geq \tau_0$. As $\tau_0 \prec_{\mathfrak{A}} \tau_1$ then $inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\} = \tau_0$, and hence, $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group (G, τ_0).

As $\tau_0 \leq \tau$ then $U = V_1 \cdot W_1$ is a neighborhood of unit in the topological group (G, τ) too.

If now $b \in U = V_1 \cdot W_1$ then there exist such elements $v \in V_1$ and $w \in W_1$ that $b = v \cdot w$. Then

$$b \cdot a \cdot b^{-1} = v \cdot w \cdot a \cdot w^{-1} \cdot v^{-1} \in V_1 \cdot (w \cdot V_1 \cdot w^{-1}) \cdot V_1^{-1} \subseteq V_0 \cdot V_0 \cdot V_0^{-1} \subseteq V.$$

The statement 4.4.3 is proved.

4.4.4. Assume the contrary, i.e. that $sup_{\mathfrak{A}}\{\tau, \tau_1\} < \tau' < \tau$ for some group topology $\tau' \in \mathfrak{A}$.

Consider the set $\mathfrak{A}' = \{\tau_0, \tau_1, \tau, \tau', sup_{\mathfrak{A}}\{\tau, \tau_1\}\}$. We shall check that $\mathfrak{A}'$ is a sublattice of the lattice $\mathfrak{A}$.

Let $\tau_2, \tau_3 \in \mathfrak{A}'$.

The following two cases are possible:

- These topology are comparable among themselves;
- These topology are not comparable among themselves.

Let topologies τ_2 and τ_3 be comparable among themselves, and let $\tau_2 \leq \tau_3$. Then $\inf_{\mathfrak{A}}\{\tau_2, \tau_3\} = \tau_2 \in \mathfrak{A}'$ and $\sup_{\mathfrak{A}}\{\tau_2, \tau_3\} = \tau_3 \in \mathfrak{A}'$.

Let now topologies τ_2 and τ_3 be not comparable among themselves. As $\tau_0 < \tau_1 < sup_{\mathfrak{A}}\{\tau, \tau_1\}$ and $\tau_0 < \tau < \tau' < sup_{\mathfrak{A}}\{\tau, \tau_1\}$, then either $\tau_2 = \tau_1$ and $\tau_3 \in \{\tau, \tau'\}$, or $\tau_3 = \tau_1$ and $\tau_2 \in \{\tau, \tau'\}$.

We assume, for definiteness, that $\tau_2 = \tau_1$, and $\tau_3 \in \{\tau, \tau'\}$.

As $\sup_{\mathfrak{A}}\{\tau, \tau_1\} < \tau' < \tau$, then $\sup_{\mathfrak{A}}\{\tau, \tau_1\} \leq \sup_{\mathfrak{A}}\{\tau', \tau_1\} \leq \sup_{\mathfrak{A}}\{\tau, \tau_1\}$. Then $\sup_{\mathfrak{A}}\{\tau', \tau_1\} = \sup_{\mathfrak{A}}\{\tau, \tau_1\}$, and hence, $\sup_{\mathfrak{A}}\{\tau_2, \tau_3\} = \sup_{\mathfrak{A}}\{\tau, \tau_1\} \in \mathfrak{A}'$.

Moreover, as τ_2 and τ_3 are not comparable, then $\tau_0 \leq \inf_{\mathfrak{A}}\{\tau_2, \tau_3\} < \tau_2 = \tau_1$, and as $\tau_0 \prec_{\mathfrak{A}} \tau_1$ then $\inf_{\mathfrak{A}}\{\tau_2, \tau_3\} = \tau_0 \in \mathfrak{A}'$.

So, we have proved that $\mathfrak{A}'$ is a sublattice of the lattice $\mathfrak{A}$, and hence, $\mathfrak{A}'$ is a sublattice of the lattice $\mathfrak{M}$ for the case when $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{M}$.

Now show that $\mathfrak{A}'$ is a sublattice of the lattice $\mathfrak{M}$ also for the case when $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{G}$.

In the proof of the statement 3.7.2 it has been proved that $\sup_{\mathfrak{M}} S = \sup_{\mathfrak{G}} S$, for any subset $S \subseteq \mathfrak{G}$, and as $\mathfrak{A}$ is a sublattice of the lattice $\mathfrak{G}$ then

$\sup_{\mathfrak{A}'}\{\tau_2, \tau_3\} = \sup_{\mathfrak{A}}\{\tau_2, \tau_3\} = \sup_{\mathfrak{G}}\{\tau_2, \tau_3\} = \sup_{\mathfrak{M}}\{\tau_2, \tau_3\}$.

Moreover:

If topologies τ_2 and τ_3 are comparable among themselves and $\tau_2 \leq \tau_3$, then $\inf_{\mathfrak{A}}\{\tau_2, \tau_3\} = \tau_2 = \inf_{\mathfrak{M}}\{\tau_2, \tau_3\}$.

If topologies τ_2 and τ_3 are not comparable among themselves as it was been proved above, $\tau_2 = \tau_1$, and $\tau_3 \in \{\tau, \tau'\}$. As $\tau_0 < \tau_3$, then from the statement 4.4.3 and Proposition 4.1 it follows that $\inf_{\mathfrak{M}}\{\tau_2, \tau_3\} \in \mathfrak{G}$, and hence, $\inf_{\mathfrak{M}}\{\tau_2, \tau_3\} = \inf_{\mathfrak{G}}\{\tau_2, \tau_3\} = \inf_{\mathfrak{A}'}\{\tau_2, \tau_3\}$.

So, we have proved that $\mathfrak{A}'$ is a sublattice of the lattice $\mathfrak{M}$ in both cases.

Now show that $(\mathfrak{A}', <)$ is a modular lattice.

For this purpose, as it agrees with Proposition 4.2 we need to check that for any two topologies $\tau_2, \tau_3 \in \mathfrak{A}'$ the statement 3.11.1 is true.

So, let $\tau_2, \tau_3 \in \mathfrak{A}'$.

If these topologies are comparable among themselves and $\tau_2 \leq \tau_3$, then for any neighborhoods U_0 and V_0 of unit in topological groups (G, τ_2) and (G, τ_3), accordingly, there exist such neighborhoods U_1 and V_1 in topological groups (G, τ_2) and (G, τ_3), accordingly, that $U_1 \cdot U_1 \subseteq U_0$ and $V_1 \subseteq U_1$. Then $V_1 \cdot U_1 \subseteq U_1 \cdot U_1 \subseteq U_0 \subseteq U_0 \cdot V_0$, i.e. in this case the statement 3.11.1 is true.

If topologies τ_2 and τ_3 are not comparable between themselves, then one of them is equal to τ_1, and the second belongs to the set $\{\tau, \tau'\}$.

We admit, for definiteness, that $\tau_2 = \tau_1$ and $\tau_3 \in \{\tau, \tau'\}$. Then $\tau_3 > \tau_0$, and for the pair of topologies τ_2 and τ_3 applying both the statement 4.4.3 and Proposition 3.12, we shall receive, that in this case for topologies τ_2 and τ_3 the statement 3.11.1 is true.

So, we have received that for any pair topologies from $\mathfrak{A}'$ the statement 3.11.1 is true, and, according to Proposition 4.2, the lattice $\mathfrak{A}'$ is modular. Then

$$\tau' = inf_{\mathfrak{A}'}\{sup_{\mathfrak{A}'}\{\tau_1, \tau\}, \tau'\} = sup_{\mathfrak{A}'}\{\tau, inf\{\tau_1, \tau'\}\} = sup_{\mathfrak{A}'}\{\tau, \tau_0\} = \tau.$$

We have obtained the contradiction with the choice of the topology τ'.

The statement 4.4.4 is proved.

4.4.5. Let τ be a group topology from $\mathfrak{A}$.

As $\tau_0 \leq sup\{\tau, \tau_0\}$, then taking into account the statement 4.4.4, we receive that

$$sup_{\mathfrak{A}}\{\tau, \tau_1\} = sup_{\mathfrak{A}}\{sup_{\mathfrak{A}}\{\tau, \tau_0\}, \tau_1\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{sup_{\mathfrak{A}}\{\tau, \tau_0\}, \tau_0\} = sup_{\mathfrak{A}}\{\tau, \tau_0\}.$$

The theorem is completely proved.

4.5. Corollary. Let G be a nilpotent group, $\mathfrak{A}$ be a sublattice of the lattice $\mathfrak{M}$ of all group topologies, or it be a sublattice of the lattice $\mathfrak{G}$ of all group topologies in each of which the group G possesses a basis of neighborhoods of unit consisting of subgroups. If $\tau(G_{[i]}) \in \mathfrak{A}$ for any $i \in \mathbb{N}$, τ_0 and τ_1 are such group topologies from $\mathfrak{A}$, that $\tau_0 \prec_{\mathfrak{A}} \tau_1$ then the following statements are true:

4.5.1. If $n = min\{k|\ (\tau_0)_{G_{[k]}} = (\tau_1)_{G_{[k]}}\}$, then $\tau_0 = inf_{\mathfrak{A}}\{\tau_1, (\tau_0)_{G_{[n-1]}}\}$;

4.5.2. If τ is a group topology on the group G such that $\tau_0 \leq \tau$, then for any neighborhood V of unit in topological group (G, τ_1) there exist neighborhoods V_1 and U_1 of unit in topological groups (G, τ_1) and (G, τ), accordingly, such that $g \cdot V_1 \cdot g^{-1} \subseteq V$ for any element $g \in U_1$;

4.5.3. If τ is a group topology from $\mathfrak{A}$ such that $\tau_0 \leq \tau$ then $sup_{\mathfrak{A}}\{\tau, \tau_1\} \preceq_{\mathfrak{A}} \tau$;

4.5.4. If τ is a group topology from $\mathfrak{A}$, then $sup_{\mathfrak{A}}\{\tau, \tau_1\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{\tau, \tau_0\}$.

Really, as G is a nilpotent group, then $G_{[k]} = \{e\}$ for some natural number k. Then $(\tau_0)_{G_{[k]}} = \{e\} = (\tau_1)_{G_{[k]}}$. Then from Theorem 4.4 the truth of the present corollary follows.

4.6. Theorem. Let:

– G be a group;

– $\mathfrak{A}$ be a sublattice of the lattice $\mathfrak{M}$ of all group topologies, or it be a sublattice of the lattice $\mathfrak{G}$ of all group topologies, in each of which the group G possesses a basis of neighborhoods of unit consisting of subgroups;

– $\tau(G_{[i]}) \in \mathfrak{A}$ for all $i \in \mathbb{N}$;

– $\tau_0 \prec_{\mathfrak{A}} \tau_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ (i.e. this chain is a unrefinable chain of group topologies in $\mathfrak{A}$);

– $\tau'_0 < \tau'_1 < \ldots < \tau'_m$ is a chain of group topologies from $\mathfrak{A}$ such that $\tau_0 = \tau'_0$ and $\tau'_m = \tau_n$.

If $(\tau_0)_{G_{[k]}} = (\tau_n)_{G_{[k]}}$ for some $k \in \mathbb{N}$, then $m \leq n$, and $m = n$ if only if $\tau'_0 \prec_{\mathfrak{A}} \tau'_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau'_m$.

Proof. In the beginning we shall prove that $n \leq m$.

Assume the contrary, i.e. that $m > n$, and let n be least of the natural numbers for which there exist such chains of topologies.

As $\tau_0 \prec_{\mathfrak{A}} \tau_1$ then $n > 1$, and hence, $m > 2$.

Then $\tau_1 \not\leq \tau'_1$, for otherwise the chain $\tau_1 \prec_{\mathfrak{A}} \tau_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ has length $n-1$, and the chain $\tau_1 \leq \tau'_1 < \tau'_1 < \ldots < \tau'_m = \tau_n$ has length not less than $m-1 > n-1$, and it contradicts the choice of number n.

Moreover, as $\tau_0 \prec_{\mathfrak{A}} \tau_1$ and $\tau_0 = \tau'_0 < \tau'_1$ then $\tau'_1 \not< \tau_1$, and hence, topologies τ_1 and τ'_1 are not comparable.

For each integer $0 \leq j \leq n$ by induction we shall define a topology $\tau''_j \in \mathfrak{A}$ as follows:

Put $\tau''_0 = \tau'_1$ and $\tau''_{i+1} = sup_{\mathfrak{A}}\{\tau_{i+1}, \tau''_i\}$.

As $\tau_i \leq \tau''_i \leq \tau_n$ for any $0 \leq i \leq n$ then $\tau''_n = \tau_n$. Then, according to the statement 4.4.5, $\tau''_{i+1} = sup_{\mathfrak{A}}\{\tau_{i+1}, \tau''_i\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{\tau_i, \tau''_i\} = \tau''_i$ for any $0 \leq i \leq n-1$.

If $\tau''_s = \tau''_{s+1}$ for some integer $0 \leq s < n-1$ then the chain

$$\tau'_1 = \tau''_0 \preceq_{\mathfrak{A}} \ldots \preceq_{\mathfrak{A}} \tau''_s \preceq_{\mathfrak{A}} \tau''_{s+2} \preceq_{\mathfrak{A}} \ldots \preceq_{\mathfrak{A}} \tau''_n = \tau_n = \tau'_m$$

has length which does not surpass number $n-1$, and the chain $\tau'_1 < \ldots < \tau'_m$ has length $m-1$.

This contradicts the choice of number n.

If $s = n-1$, then $\tau''_{n-1} = \tau_n = \tau'_m$ and hence the chain

$$\tau'_1 = \tau''_0 \preceq_{\mathfrak{A}} \ldots \preceq_{\mathfrak{A}} \tau''_s \preceq_{\mathfrak{A}} \tau''_{s+2} \preceq_{\mathfrak{A}} \ldots \preceq_{\mathfrak{A}} \tau''_{n-1} = \tau_n = \tau'_m$$

has length which does not surpass number $n-1$, and the chain $\tau'_1 < \ldots < \tau'_m$ has length $m-1$.

This contradicts the choice of number n in this case, too.

Hence, $\tau''_j \neq \tau''_{j+1}$ for any $0 \leq j \leq n-1$.

As topologies τ_1 and τ'_1 are not comparable, then $\tau_1 < \tau''_1$, and hence, we have received the chain $\tau_1 \prec_{\mathfrak{A}} \tau_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ which has length $n-1$ and the chain $\tau_1 < \tau''_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau''_n$ which has length n.

This contradicts the choice of number n. Hence, $m \leq n$.

Let now $m = n$ and assume that $\tau'_l \not\prec_{\mathfrak{A}} \tau'_{l+1}$ for some number $0 \leq l \leq m-1$.

Then there is such topology $\tau'' \in \mathfrak{A}$ that $\tau'_l < \tau'' < \tau'_{l+1}$.

Then the chain of topologies $\tau'_0 < \ldots < \tau'_l < \tau'' < \tau'_{l+1} < \ldots < \tau'_m$ has length $m+1 > n$.

We have received the contradiction with earlier proved.

The theorem is completely proved.

4.7. Corollary. Let:
– G be a nilpotent group;
– $\mathfrak{A}$ be a sublattice of the lattice $\mathfrak{M}$ of all group topologies or it be a sublattice of the lattice $\mathfrak{G}$ of all group topologies in each of which the group G possesses basis of neighborhoods of unit consisting of subgroups;
– $\tau(G_{[i]}) \in \mathfrak{A}$ for all $i \in \mathbb{N}$;
$\tau_0 \prec_{\mathfrak{A}} \tau_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ is unrefinable chain of group topologies in $\mathfrak{A}$:
$\tau'_0 < \tau'_1 < \ldots < \tau'_m$ is some chain of topologies from $\mathfrak{A}$.
If $\tau_0 = \tau'_0$ and $\tau'_m = \tau_n$, then $m \leq n$, and $m = n$ if and only if $\tau'_0 \prec_{\mathfrak{A}} \tau'_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau'_m$.

Really, as G is a nilpotent group, then $G_{[k]} = \{e\}$ for some natural number k, and hence, $(\tau_0)_{G_{[k]}} = e\} = (\tau_1)_{G_{[k]}}$. Then from Theorem 4.6 the truth fidelity of the present corollary follows.

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Received March 3, 2010