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On subsemimodules of semimodules

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Abstract. P. J. Allen [1] introduced the notion of a Q-ideal and a construction process was presented by which one can build the quotient structure of a semiring modulo a Q-ideal. Here we introduce the notion of Q_M-subsemimodule N of a semimodule M over a semiring R and construct the factor semimodule M/N. It is shown that this notion inherits most of the essential properties of the factor modules over a ring.

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1 Introduction

Just as the study of rings inevitably involves the study of modules over them, so the study of semirings inevitably involves the study of semimodules over them. Since semirings (resp. semimodules) are just "rings (resp. modules) without subtraction", to extend the properties of factor module and submodule from the category of modules to the category of semimodules has stimulated several authors to show that many, but not all, of the results in the theory of modules are also valid for semimodules. Consequently, semiring theory has emerged as a potential area of interdisciplinary research and semimodule theory is of recent interest [4, 5]. So semimodules constitute a fairly natural generalization of modules, with broad applications in the mathematical foundations of computer science [7]. The main part of this paper is devoted to stating and proving analogues to several well-known theorems in the theory of factor modules and prime submodules.

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring we mean an algebraic system $R = (R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab+ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r+0 = r$ and $r.0 = 0.r = 0$ for all $r \in R$. Throughout this paper let R be a commutative semiring. A semiring R is said to be semidomain whenever $a, b \in R$ with $ab = 0$ implies that either $a = 0$ or $b = 0$. A semifield is a semiring in which non-zero elements form a group under multiplication. A (left) semimodule M over a semiring R is a commutative additive semigroup which has a zero element, together with a mapping from $R \times M$ into M (sending (r, m) to rm) such that $(r + s)m = rm + sm$, $r(m + p) = rm + rp$, $r(sm) = (rs)m$ and $0m = r0_M = 0_M$ for all $m, p \in M$ and $r, s \in R$. Let M be a semimodule over the semiring R, and let N be a subset of M. We say that N is a subsemimodule of M, or an R-subsemimodule of M, percisely when N is itself an R-semimodule with respect to the operations for M (so $0_M \in N$). It is easy to see that if $r \in R$, then $rM = \{rm : m \in M\}$ is a subsemimodule of M. The semiring R is considered to be also a semimodule over itself. In this case, the subsemimodules of R are called ideals of R.

Let M be a semimodule over a semiring R. A subtractive subsemimodule ($= k$-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M). A prime subsemimodule of M is a proper subsemimodule P of M in which $x \in P$ or $rM \subseteq P$ whenever $rx \in P$. We define k-ideals and prime ideals of a semiring R in a similar fashion.

2 Factor semimodules

Allen in [1] has presented the notion of a Q-ideal I in the semiring R and constructed the quotient semiring $R/I = \{q + I : q \in Q\}$. Similarly, our aim here is to construct factor semimodules.

Definition 1. A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule ($= Q_M$-subsemimodule) if there exists a non-empty subset Q_M of M such that

(1) $RQ_M \subseteq Q_M$, where $RQ_M = \{rq : r \in R, q \in Q_M\}$;
(2) $M = \cup\{q + N : q \in Q_M\}$;
(3) If $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

It is easy to see that if $M = Q_M$, then $\{0\}$ is a Q_M-subsemimodule of M.

Remark 1. (The construction of factor semimodules.) Let M be a semimodule over a semiring R, and let N be a Q_M-subsemimodule of M. We put $M/N = \{q+N : q \in Q_M\}$. Then M/N forms a commutative additive semigroup which has zero element under the binary operation $\oplus$ defined as follows: $(q_1+N)\oplus(q_2+N) = q_3+N$ where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$. By the definition of Q_M-subsemimodule, there exists a unique $q_0 \in Q_M$ such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of M/N. But, for every $q \in Q$ from (1) one obtains $0_M = 0_R q \in Q$; hence $q_0 = 0_M$.

Now let $r \in R$ and suppose that $q_1 + N, q_2 + N \in M/N$ are such that $q_1 + N = q_2 + N$ in M/N. Then $q_1 = q_2$, we must have $rq_1 + N = rq_2 + N$ by Definition 1. Hence we can unambiguously define a mapping from $R \times M/N$ into M/N (sending $(r, q_1 + N)$ to $rq_1 + N$) and it is routine to check that this turns the commutative semigroup M/N into an R-semimodule. We call this R-semimodule the residue class semimodule or factor semimodule of M modulo N. The following example shows that the theory of Q-subsemimodules as developed in this paper is not superfluous.

Example 1. Let $R = \{0, 1, ..., n\}$ and define $x+y = \max\{x, y\}$ and $xy = \min\{x, y\}$ for each $x, y \in R$. R together with the two defined operations forms a semiring. Define $a + b = \max\{a, b\}$ for each $a, b \in M$, where M is the set of all nonnegative integers. Then $(M, +)$ is a commutative additive semigroup with the zero element 0. Define a mapping from $R \times M$ into M (sending (r, m) to $\max\{r, m\}$) and it is clear that M is an R-semimodule. One can easily show that $N = R$ is an R-subsemimodule of M with $N \neq 0$, $0 + N = N$ and $k + N = \{k\}$ for each $n < k$. Thus N is a Q_M-subsemimodule of M when $Q_M = \{0\} \cup \{k \in M : n < k\}$. Therefore, M/N is a semimodule over R.

Let M and N be semimodules over the semiring R. A mapping $f : M \to N$ is said to be a homomorphism of R-semimodules if $f(a + b) = f(a) + f(b)$ and $f(ra) = rf(a)$ for all $a, b \in M$ and $r \in R$. In this case, the kernel of f, denoted by $\mathrm{Ker}(f)$, is the set $\{x \in M : f(x) = 0\}$. The proof of the following lemma is straightforward.

Lemma 1. *Assume that M and N are semimodules over the semiring R and let $f : M \to N$ be a homomorphism of R-semimodules. Then* $\mathrm{Ker}(f)$ *is a k-subsemimodule of M.*

Remark 2. Let R be a semiring. If N is a Q_M-subsemimodule of an R-semimodule M, then N is a k-subsemimodule of M.

Proof. Let $a, a + b \in N$. Then $b = q + z$ for some $q \in Q_M$ and $z \in N$, so $a + b \in (q + N) \cap (0_M + N)$, so $q = 0_M$; hence $b \in N$, as required. □

Theorem 1. *Let M be a semimodule over a semiring R, N a Q_M-subsemimodule of M and L a k-subsemimodule of M with $N \subseteq L$. Then $L/N = \{q+N : q \in L \cap Q_M\}$ is a k-subsemimodule of M/N.*

Proof. Since $0_M + N = N \subseteq L$, we must have $0_M + N \in L/N$. Suppose that $\bar{q_1} = q_1 + N$, $\bar{q_2} = q_2 + N \in L/N$ where $q_1, q_2 \in L \cap Q_M$. There is a unique element $q_3 \in Q_M$ with $\bar{q_1} \oplus \bar{q_2} = q_3 + N$ and $q_1 + q_2 + N \subseteq q_3 + N$, so $q_1 + q_2 + a = q_3 + b \in L$ for some $a, b \in N$; thus $q_3 \in Q_M \cap L$ since L is a k-subsemimodule. Therefore, $\bar{q_1} \oplus \bar{q_2} \in L/N$. Now it is enough to show that if $r \in R$ and $a + N \in L/N$ (where $a \in L \cap Q_M$), then $r(a + N) = ra + N \in L/N$. Since by Definition 1, $ra \in L \cap Q_M$, we must have $r(a + N) \in L/N$. Thus L/N is a subsemimodule of M/N.

Finally, assume that $t + N \in L/N$ and $(t + N) \oplus (s + N) = u + N \in L/N$ where $t, u \in L \cap Q_M$, $s \in Q_M$ and $t + s + N \subseteq u + N$. Then $t + s + e = u + f \in L$ for some $e, f \in N$; hence $s \in L \cap Q_M$ since L is a k-subsemimodule of M. Therefore, $s + N \in L/N$, as needed. □

Theorem 2. *Let M be a semimodule over a semiring R, N a Q_M-subsemimodule of M and L a k-subsemimodule of M/N. Then $L = T/N$ for some k-subsemimodule T of M.*

Proof. Set $T = \{m \in M : m \in q + N \in L$ for some $q \in Q_M\}$. We show that T is a k-subsemimodule of M and $L = T/N$. We split the proof into three cases. of steps.

1) Clearly, $N \subseteq T$.

2) T is a subsemimodule of M. For if x and y are in T, then there are elements $q_1, q_2 \in Q_M$ such that $q_1 + N, q_2 + N \in L$, $x = q_1 + c$ and $y = q_2 + d$ for some $c, d \in N$, so $(q_1 + N) \oplus (q_2 + N) = q_3 + N \in L$ where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N \in L$; hence $x + y \in q_1 + q_2 + N \subseteq q_3 + N \in L$. Thus, $x + y \in T$. It suffices to show that if $x \in T$ and $t \in R$, then $tx \in T$. There are elements $q_4 \in Q_M$ and $u \in N$ such that $x = q_4 + u$ and $t(q_4 + N) = tq_4 + N \in L$, so $tq_4 \in T$; hence $tx = tq_4 + tu \in T$.

3) T is a k-subsemimodule of M. Let $a, a+b \in T$. Then there are elements q_1, q_2 and q_3 of Q_M such that $a \in q_1 + N$, $a + b \in q_2 + N$ and $b \in q_3 + N$, so $a = q_1 + c$, $a+b = q_2+d$ and $b = q_3+e$ for some $c, d, e \in N$; hence $a+b \in (q_1+q_3+N)\cap(q_2+N)$. There is a unique element $q_4 \in Q_M$ such that $(q_1 + N) \oplus (q_3 + N) = q_4 + N$ where $q_1+q_3+N \subseteq q_4+N$, so $a+b \in (q_2+N)\cap(q_4+N)$; hence $q_2 = q_4$. Therefore, $q_3+N \in L$ since L is a k-subsemimodule; hence $b \in T$. Therefore, T is a k-subsemimodule of M. Finally, it is easy to see that $L = T/N = \{q + N : q \in Q_M \cap T\}$. □

Lemma 2. *Let M be a semimodule over a semiring R. Then the following hold:*

(i) If N and T are k-subsemimodules of M, then $N+T = \{a+b : a \in N, b \in T\}$ is a k-subsemimodule of M.

(ii) An intersection of a family of k-subsemimodules of M is a k-subsemimodule of M.

(iii) If I is an ideal of R, then the set IM consisting of all finite sums of elements $r_i m_i$ with $r_i \in R$ and $m_i \in M$ is a subsemimodule of M.

Proof. The proof is completely straightforward. □

Theorem 3. *Assume that N is a Q_M-subsemimodule of a semimodule M over a semiring R and let T be a k-subsemimodule of M. Then $(N + T)/N$ is a k-subsemimodule of M/N.*

Proof. By Remark 2 and Lemma 2, we must have $N + T$ is a k-subsemimodule of M; hence $(N + T)/N$ is a k-subsemimodule of M/N by Theorem 1. □

Theorem 4. *Assume that N is a Q_M-subsemimodule of a semimodule M over a semiring R and let T, L be k-subsemimodules of M containing N. Then $T/N = L/N$ if and only if $T = L$.*

Proof. Let $a \in T$. Then $a \in q_1 + N$ for some $q_1 \in Q_M$, so there is an element $c \in N \subseteq T$ such that $a = q_1 + c$; hence T k-subsemimodule gives $q_1 \in T$. Thus $q_1 + N \in T/N = L/N$ by Theorem 1, so $q_1 + N = q_2 + N$ for some $q_2 \in Q_M \cap L$. It follows that $q_1 \in L$. Therefore, $a \in L$, and so $T \subseteq L$. Similarly, $L \subseteq T$, and we have equality. □

Let R be a semiring with identity 1, M an R-semimodule and N an R-subsemimodule of M. N is a maximal (resp. k-maximal) subsemimodule of M if $M \neq N$ and there is no subsemimodule (resp. k-subsemimodule) T of M such that $N \subsetneq T \subsetneq M$. Also, we say that M is simple if it has only two k-subsemimodules $\{0_M\}$ and M.

Theorem 5. *Assume that N is a Q_M-subsemimodule of a semimodule M over a semiring R. Then N is a k-maximal subsemimodule of M if and only if M/N is a simple R-semimodule.*

Proof. First, suppose that N is a k-maximal subsemimodule of M and let L be a k-subsemimodule of M/N such that $L \neq 0_{M/N}$ and $M/N \neq L$. Then by Theorem 1 and Theorem 4, there is a k-subsemimodule T of M such that $N \subsetneq T \subsetneq M$ which is a contradiction. Next, if M/N is simple, then by a similar argument N is a k-maximal subsemimodule. □

If M is a semimodule over a semiring R, then M is Noetherian (resp. Artinian) if any non-empty set of k-subsemimodules of M has maximal member (resp. minimal member) with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on k-subsemimodules of M.

Theorem 6. *Let M be a semimodule over a semiring R, and let N be a Q_M-subsemimodule of M. Then the following hold:*

(i) The R-semimodule M is Noetherian if and only if both N and M/N are Noetherian.

(ii) The R-semimodule M is Artinian if and only if both N and M/N are Artinian.

Proof. (i) First, suppose that M is Noetherian. Since every k-subsemimodule of N is a k-subsemimodule of M it is clear from the definition of Noetherian R-semimodule that N is Noetherian. By Theorem 2, an ascending chain of k-subsemimodules of M/N must have the form

$$T_1/N \subseteq T_2/N \subseteq ... \subseteq T_n/N \subseteq T_{n+1}/N \subseteq ...$$

where

$$T_1 \subseteq T_2 \subseteq ... \subseteq T_n \subseteq T_{n+1} \subseteq ...$$

is an ascending chain of k-subsemimodules of M all of which contain N by Theorem 4. Since the latter chain eventually becomes stationary, so must the former by Theorem 4. Conversely, assume that both N and M/N are Noetherian. Let

$$T_1 \subseteq T_2 \subseteq ... \subseteq T_n \subseteq T_{n+1} \subseteq ...$$

be an ascending chain of k-subsemimodules of M. The Remark 2 and Lemma 2 give

$$T_1 \cap N \subseteq T_2 \cap N \subseteq ... \subseteq T_n \cap N \subseteq T_{n+1} \cap N \subseteq ...$$

is an ascending chain of k-subsemimodules of N, and so there is a positive integer s such that $T_s \cap N = T_{s+i} \cap N$ for all positive integers i. By Theorem 3,

$$(T_1 + N)/N \subseteq (T_2 + N)/N \subseteq ... \subseteq (T_n + N)/N \subseteq (T_{n+1} + N)/N \subseteq ...$$

is a chain of k-subsemimodules of M/N. Thus there exists a positive integer t such that $(T_t + N)/N = (T_{t+i} + N)/N$ for all positive integer i, so that $N + T_t = N + T_{t+i}$ for all i by Theorem 4. Let $u = \max\{s, t\}$. We show that, for each positive integer i, $T_u = T_{u+i}$. Since the inclusion $T_u \subseteq T_{u+i}$ is trivial, we will prove the reverse inclusion. Let $x \in T_{u+i}$. Since $x \in N + T_{u+i} = N + T_u$, we must have $x = a + b$ for some $a \in N$ and $b \in T_u \subseteq T_{u+i}$. Hence $a \in T_{u+i}$ since it is a k-subsemimodule of M. It follows that $a \in N \cap T_{u+i} = N \cap T_u$; hence both a and b belong to T_u and $x \in T_u$, as needed.

(ii) This can be proved in a very similar manner to the way in which (i) was proved above, and we omit it. □

3 Prime subsemimodules

In this section we generalize some of the basic results from prime submodules of a module to prime subsemimodules of a semimodule.

Lemma 3. *Let R be a semiring, M an R-semimodule and N, L subsemimodules of M such that N is a k-subsemimodule, and let $m \in M$. Then the following hold:*

(i) $(N :_R L) = \{r \in R : rL \subseteq N\}$ is a k-ideal of R.

(ii) $(0 :_R M) = \{r \in R : rM = 0\}$ and $\{r \in R : rm \in N\}$ are k-ideals of R.

Proof. Clearly, $(N :_R L)$ is an ideal of R. Let $a, a + b \in (N :_R L)$; we show that $b \in (N :_R L)$. It suffices to show that $bc \in N$ for every $c \in L$. By assumption, $ac + bc, ac \in N$, so $bc \in N$ since N is a k-subsemimodule, and the proof is complete. (ii) follows from (i). □

Lemma 4. *Let R be a semiring with identity, M an R-semimodule and N a prime subsemimodules of M. Then $(N :_R M)$ is a prime ideal of R.*

Proof. Since N is a proper subsemimodule, we must have $(N :_R M) \neq R$. Let $ab \in (N :_R M)$. We may assume that there exists $m \in M$ such that $bm \notin N$. As $abm \in N$, N prime gives $aM \subseteq N$, as needed. □

Theorem 7. *Let R be a semiring with identity, M an R-semimodule and N an R-subsemimodule of M. Then the following assertions are equivalent.*

(i) N is a prime subsemimodule of M.

(ii) If whenever $IT \subseteq N$ with I an ideal of R and T a subsemimodule of M implies that $I \subseteq (N :_R M)$ or $T \subseteq N$.

Proof. $(i) \to (ii)$. Assume that N is prime in M and let $IT \subseteq N$ with $x \in T - N$. We show that $I \subseteq (N :_R M)$. Let $c \in I$. Then N prime and $cx \in N$ gives $c \in (N :_R M)$.

$(ii) \to (i)$. Suppose that $ay \in N$ where $a \in R$ and $y \in M$. Set $I = Ra$ and $T = Ry$. Then $IT \subseteq N$. By assumption, either $y \in N$ or $a \in (N :_R M)$. So N is prime. □

Lemma 5. *Let N be a Q_M-subsemimodule of a semimodule M over a semiring R, and let T be a k-subsemimodule of M with $N \subseteq T$. Then $(T :_R M) = (T/N :_R M/N)$.*

Proof. Assume that $r \in (T :_R M)$ and let $q_1 + N \in M/N$ where $q_1 \in Q_M$. Since $r(q_1 + N) = rq_1 + N$ and $rq_1 \in rM \subseteq N$, we must have $rq_1 \in T \cap Q_M$; hence $r(q_1 + N) \in T/N$. It follows that $(N :_R M) \subseteq (T/N :_R M/N)$. For the reverse inclusion, assume that $a(M/N) \subseteq T/N$ and let $x \in M$. There are elements $q_2 \in Q_M$ and $y \in N$ such that $x = q_2 + y$, so $ax = aq_2 + ay \in aq_2 + N = a(q_2 + N) \in a(M/N) \subseteq T/N$; hence $ax \in T$. Thus, $a \in (N :_R M)$, and the proof is complete. □

Theorem 8. *Assume that N is a Q_M-subsemimodule of a semimodule M over a semiring R with identity and let T be a k-subsemimodule of M with $N \subseteq T$. Then T is a prime R-subsemimodule of M if and only if T/N is a prime R-subsemimodule of M/N.*

Proof. First, suppose that T is a prime subsemimodule of M. By Theorem 4, $T/N \neq M/N$. To see that T/N is a prime subsemimodule of M/N, let $r(q_1 + N) = rq_1 + N \in T/N$ where $q_1 \in Q_M$. It follows from Theorem 1 that $rq_1 \in T$, so Lemma 5 and T prime gives either $q_1 + N \in T/N$ or $r \in (T/N :_R M/N)$. Thus T/N is a prime R-subsemimodule of M/N. The proof of the reverse implication is similar. □

Let M be a semimodule over a semiring R with identity. M is called a cancellative semimodule if whenever $rm = sm$ for elements $m \in M$ and $r, s \in R$, then $r = s$. A semiring R is called a cancellative semiring if it is a cancellative semimodule over itself.

Proposition 1. *Let R be a semiring with identity, M a cancellative R-semimodule and N a proper Q_M-subsemimodule of M. Then $I = (N :_R M)$ is a Q-ideal of R.*

Proof. Suppose that $Q = (R - I) \cup \{0\}$; we show that I is a Q-ideal of R. It is easy to see that $R = \cup\{q + I : q \in Q\}$. Let $(r + I) \cap (s + I) \neq \emptyset$ where $r, s \in Q$. Then there are elements $a, b \in I$ with $r + a = s + b$. We may assume that $r \neq 0$. There exists $m = q_1 + n \in M$ such that $rm \notin N$ for some $q_1 \in Q_M$ and $n \in N$, so $rq_1 + rn + aq_1 + an \in (rq_1 + N) \cap (sq_1 + N)$; hence $rq_1 = sq_1$ since N is a Q_M-subsemimodule. Since M is cancellative, we must have $r = s$. Thus I is a Q-ideal of R. □

In [3, Theorem 2.6], it is shown that if I is a proper Q-ideal of a semiring R, then I is prime if and only if R/I is a semidomain. Now by Lemma 4 and Proposition 1 we have the following theorem:

Theorem 9. *Let R be a semiring with identity, M a cancellative R-semimodule and N a prime Q_M-subsemimodule of M. Then $R/(N :_R M)$ is a semidomain*

Remark 3. (Change of semirings) Assume that I is a Q-ideal of a semiring R and let M be an R-semimodule. We show now how M can be given a natural structure as a semimodule over R/I. Let $q_1, q_2 \in Q$ such that $q_1+I = q_2+I$, and let $m \in M$. Then $q_1 = q_2$, and $q_1m = q_2m$. Hence we can unambiguously define a mapping $R/I \times M$ into M (sending $(q_1 + I, m)$ to q_1m) and it is routine to check that this turns the commutative additive semigroup with a zero element M into an R/I-semimodule. It should be noted that a subset of M is an R-subsemimodule if and only if it is an R/I-subsemimodule.

Lemma 6. *Let I be a Q-ideal of a semiring R, M an R-semimodule and N an R-subsemimodule of M. Then $(N :_R M) = (N :_{R/I} M)$.*

Proof. The proof is straightforward by Remark 3. □

Theorem 10. *Let R be a semiring with identity, M a cancellative R-semimodule and N a prime Q_M-subsemimodule of M with $P = (N :_R M)$. Then there is a one-to-one correspondence between prime subsemimodules of R/P-semimodule M/N and prime k-subsemimodules of M containing N.*

Proof. Let T be a prime k-subsemimodule of M containing N. It then follows from Theorem 4, Proposition 1 and Remark 3 that T/N is a proper R/P-susemimodule of M/N. Let $(a + P)(q_1 + N) = aq_1 + N \in T/N$ where $q_1 \in Q_M$ and $a \in Q = (R - P) \cup \{0\}$. We may assume that $a \neq 0$. There exists $q_2 \in Q_M \cap T$ such that $aq_1 = q_2 \in T$. Then T prime gives either $q_1 \in N$ (so $q_1 + N \in T/N$) or $(a + P) \in (T/N :_{R/P} M/N)$ by Lemma 5 and Lemma 6. Thus, T/N is a prime subsemimodule of M/N. To show that T is a prime subsemimodule of M, suppose that $rm \in T$ where $r \in R$ and $m \in M$. We may assume that $r \neq 0$. There are elements $s \in Q$, $q \in Q_M$, $p \in P$ and $n \in N$ such that $r = s + p$ and $m = q + n$, so $rm = sq + sn + pq + pn \in T$; hence $sq \in T \cap Q_M$ since T is a k-subsemimodule. Therefore, $(s+P)(q+N) = sq+N \in T/N$. So T/N prime gives either $q+N \in T/N$ (so $m \in T$) or $(s+P) \in (T/N :_{R/P} M/N)$ (so $r \in (T :_R M)$ by Lemma 5 and Lemma 6), and the proof is complete. □

Let M be a semimodule over a semiring R with identity. M is called a M-cancellative semimodule if whenever $rm = rn$ for elements $m, n \in M$ and $r \in R$, then $n = m$. A semiring R is called a R-cancellative semiring if it is a R-cancellative semimodule over itself. We say that a subsemimodule N of M is pure if $aN = N \cap aM$ for every $a \in R$.

Theorem 11. *Let R be a semiring with identity, M a M-cancellative R-semimodule and N a proper subsemimodule of M. Then N is a pure subsemimodule of M if and only if it is a prime subsemimodule of M with $(N :_R M) = 0$*

Proof. First, assume that N is pure in M and let $rm \in N$ with $r \notin (N :_R M)$ where $r \in R$ and $m \in M$. Then $rm \in rM \cap N = rN$, so $rm = rn$ for some $n \in N$; hence $m = n \in N$ since M is M-cancellative. Thus N is prime. Next, suppose that $a \in (N :_R M)$ with $a \neq 0$. Since $N \neq M$, there is an element $x \in M - N$ with $ax \in N \cap aM = aN$, so there exists $y \in N$ such that $ax = ay$; hence $x = y$ which is a contradiction. Thus $(N :_R M) = 0$.

Conversely, assume that N is prime in M. It suffices to show that $aM \cap N \subseteq aN$ for every $a \in R$. Let $az \in aM \cap N$ where $z \in M$. We may assume that $a \neq 0$. Then N prime gives $z \in N$, which is required. □

If R is a semiring (not necessarily a semidomain) and M is an R-semimodule, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in M\}$. It is clear that if R is a semidomain, then $T(M)$ is a subsemimodule of M.

Proposition 2. *Let M be a semimodule over a semidomain R with identity such that $T(M) \neq M$. Then $T(M)$ is a prime subsemimodule of M with $(T(M) :_R M) = 0$.*

Proof. Let $am \in T(M)$ with $a \notin (T(M) :_R M)$ where $a \in R$ and $m \in M$. Then $abm = 0$ for some non-zero element b of R. If $am = 0$, then $m \in T(M)$. So we may assume that $am \neq 0$. Therefore, $ab \neq 0$; hence $m \in T(M)$. Thus $T(M)$ is prime. So by Lemma 4, $P = (N :_R M)$ is a prime ideal of R; we show that $P = 0$. Let $c \in P$ with $c \neq 0$. By assumption, there exists $x \in M - N$ with $cx \in T(M)$; thus $cdx = 0$ for some non-zero element d of R which is a contradiction. Thus $P = 0$. □

Theorem 12. *Let M be a non-zero cancellative semimodule over a cancellative semiring R with identity. Then the following hold:*

(i) R is a semifield if and only if every proper ideal of R is a prime ideal.

(ii) R is a semifield if and only if every proper subsemimodule of M is a prime subsemimodule and $T(M) \neq M$.

Proof. (i) It is enough to show that if every proper ideal of R is prime, then R is a semifield. Let a be a non-zero element of R. By assumption $Ra^2 \neq 0$ is a prime ideal of R, so $a^2 \in Ra^2$ gives $a1_R = a(ra)$ for some $r \in R$, and since R is a cancellative semiring, we can cancel a, showing that a is an unit. Thus R is a semifield.

(ii) It suffices to show that if every proper subsemimodule of M is prime, then R is a semifield. Let $a \in M - T(M)$, so $(0 :_R a) = 0$. Note that since $\{0\}$ is a Q-ideal of R with $Q = R$, we must have $R \cong Ra \cong R/\{0\}$ as R-semimodules. In view of the assumption, it is easy to see that every proper subsemimodule of the R-semimodule Ra is a prime subsemimodule; hence R is a semifield by (i). □

Let M be a semimodule over a semiring R. We say that M is a torsion-free R-semimodule whenever $r \in R$ and $m \in M$ with $rm = 0$ implies that either $m = 0$ or $r = 0$.

Theorem 13. *Let R be a semiring with identity, M an R-semimodule and N a Q_M-subsemimodule of M with $P = (N :_R M)$. Then N is a prime subsemimodule of M if and only if P is a prime ideal of R and M/N is a torsion-free R/P-semimodule.*

Proof. First, suppose that N is a prime subsemimodule of M. Then by Lemma 4, P is a prime ideal of R and M/N is an R/P-semimodule by Proposition 1 and Remark 3. Let $(p+P)(q+N) = pq+N = 0_M+N$ where $q \in Q_M$ and $p \in Q = (R-P)\cup\{0\}$, so $pq \in N$. Therefore, N prime gives either $p \in P$ or $q \in N$. If $p = 0$, then $p + P$ is the zero in R/P (otherwise, $q + N$ is the zero in M/N). Thus M/N is torsion-free semimodule as an R/P-semimodule.

Conversely, since P is a prime ideal of R, we must have $N \neq M$. To see that N is prime, assume that $am \in N$ where $a \in R$ and $m \in M$. There are elements $s \in Q$, $a \in P$, $q \in Q_M$ and $n \in N$ such that $a = s + a$ and $m = q + n$, so $am = sq + sn + aq + an \in N$; hence $sq \in N$ since N is a k-subsemimodule by Remark 2. It follows that $(s+P)(q+N) = sq+N = q_0+N$; thus either $s+P$ is the zero in R/P (so $a \in P$) or $q + N$ is the zero in M/N (so $m \in N$), as required. □

Let M be a semimodule over a semiring R. We say that an element $r \in R$ is a zero-divisor on M if $rm = 0$ for some $0 \neq m \in M$.

Lemma 7. *Let M be a simple semimodule over a semiring R with identity. Then every zero-divisor on M is an annihilator of M.*

Proof. Let r be an arbitrary zero-divisor on M. Then there exists $0 \neq m \in M$ such that $rm = 0$. Since M is simple, we must have $rM = r(Rm) = (Rr)m = R(rm) = 0$. □

Theorem 14. *Let R be a semiring with identity, M an R-semimodule and N a Q_M-subsemimodule of M. If N is a k-maximal subsemimodule of M, then it is a prime subsemimodule of M.*

Proof. Let $ax \in N$ where $a \in R$ and $x \in M - N$. There are elements $q_1 \in Q_M$ and $n \in N$ such that $m = q_1 + n$, so $q_1 + N \neq 0_M + N$; hence $rq_1 \in N$. Therefore, $r(q_1+N) = 0_M+N$. It then follows from Lemma 7 that r is a zero-divisor on simple semimodule M/N. Let $ry \in rM$ where $y \in M$, so $y = q_2 + a$ for some $q_2 \in Q_M$, $a \in N$. Then Lemma 7 gives $r(q_2+N) = rq_2+N = 0_M+N$; thus $ry = rq_2+ra \in N$ since N is a k-subsemimodule by Remark 2 which is required. □

References

[1] Allen P. J. *A fundamental theorem of homomorphisms for simirings.* Proc. Amer. Math. Soc, 1969, **21**, 412–416.

[2] Ebrahimi Atani S. *The ideal theory in quotients of commutative semirings.* Glasnik Matematicki, 2007, **42**, 301–308.

[3] Ebrahimi Atani R., Ebrahimi Atani S. *Ideal theory in commutative semirings.* Buletinul Acad. Sci. Rep. Moldova, ser. Math., 2008, No. 2(57), 14–23.

[4] Golan J. S. *The theory of semirings with applications in mathematics and theoretical computer Science,* Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific and Technical, Harlow UK, 1992.

[5] Hebisch U., Weinert U. J. *Halbringe – Algebraische Theorie und Anwendungen in der Informatik,* Teubner, Stuttgart 1993.

[6] Lu C. P. *Prime submodules of modules.* Comment. Univ. St. Pauli 1984, **33**, 61–69.

[7] Maze G., Monico C., Rosenthal J. *Public key cryptography on semigroup actions,* arXiv:cs. CR/0501017v2 28 Jan 2005.

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