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On the Diophantine equation $x^x \cdot y^{y^{k(y)}} = z^{z^p}$

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Abstract. In this paper the existence and the ways of finding some positive integer solutions x, y, z for the equation $x^x \cdot y^{y^{k(y)}} = z^{z^p}$ are studied.

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It is known that in 1940 the Chinese mathematician Chao-Ko proved that the equation

$$x^x \cdot y^y = z^z \tag{1}$$

has infinitely many solutions in positive integers x, y, z.

The above equation was generalized in several forms as can be seen in the papers [4–9].

Let us have the equation

$$x^x \cdot y^{y^{k(y)}} = z^{z^p} \tag{2}$$

where $p > 1$ is a natural number, $k(y)$ is a certain function and x, y, z are unknowns. First we tackle equation (2) for $k(y) = m$, a natural number such that $m + p > 2$. Thus, for $x = uz^p$, $y = v^{\frac{1}{m}} \cdot z^{\frac{p}{m}}$, the given equation becomes

$$z = u^{\frac{u}{1-pu-\frac{p}{m}v}} \cdot v^{\frac{\frac{v}{m}}{1-pu-\frac{p}{m}v}} \tag{3}$$

Denoting

$$\frac{u}{1-pu-\frac{p}{m}v} = -n_1\,, \quad \frac{\frac{v}{m}}{1-pu-\frac{p}{m}v} = -n_2\,,$$

it results

$$u = \frac{n_1}{n_1p+n_2p-1}\,, \quad v = \frac{n_2m}{n_1p+n_2p-1}\,, \tag{4}$$

whence

$$\begin{cases} x = \left(\dfrac{n_1p+n_2p-1}{n_1}\right)^{n_1p-1} \cdot \left(\dfrac{n_1p+n_2p-1}{n_2m}\right)^{n_2p}, \\ y = \left(\dfrac{n_1p+n_2p-1}{n_1}\right)^{n_1\cdot\frac{p}{m}} \cdot \left(\dfrac{n_1p+n_2p-1}{n_2m}\right)^{\frac{n_2p}{m}-\frac{1}{m}}, \\ z = \left(\dfrac{n_1p+n_2p-1}{n_1}\right)^{n_1} \cdot \left(\dfrac{n_1p+n_2p-1}{n_2m}\right)^{n_2}. \end{cases} \tag{5}$$

The numbers n_1, n_2 can be chosen such that solutions (5) be natural numbers.

Now we prove the following

Theorem 1. *The equation* $x^x \cdot y^{y^{k(y)}} = z^{z^p}$, *with* $m + p \geq 2$, *m and p positive integers, has infinitely many solutions x, y, z in positive integers.*

Proof. For $m + p = 2$, that is $m = p = 1$, Theorem 1 is obviously proved according to the Chao-Ko result. Let $m + p > 2$. Taking $n_2 = \dfrac{1}{p}$ and $n_1 = m \cdot p^{p-2} \cdot n^p$ in (5), where n is an arbitrary natural number, the following solutions are obtained:

$$\begin{cases} x_n = n^p \cdot p^{m \cdot p^{p-1} \cdot n^p + p - 1}, \\ y_n = p^{p^{p-1} \cdot n^p}, \\ z_n = n \cdot p^{m \cdot p^{p-2} \cdot n^p + 1}. \end{cases} \tag{6}$$

□

For example, the equation $x^x \cdot y^{y^2} = z^{z^2}$ is a particular case of the equation discussed above in the case $m = p = 2$, hence it has as solutions

$$x_n = n^2 \cdot 2^{4n^2+1}, \quad y_n = 2^{2n^2}, \quad z_n = n \cdot 2^{2n^2+1}, \quad n \in \mathbb{N}^*.$$

Now, returning to equation (2), we can choose functions $k(y)$ such that the given equation have infinitely many natural solutions x, y, z. Thus, for example, $k(y) = y^s$, $s \in \mathbb{N}^*$, $k(y) = \sqrt[r]{y^s}$, $r, s \in \mathbb{N}^*$, $k(y) = y,\ y^y,\ y^{y^y}, \ldots$, and also linear combinations of such functions $k(y) = a_1 k_1(y) + a_2 k_2(y) + \cdots + a_t k_t(y)$, where $a_1, a_2, \ldots, a_t$ are natural numbers.

Theorem 2. *There are infinitely many functions $k(y)$ for which equation* (2) *has infinitely many solutions in positive integers x, y, z.*

Proof. The assertion of Theorem 2 follows from the fact that for a given function $k(y)$ as like as mentioned above, we put in (6) $m = k\left(p^{p^{p-1} \cdot n^p}\right)$ and n will be chosen such that $m \in \mathbb{N}^*$. □

Now, we will consider some applications:

Example 1. Consider the equation from [8]

$$x^x \cdot y^{y^y} = z^{z^2}. \tag{7}$$

Using formulas (6), the following solution results:

$$x_n = n^2 \cdot 2^{n^2 \cdot 2^{2n^2+1}+1}, \quad y_n = 2^{2n^2}, \quad z_n = n \cdot 2^{n^2 \cdot 2^{2n^2}+1}, \quad n \in \mathbb{N}^*. \tag{8}$$

For $n = 3$, the solution

$$x = 3^2 \cdot 2^{3^2 \cdot 2^{19}+1}, \quad y = 2^{18}, \quad z = 3 \cdot 2^{3^2 \cdot 2^{18}+1}$$

is obtained.

Example 2. Consider the equation

$$x^x \cdot y^{y^{y^2}} = z^{z^2}. \tag{9}$$

For this equation $p = 2$ and $k(y) = y^2$. Then

$$m = k\left(p^{p^{p-1}\cdot n^p}\right) = p^{2p^{p-1}\cdot n^p} = 2^{4n^2},$$

whence by (6) it results that

$$x_n = n^2 \cdot 2^{2^{4n^2+1}\cdot n^2+1},\ y_n = 2^{2n^2},\ z_n = n \cdot 2^{2^{4n^2}\cdot n^2+1},\ n \in \mathbb{N}^*. \tag{10}$$

Example 3. Consider the equation

$$x^x \cdot y^{y^{\sqrt[3]{y}}} = z^{z^2}. \tag{11}$$

In this case $p = 2$ and $k(y) = \sqrt[3]{y}$. Then

$$m = k\left(p^{p^{p-1}\cdot n^p}\right) = p^{\frac{1}{3}p^{p-1}\cdot n^p}.$$

Since p does not divide by 3, we take $n = 3r$, $r \in \mathbb{N}^*$, and get $m = 2^{6r^2}$. Then by (6) we obtain the following solutions:

$$\begin{cases} x_r = 9r^2 \cdot 2^{2^{6r^2+1}\cdot 9r^2+1}, \\ y_r = 2^{18r^2}, \\ z_r = 3r \cdot 2^{2^{6r^2}\cdot 9r^2+1}. \end{cases} \quad r \in \mathbb{N}^*. \tag{12}$$

Remark. For $k(y) = \sqrt[s]{y}$ we choose $n = r \cdot s$, $r \in \mathbb{N}^*$, in order to obtain the solutions of equation (2) given by (6).

References

[1] Andreescu T., Andrica D. *An Introduction to Diophantine Equations*. Gil Publishing House, 2002.

[2] Mordell L. J. *Diophantine Equations*. Academic Press, London and New York, 1969.

[3] Sierpinski W. *What we know and what we don't know about prime numbers*. Bucharest, 1966 (in Romanian).

[4] Tudor Gh. M. *L'équation Diophantienne* $x_1^{x_1} \cdot x_2^{x_2} \cdot \ldots \cdot x_k^{x_k} = y_k^{y_k}$ *(I)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2003, Nr. 48(62), **2**, 13–18.

[5] Tudor Gh. M. *L'équation Diophantienne* $(x_1^{x_1})^{m_1} \cdot (x_2^{x_2})^{m_2} \cdot \ldots \cdot (x_k^{x_k})^{m_k} = y_k^{y_k}$ *(II)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2004, Nr. 49(63), **1**, 13–19.

[6] Tudor Gh. M. *L'équation Diophantienne* $x_1^{x_1} \cdot x_2^{x_2} \cdot \ldots \cdot x_k^{x_k} = y_1^{y_1} \cdot y_2^{y_2} \cdot \ldots \cdot y_k^{y_k}$ *(III)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2004, Nr. 49(63), **1**, 1–8.

[7] Tudor Gh. M. *L'équation Diophantienne* $x_1^{x_1} \cdot x_2^{x_2} \cdot \ldots \cdot x_k^{x_k} = w^{w^p}$ *(IV)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2005, Nr. 50(64), **1**, 12–18.

[8] Tudor Gh. M. *L'équation Diophantienne* $x^x \cdot y^{y^q} = z^{z^p}$ *(V)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2005, Nr. 50(64), **2**, 18–24.

[9] Tudor Gh. M., Bînzar T. *On some exponential Diophantine equations*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2005, Nr. 50(64), **2**, 47–52.

[10] Tudor Gh. M., Bînzar T. *On the Diophantine equation* $x_1^{x_2^\alpha} \cdot x_2^{x_3^\alpha} \cdot \ldots \cdot x_k^{x_1^\alpha} = w^{w^\alpha}$, $\alpha = \pm 1$ *(II)*. Bul. Şt. al Univ. "Politehnica" din Timişoara, Seria Mat.-Fiz., 2008, Nr. 53(67), **1**, 44–49.

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