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New Inequalities of Hardy-Hilbert Type

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Abstract. In this paper, we establish a new inequality of Hardy-Hilbert type. As applications, some particular results and an equivalent form are derived. The integral analogues of the main results are also given.

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1 Introduction

If $p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, $a_n, b_n \geq 0$ satisfy $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}} \tag{1}$$

and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=1}^{\infty} a_n^p, \tag{2}$$

where the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. Inequality (1) is called Hardy-Hilbert's inequality (see [1]) and is important in analysis and its applications (cf. Mitrinovic et al. [5]). Recently many generalization and refinements of these inequalities have been also obtained, see [4, 12] and the references cited therein.

Hardy et al. [1] gave an inequality, under the same condition of (1), similar to (1) as :

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}} \tag{3}$$

and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{max\{m,n\}}\right)^p < [pq]^p \sum_{n=1}^{\infty} a_n^p, \tag{4}$$

where the constant factors pq and $(pq)^p$ are the best possible. The integral analogue of the inequality is:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}} \tag{5}$$

and an equivalent form is given by

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{\max\{x,y\}} dx \right)^p dy < (pq)^p \int_0^\infty f^p(x)dx, \tag{6}$$

where the constant factor pq and $(pq)^p$ are the best possible. Recently some generalizations of these Hilbert type inequalities were obtained. For details see [9, 13–15].

In the recent years, many new inequalities similar to (1) have been established, see [7,8]. Recently Das and Sahoo [11] have given a new inequality similar to Hardy–Hilbert inequality (1) as follows:

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 4$, $0 < r, s < \lambda$ if $\lambda \leq 2$, $0 < r, s \leq 2$ if $\lambda > 2$, $r + s = \lambda$, $a_n, b_n \geq 0$, $A_n = \sum_{k=1}^n a_k$, $B_n = \sum_{k=1}^n b_k$. If $0 < \sum_{n=1}^\infty a_n^p < \infty$ and $0 < \sum_{n=1}^\infty b_n^q < \infty$, then the following two inequalities holds:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{(m+n)^\lambda} A_m B_n < pqB(r,s) \left(\sum_{n=1}^\infty a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty b_n^q \right)^{\frac{1}{q}}; \tag{7}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{(m+n)^\lambda} A_m \right)^p < (qB(r,s))^p \sum_{n=1}^\infty a_n^p, \tag{8}$$

where the constant factors $pqB(r,s)$ and $(qB(r,s))^p$ are the best possible.

Sulaiman [10, Theorem 1] derived a new integral inequality similar to (5) as follows:

Let $f,\ g \geq 0$, $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $p = \lambda - \alpha - 1 > 1$, $q = \lambda - \beta - 1 > 1$, $\alpha,\ \beta > -1$. Then

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{x^{\frac{\beta}{q}} y^{\frac{\alpha}{p}}}{\max\{x^\lambda, y^\lambda\}} F(x)G(y)dxdy \\ &\leq \frac{\lambda}{(\alpha+1)^{\frac{1}{p}}(\beta+1)^{\frac{1}{q}}} \frac{p^{1-\frac{1}{p}} q^{1-\frac{1}{q}}}{(p-1)(q-1)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}. \end{aligned} \tag{9}$$

In that paper Sulaiman does not prove whether the constant factor is best possible or not.

In this paper we obtain a generalization of the inequality (9), given by Sulaiman [10, Theorem 1] and the constant factor obtained is the best possible. First we prove the discrete version of the inequality and some particular results. Then we prove the integral analogue of the inequality.

We need the following two inequalities, which are well-known as Hardy's inequalities (cf. Hardy et al. [1]).

Lemma 1. *If $p > 1,\ a_n \geq 0$ and $A_n = a_1 + a_2 + ... + a_n$, then*

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{10}$$

unless all the $a_n = 0$. The constant is the best possible.

Lemma 2. *If $p > 1,\ f \geq 0$ and $F(x) = \int_0^x f(t)dt$, then*

$$\int_0^{\infty} \left(\frac{F(x)}{x} \right)^p dx < \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x)dx, \tag{11}$$

unless $f \equiv 0$. The constant is the best possible.

2 Main Results

In this section we prove our main result and derive some particular cases.

Theorem 1. *Let $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ 0 < r, s \leq 1,\ r + s = \lambda,\ a_n, b_n \geq 0$, $A_n = \sum_{k=1}^{n} a_k,\ B_n = \sum_{k=1}^{n} b_k$. If $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the following two inequalities holds:*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{\max\{m^\lambda, n^\lambda\}} A_m B_n < \frac{pq\lambda}{rs} \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}; \tag{12}$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{m^\lambda, n^\lambda\}} A_m \right)^p < \left(\frac{q\lambda}{rs} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{13}$$

where the constant factors $\frac{pq\lambda}{rs}$ and $\left(\frac{q\lambda}{rs} \right)^p$ are the best possible.

Proof. By Hölder's inequality with weight (cf. Kuang [3]), we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{\max\{m^\lambda, n^\lambda\}} A_m B_n =$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(n^{\frac{s-1}{p}} m^{\frac{r}{p}-1} A_m \right) \left(m^{\frac{r-1}{q}} n^{\frac{s}{q}-1} B_n \right) \leq$$

$$\leq \left\{ \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \frac{n^{s-1}}{\max\{m^\lambda, n^\lambda\}} \right] m^{r-p} A_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{m^{r-1}}{\max\{m^\lambda, n^\lambda\}} \right] n^{s-q} B_n^q \right\}^{\frac{1}{q}}.$$

For $\alpha = r, s$, as $0 < \alpha \leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{\max\{m^{\lambda}, n^{\lambda}\}} < \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{t^{\alpha-1}}{\max\{m^{\lambda}, t^{\lambda}\}} dt = \\ = \int_{0}^{\infty} \frac{t^{\alpha-1}}{\max\{m^{\lambda}, t^{\lambda}\}} dt = \frac{\lambda}{rs} m^{\alpha-\lambda}. \tag{14}$$

Hence

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{\max\{m^{\lambda}, n^{\lambda}\}} A_m B_n < \frac{\lambda}{rs} \left\{ \sum_{m=1}^{\infty} \left(\frac{A_m}{m} \right)^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left(\frac{B_n}{n} \right)^q \right\}^{\frac{1}{q}}.$$

Then by Hardy inequality (1), (12) is valid.

Again by Hölder's inequality and (14), we get

$$\sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{m^{\lambda}, n^{\lambda}\}} A_m = \sum_{m=1}^{\infty} \frac{1}{\max\{m^{\lambda}, n^{\lambda}\}} \left(n^{\frac{s-1}{p}} m^{\frac{r}{p}-1} A_m \right) \left(m^{\frac{r-1}{q}} n^{\frac{s}{q}} \right) \leq \\ \leq \left\{ \sum_{m=1}^{\infty} \frac{n^{s-1}}{\max\{m^{\lambda}, n^{\lambda}\}} m^{r-p} A_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{m=1}^{\infty} \frac{m^{r-1}}{\max\{m^{\lambda}, n^{\lambda}\}} n^s \right\}^{\frac{1}{q}} < \\ < \left(\frac{\lambda}{rs} \right)^{\frac{1}{q}} \left\{ \sum_{m=1}^{\infty} \frac{n^{s-1}}{\max\{m^{\lambda}, n^{\lambda}\}} m^{r-p} A_m^p \right\}^{\frac{1}{p}}.$$

Hence, again applying (14), we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{m^{\lambda}, n^{\lambda}\}} A_m \right)^p < \left(\frac{\lambda}{rs} \right)^{\frac{p}{q}} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \frac{n^{s-1}}{\max\{m^{\lambda}, n^{\lambda}\}} \right] m^{r-p} A_m^p < \\ < \left(\frac{\lambda}{rs} \right)^p \sum_{m=1}^{\infty} \left(\frac{A_m}{m} \right)^p,$$

then by Hardy inequality (1), (13) is valid.

For $0 < \varepsilon < \min\left\{ qs, \frac{1}{p-1}, \frac{1}{q-1} \right\}$, take $\tilde{a}_n = n^{-\frac{1+\varepsilon}{p}}$, $\tilde{b}_n = n^{-\frac{1+\varepsilon}{q}}$ for $n \geq 1$. Then

$$\left(\sum_{n=1}^{\infty} \tilde{a}_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \tilde{b}_n^q \right)^{\frac{1}{q}} < 1 + \frac{1}{\varepsilon}. \tag{15}$$

Since $\tilde{A}_1 = \tilde{a}_1 = 1$ and for $n > 1$,

$$\tilde{A}_m = \sum_{k=1}^{m} \tilde{a}_k > \sum_{k=1}^{m-1} \int_{k}^{k+1} x^{-\frac{1+\varepsilon}{p}} dx = \int_{1}^{m} x^{-\frac{1+\varepsilon}{p}} dx = \frac{q}{1-\varepsilon(q-1)} \left(m^{\frac{1}{q}-\frac{\varepsilon}{p}} - 1 \right).$$

Hence

$$\tilde{A}_m > \frac{q}{1-\varepsilon(q-1)}\left(m^{\frac{1}{q}-\frac{\varepsilon}{p}}-1\right), \text{ for } m \geq 1.$$

Similarly

$$\tilde{B}_n > \frac{p}{1-\varepsilon(p-1)}\left(n^{\frac{1}{p}-\frac{\varepsilon}{q}}-1\right), \text{ for } n \geq 1.$$

Taking $\phi(\varepsilon) = \dfrac{pq}{\{1-\varepsilon(p-1)\}\{1-\varepsilon(q-1)\}}$, we have $\lim\limits_{\varepsilon\to 0^+}\phi(\varepsilon) = pq$ and for $m, n \geq 1$,

$$\tilde{A}_m\tilde{B}_n > \phi(\varepsilon)\left(m^{\frac{1}{q}-\frac{\varepsilon}{p}}n^{\frac{1}{p}-\frac{\varepsilon}{q}} - n^{\frac{1}{p}-\frac{\varepsilon}{q}} - m^{\frac{1}{q}-\frac{\varepsilon}{p}}\right).$$

Then

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{\max\{m^\lambda, n^\lambda\}}\tilde{A}_m\tilde{B}_n > \\
&> \phi(\varepsilon)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^{r-\frac{\varepsilon}{p}-1}n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}} - \frac{m^{r-\frac{1}{q}-1}n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}} - \frac{m^{r-\frac{\varepsilon}{p}-1}n^{s-\frac{1}{p}-1}}{\max\{m^\lambda, n^\lambda\}}\right) = \\
&= \phi(\varepsilon)\left(\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\right) \text{ (say).}
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}} &> \int_1^\infty \frac{t^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, t^\lambda\}}dt = \\
&= \frac{\lambda}{\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)}m^{-r-\frac{\varepsilon}{q}} - \frac{1}{\left(s-\frac{\varepsilon}{q}\right)}m^{-\lambda},
\end{aligned}$$

so, we have

$$\begin{aligned}
\sum\nolimits_1 &:= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{\varepsilon}{p}-1}n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}} = \sum_{m=1}^{\infty}m^{r-\frac{\varepsilon}{p}-1}\left(\sum_{n=1}^{\infty}\frac{n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}}\right) > \\
&> \frac{\lambda}{\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)}\sum_{m=1}^{\infty}m^{-\varepsilon-1} - \frac{1}{\left(s-\frac{\varepsilon}{q}\right)}\sum_{m=1}^{\infty}m^{-s-\frac{\varepsilon}{p}-1} > \\
&> \frac{\lambda}{\varepsilon\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)} - \frac{1}{\left(s-\frac{\varepsilon}{q}\right)}\sum_{m=1}^{\infty}m^{-s-\frac{\varepsilon}{p}-1},
\end{aligned}$$

where $\sum_{m=1}^{\infty}m^{-s-\frac{\varepsilon}{p}-1} < \infty$. Again

$$\sum_{n=1}^{\infty}\frac{n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, n^\lambda\}} < \int_0^\infty \frac{t^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda, t^\lambda\}}dt = \frac{\lambda}{\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)}m^{-r-\frac{\varepsilon}{q}},$$

so, we obtain

$$\sum\nolimits_2 := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{\varepsilon}{q}-1}}{\max\{m^\lambda,n^\lambda\}} < \frac{\lambda}{\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)}\sum_{m=1}^{\infty} m^{-\frac{\varepsilon+1}{q}-1} < \infty.$$

Similarly, we get $\sum\nolimits_3 < \infty$. Hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{\max\{m^\lambda,n^\lambda\}}\tilde{A}_m\tilde{B}_n > \phi(\varepsilon)\left\{\frac{\lambda}{\varepsilon\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)} - \bigcirc(1)\right\}. \tag{16}$$

If the constant factor $\frac{pq\lambda}{rs}$ in (12) is not the best possible, then there exists a positive constant K such that $K < \frac{pq\lambda}{rs}$ and (12) still remains valid if $\frac{pq\lambda}{rs}$ is replaced by K. In particular by (15) and (16), we have

$$\phi(\varepsilon)\left\{\frac{\lambda}{\left(r+\frac{\varepsilon}{q}\right)\left(s-\frac{\varepsilon}{q}\right)} - \varepsilon\bigcirc(1)\right\} < \varepsilon\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{\max\{m^\lambda,n^\lambda\}}\tilde{A}_m\tilde{B}_n <$$

$$< \varepsilon K\left(\sum_{n=1}^{\infty}\tilde{a}_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\tilde{b}_n^q\right)^{\frac{1}{q}} < K(\varepsilon+1).$$

Then $\frac{pq\lambda}{rs} \leq K$ as $\varepsilon \to 0^+$. This contradiction shows that the constant factor $\frac{pq\lambda}{rs}$ in (12) is the best possible.

If the constant factor $\left(\frac{q\lambda}{rs}\right)^p$ in (13) is not the best possible, then there exists a positive constant $\tilde{K}$ such that $\tilde{K} < \frac{q\lambda}{rs}$ and (13) still remains valid if $\left(\frac{q\lambda}{rs}\right)^p$ is replaced by $\tilde{K}^p$. Then by Hölder's inequality, (13) and Hardy inequality (1), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{\max\{m^\lambda,n^\lambda\}}A_mB_n = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{\max\{m^\lambda,n^\lambda\}}A_m\right)\left(\frac{B_n}{n}\right) \leq$$

$$\leq \left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{\max\{m^\lambda,n^\lambda\}}A_m\right)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right\}^{\frac{1}{q}} <$$

$$< p\tilde{K}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}},$$

which gives that the constant factor $\frac{pq\lambda}{rs}$ in (12) is not the best possible. This contradiction shows that the constant factor $\left(\frac{q\lambda}{rs}\right)^p$ in (13) is the best possible. This proves the theorem. $\square$

Now we discuss some particular cases of (12) and (13). Taking $\lambda = \frac{1}{2}, 1, 2$, in Theorem 1, we get the following results, respectively.

Corollary 1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r, s > 0$, $r + s = \frac{1}{2}$, $a_n, b_n \geq 0$, $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$. *If* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then the following two inequalities holds:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{\max\{\sqrt{m}, \sqrt{n}\}} A_m B_n < \frac{pq}{2rs} \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{17}$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{\sqrt{m}, \sqrt{n}\}} A_m\right)^p < \left(\frac{q}{2rs}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{18}$$

where the constant factors $\frac{pq}{2rs}$ *and* $\left(\frac{q}{2rs}\right)^p$ *are the best possible. In particular*

(i) *for* $r = \frac{1}{2q}$ *and* $s = \frac{1}{2p}$, *we obtain the following two inequalities:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_m B_n}{m^{\frac{1}{2q}+1} n^{\frac{1}{2p}+1} \max\{\sqrt{m}, \sqrt{n}\}} < 2(pq)^2 \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{m=1}^{\infty} \frac{A_m}{m^{\frac{1}{2q}+1} \max\{\sqrt{m}, \sqrt{n}\}}\right)^p < \left(2pq^2\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{20}$$

(ii) *for* $r = s = \frac{1}{4}$ *and* $p = q = 2$, *we obtain the following two inequalities:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_m B_n}{m^{\frac{5}{4}} n^{\frac{5}{4}} \max\{\sqrt{m}, \sqrt{n}\}} < 32 \left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}}; \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{m=1}^{\infty} \frac{A_m}{m^{\frac{5}{4}} \max\{\sqrt{m}, \sqrt{n}\}}\right)^2 < 256 \sum_{n=1}^{\infty} a_n^2. \tag{22}$$

Corollary 2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r, s > 0$, $r + s = 1$, $a_n, b_n \geq 0$, $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$. *If* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then the following two inequalities holds:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{\max\{m, n\}} A_m B_n < \frac{pq}{rs} \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{23}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{\max\{m,n\}}A_m\right)^p < \left(\frac{q}{rs}\right)^p\sum_{n=1}^{\infty}a_n^p, \tag{24}$$

where the constant factors $\frac{pq}{rs}$ *and* $\left(\frac{q}{rs}\right)^p$ *are the best possible. In particular*

(i) *for* $r=\frac{1}{q}$ *and* $s=\frac{1}{p}$ *, we obtain the following two inequalities:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{mn\max\{m,n\}} < (pq)^2\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{25}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{A_m}{m\max\{m,n\}}\right)^p < \left(pq^2\right)^p\sum_{n=1}^{\infty}a_n^p, \tag{26}$$

(ii) *for* $r=s=\frac{1}{2}$ *and* $p=q=2$ *, we obtain the following two inequalities:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{mn\max\{m,n\}} < 16\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{27}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{A_m}{m\max\{m,n\}}\right)^2 < 64\sum_{n=1}^{\infty}a_n^2. \tag{28}$$

Taking $\lambda=2,\ r=s=1$ in Theorem 1, we get the following result.

Corollary 3. *Let* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ a_n,b_n\geq 0,\ A_n=\sum_{k=1}^n a_k,\ B_n=\sum_{k=1}^n b_k$. *If* $0<\sum_{n=1}^{\infty}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}b_n^q<\infty$, *then the following two inequalities holds:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{m^{\frac{1}{q}}n^{\frac{1}{p}}\max\{m^2,n^2\}} < 2pq\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{29}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{n^{\frac{1}{q}}}{m^{\frac{1}{q}}\max\{m^2,n^2\}}A_m\right)^p < (2q)^p\sum_{n=1}^{\infty}a_n^p, \tag{30}$$

where the constant factors $2pq$ *and* $(2q)^p$ *are the best possible. In particular for* $p=q=2$, *we obtain the following two inequalities:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{\sqrt{mn}\max\{m^2,n^2\}} < 8\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{31}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\sqrt{n}A_m}{\sqrt{m}\max\{m^2,n^2\}}\right)^2 < 16\sum_{n=1}^{\infty}a_n^2. \tag{32}$$

3 Integral Analogues

In this section we present integral analogues of the inequalities given in Theorem 1, which in fact are similar to the integral analogues of the Hilbert's inequality.

Theorem 2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, r, s > 0$, $r + s = \lambda$, $f, g \geq 0$ *and* $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$. *If* $0 < \int_0^\infty f^p(x)dx < \infty$ *and* $0 < \int_0^\infty g^q(x)dx < \infty$ *then the following two inequalities holds:*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} F(x)G(y)dxdy < \frac{pq\lambda}{rs} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}; \quad (33)$$

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}}}{\max\{x^\lambda, y^\lambda\}} F(x)dx\right)^p dy < \left(\frac{q\lambda}{rs}\right)^p \int_0^\infty f^p(x)dx, \quad (34)$$

where the constant factors $\frac{pq\lambda}{rs}$ *and* $\left(\frac{q\lambda}{rs}\right)^p$ *are the best possible.*

Proof. Using Hölder's inequality, Hardy inequality (11) and proceeding as in the proof of Theorem 1, we get (33) and (34) are valid. For the best constant factor, let $0 < \varepsilon < \min\left\{qs, \frac{1}{p-1}, \frac{1}{q-1}\right\}$,

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ x^{-\frac{1+\varepsilon}{p}} & \text{if } x \in [1,\infty). \end{cases}$$

$$g_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in (0,1), \\ y^{-\frac{1+\varepsilon}{q}} & \text{if } y \in [1,\infty). \end{cases}$$

Then

$$\left(\int_0^\infty f_\varepsilon^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g_\varepsilon^q(x)dx\right)^{\frac{1}{q}} = \frac{1}{\varepsilon}, \quad (35)$$

and

$$F_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ \frac{q}{1-\varepsilon(q-1)} \left(x^{\frac{1}{q}-\frac{\varepsilon}{p}} - 1\right) & \text{if } x \in [1,\infty). \end{cases}$$

$$G_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in (0,1), \\ \frac{p}{1-\varepsilon(p-1)} \left(y^{\frac{1}{p}-\frac{\varepsilon}{q}} - 1\right) & \text{if } x \in [1,\infty). \end{cases}$$

Denote $\phi(\varepsilon) = \frac{pq}{(1-\varepsilon(p-1))(1-\varepsilon(q-1))}$. Then $\phi(\varepsilon) \to pq$, as $\varepsilon \to 0^+$ and for $x, y \geq 1$

$$F_\varepsilon(x)G_\varepsilon(y) > \phi(\varepsilon)\left(x^{\frac{1}{q}-\frac{\varepsilon}{p}} y^{\frac{1}{p}-\frac{\varepsilon}{q}} - y^{\frac{1}{p}-\frac{\varepsilon}{q}} - x^{\frac{1}{q}-\frac{\varepsilon}{p}}\right).$$

Then

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} F_\varepsilon(x) G_\varepsilon(y) dx dy >$$
$$> \phi(\varepsilon) \int_1^\infty \int_1^\infty \left(\frac{x^{r-\frac{\varepsilon}{p}-1} y^{s-\frac{\varepsilon}{q}-1}}{\max\{x^\lambda, y^\lambda\}} - \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{\varepsilon}{q}-1}}{\max\{x^\lambda, y^\lambda\}} - \frac{x^{r-\frac{\varepsilon}{p}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} \right) dx dy =$$
$$= \phi(\varepsilon) \left(I_1 - I_2 - I_3 \right) \text{ (say)}.$$

Integrating, we get

$$I_1 := \int_1^\infty \int_1^\infty \frac{x^{r-\frac{\varepsilon}{p}-1} y^{s-\frac{\varepsilon}{q}-1}}{\max\{x^\lambda, y^\lambda\}} dx dy = \frac{\lambda}{\varepsilon \left(r + \frac{\varepsilon}{q}\right)\left(s - \frac{\varepsilon}{q}\right)} - \frac{1}{\left(s + \frac{\varepsilon}{p}\right)\left(s - \frac{\varepsilon}{q}\right)};$$

$$I_2 := \int_1^\infty \int_1^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{\varepsilon}{q}-1}}{\max\{x^\lambda, y^\lambda\}} dx dy = \frac{q\lambda}{(1+\varepsilon) \left(r + \frac{\varepsilon}{q}\right)\left(s - \frac{\varepsilon}{q}\right)} - \frac{1}{\left(s + \frac{1}{q}\right)\left(s - \frac{\varepsilon}{q}\right)};$$

$$I_3 := \int_1^\infty \int_1^\infty \frac{x^{r-\frac{\varepsilon}{p}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} dx dy = \frac{p\lambda}{(1+\varepsilon) \left(r - \frac{\varepsilon}{p}\right)\left(s + \frac{\varepsilon}{p}\right)} - \frac{1}{\left(r - \frac{\varepsilon}{p}\right)\left(r + \frac{1}{p}\right)}.$$

Hence

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} F_\varepsilon(x) G_\varepsilon(y) dx dy > \phi(\varepsilon) \left\{ \frac{\lambda}{\varepsilon \left(r + \frac{\varepsilon}{q}\right)\left(s - \frac{\varepsilon}{q}\right)} - \bigcirc(1) \right\}. \quad (36)$$

If the constant factor $\frac{pq\lambda}{rs}$ in (33) is not the best possible, then there exists a positive constant K such that $K < \frac{pq\lambda}{rs}$ and (33) still remains valid if $\frac{pq\lambda}{rs}$ is replaced by K. In particular by (35) and (36), we have

$$\phi(\varepsilon) \left\{ \frac{\lambda}{\left(r + \frac{\varepsilon}{q}\right)\left(s - \frac{\varepsilon}{q}\right)} - \varepsilon \bigcirc (1) \right\} <$$
$$< \varepsilon \int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}} F_\varepsilon(x) G_\varepsilon(y) dx dy <$$
$$< \varepsilon K \left(\int_0^\infty f_\varepsilon^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g_\varepsilon^q(x) dx \right)^{\frac{1}{q}} = K.$$

Then $\frac{pq\lambda}{rs} \leq K$ as $\varepsilon \to 0^+$. This contradiction shows that the constant factor $\frac{pq\lambda}{rs}$ in (33) is the best possible.

Proceeding as in the proof for the best constant factor in (13), we prove the constant factor $\left(\frac{q\lambda}{rs}\right)^p$ in (34) is the best possible. This proves the theorem. □

Remark 1. Taking $r = \dfrac{\beta+1}{q} + 1,\ s = \dfrac{\alpha+1}{p} + 1$, in Theorem 2, we get the following generalization of (9).

Let $p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1,\ \lambda > 0,\ \alpha > -(p+1),\ \beta > -(q+1),\ \dfrac{\alpha}{p} + \dfrac{\beta}{q} = \lambda - 3,\ f, g \geq 0$ and $F(x) = \int_0^x f(t)dt,\ G(x) = \int_0^x g(t)dt$. If $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then the following inequality holds:

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{x^{\frac{\beta}{q}} y^{\frac{\alpha}{p}} F(x) G(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < \\
&< \frac{p^2 q^2 \lambda}{(\alpha+p+1)(\beta+q+1)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}
\end{aligned}
\tag{37}
$$

where the constant factors $\dfrac{p^2 q^2 \lambda}{(\alpha+p+1)(\beta+q+1)}$ is the best possible.

Remark 2. Taking the different values of the parameters $\lambda,\ r,\ s,$ as following the Theorem 1, we get the particular inequalities of (33) and (34).

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