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On the centralizers of finite subgroups in quasi-HNN groups

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Abstract. Quasi-HNN groups can be characterized as a generalization of HNN groups in the sense that there is a stable letter t such that the element t^2 is in the base. In this paper we show that under certain conditions the centralizer of a finite subgroup in a quasi-HNN group is contained in a conjugate of the base. As an application we show that the centralizer of a finite subgroup in a one-relator group is contained in a conjugate of a one-relator subgroup of shorter relator.

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1 Introduction

In [2, Theorem 2.7, p. 187] it is shown that every element of finite order in free products of groups with amalgamation is a conjugate of an element of finite order in a factor of the group. This implies that finite subgroups in free products of groups with amalgamation are contained in conjugates of the factors of the group. Then the normal form theorem for free products with amalgamation [2, Theorem 2.6, p. 187] implies that the centralizers of finite subgroups are contained in the conjugates of the factors of the group. Similarly, Theorems 2.4 and 2.5 of [2, p. 185] imply that in HNN groups the centralizers of finite subgroups are contained in conjugates of the bases. The aim of this paper is to show that under certain conditions the centralizers of finite subgroups in quasi-HNN groups defined below are contained in conjugates of the bases. A new class of groups called quasi-HNN groups obtained by Khanfar and Mahmood in [1] are defined as follows.

Let G be a group, and, I and J be two index sets such that $I \cap J = \varnothing$, and $I \cup J \neq \varnothing$. Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$, and $\{C_j : j \in J\}$ be families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism and for each $j \in J$, let α_j: $C_j \to C_j$ be an automorphism such that α_j^2 is an inner automorphism determined by $c_j \in C_j$ and c_j is fixed by α_j. That is, $\alpha_j(c_j) = c_j$ and $\alpha_j^2(c) = c_j c c_j^{-1}$ for all $c \in C_j$. The group G^* determined by the following presentation

$$G^* = \langle gen(G), t_i, t_j \mid rel\ (G),\ \ t_i A_i t_i^{-1} = B_i\ ,\ \ t_j C_j t_j^{-1}\ = C_j,\ \ t_j^2\ =\ c_j \rangle,$$
$$i \in I,\ j \in J,$$

is called a **quasi-HNN group** of base G and associated pairs of subgroups (A_i, B_i) and (C_j, C_j) of G.

The notations of the presentation of G^* can be explained as follows.

1. $\langle gen(G) \,|\, rel(G)\rangle$ stands for any presentation of G, where $gen(G)$ is a set of generating symbols of G and $rel(G)$ is the set of relations of the presentation of G.
2. $t_iA_it_i^{-1} = B_i$ stands for the set of relations $t_iw(a)t_i^{-1} = w(\phi_i(a))$, where $w(a)$ and $w(\phi_i(a))$ are words in $gen(G)$ of values a and $\phi_i(a)$, respectively, where a runs over a set of generators of A_i.
3. $t_jC_jt_j^{-1} = C_j$ stands for the set of relations $t_jw(c)t_j^{-1} = w(\alpha_j(c))$, where $w(c)$ and $w(\alpha_j(c))$ are words in $gen(G)$ of values c and $\alpha_j(c)$, respectively, where c runs over a set of generators of C_j.
4. $t_j^2 = c_j$ stands for the of relation $t_j^2 = w(c_j)$, where $w(c_j)$ is a word in $gen(G)$ of value c_j.

For more information related to quasi-HNN group we refer the readers to [3] and [4].

The following are examples of quasi-HNN groups.

Example 1. Any HNN group is a quasi-HNN group where $J = \varnothing$.

Example 2. The free product of an HNN group and cyclic groups of order 2 is a quasi HNN group.

For example if

$$G^* = \langle gen(G), t_i \mid rel\ (G),\ \ t_iA_it_i^{-1} = B_i\ \rangle, \quad i \in I,$$

is an HNN group of base G and associated pairs of subgroups (A_i, B_i) of G and $H = \{1, t\}$ is a cyclic group of order 2, then in the free product $G^* * H$ of the groups G^* and H we have

$$G^* * H = \langle gen(G), t_i, t \mid rel\ (G),\ \ t_iA_it_i^{-1} = B_i\ ,\ t^2 = 1\rangle$$

is a quasi-HNN group of base G and associated pairs of subgroups (A_i, B_i) and $(\{1\}, \{1\})$ of G where $C_j = \{1\}$.

Example 3. It is proved in [4, Theorem 5.1] that a group is a quasi-HNN if and only if there exists a tree for which the action of the group on the tree is with inversions and is transitive on the set of the vertices of the tree.

This paper is divided into 3 sections. In Section 2, we give definitions and results related to quasi-HNN groups needed in Section 3. In Section 3 we show that if H is a finite subgroup of the quasi-HNN group G^* defined above such that H is not in any conjugate of the subgroups A_i or B_i, $i \in I$, or C_j , $j \in J$ of G, then the centralizer $C_{G^*}(H)$ of H in G^* is contained in a conjugate of G.

As application, we show that if $G = \langle t, b, c, \ldots; r\rangle$ is a one-relator group, r is cyclically reduced, and contains at least two different letters, and if H is a finite subgroup of G, then the centralizer $C_G(H)$ of H in G is contained in a conjugate of a subgroup G_0 of G, where G_0 is a one-relator group whose defining relator has shorter length than r.

2 Definitions and results

Throughout this section G^* is the quasi-HNN group

$$G^* = \langle gen(G), t_i, t_j \mid rel\ (G),\ \ t_i A_i t_i^{-1} = B_i\ ,\ \ t_j C_j t_j^{-1}\ = C_j,\ \ t_j^2\ =\ c_j \rangle,$$
$$i \in I, \quad j \in J.$$

For the simplicity of notation, let $T = \{t_k^{e_k} : k \in I \cup J,\ e_k = \pm 1\}$, and for $x \in T$, define

$$G_x = \begin{cases} A_i, & if \quad x = t_i,\ i \in I, \\ B_i, & if \quad x = t_i^{-1},\ i \in I, \\ C_j, & if \quad x = t_j^{e_j},\ j \in J,\ e_j = \pm 1, \end{cases}$$

and

$$\phi_x = \begin{cases} \phi_i, & if \quad x = t_i,\ i \in I, \\ \phi_i^{-1}, & if \quad x = t_i^{-1},\ i \in I, \\ a_j, & if \quad x = t_j^{e_j},\ j \in J,\ e_j = \pm 1. \end{cases}$$

It is clear that if $x \in T$, then the mapping $\phi_x : G_x \to G_{x^{-1}}$ given by $\phi_x(g) = xgx^{-1}$ is an isomorphism. Furthermore, for all $x \in J$ we have $G_x = G_{x^{-1}}$.

Definition 1. By a **word** w of G^* we mean a sequence of the form

$$w\ = (g_0, y_1, g_1, y_2, g_2,\ \dots\ , y_n, g_n),$$

or simply

$$w\ = g_0.y_1.g_1.y_2.g_2.\ \dots\ .y_n.g_n,$$

where $g_i \in G$ for $i = 0, 1, \dots, n$, and $y_s \in T$ for $s = 1, \dots, n$.

We have the following concepts related to the word $w\ = g_0.y_1.g_1.y_2.g_2.\ \dots\ .y_n.g_n$ defined above.

(i) n is called the *length* of w and is denoted by $|w| = n$.

(ii) w is called a *trivial word* of G if $|w| = 0$, or $w\ =\ g_0$.

(iii) The *value* of w denoted $[w]$ is defined to be the element$w] =\ g_0 y_1 g_1 y_2 g_2 \dots y_n g_n$ of G^*.

(iv) The *inverse* of w denoted w^{-1} is defined to be the word $w^{-1} = g_n^{-1}.y_n^{-1}.g_{n-1}^{-1}. \dots$ $\dots .g_2^{-1}.y_2^{-1}.g_1^{-1}.y_1^{-1}.g_0^{-1}$ of G^*. It is clear that $[w^{-1}] = [w]^{-1}$.

(v) If $w' = h_0.x_1.h_1.x_2.h_2.\dots.x_m.h_m$ is a word of G^*, then $w.w'$ is defined to be the word $w.w' =\ g_0.y_1.g_1.y_2.g_2.\ \dots\ .y_n.g_n h_0.x_1.h_1.x_2.h_2.\dots.x_m.h_m$ of G^*. It is clear that $[w.w'] = [w][w']$.

(vi) w is called a *reduced word* of G^* if w contains no subword of the form $y_i.g_i.y_i^{-1}$ if $g_i \in G_{y_i}$ for $i = 1, \ldots, n$. We take trivial words to be reduced.

(vii) The *t-reduction* on w is defined to be the replacement of the subword of the form $y_i.g_i.y_i^{-1}$ if $g_i \in G_{y_i}$ by $\phi_{y_i}(g_i)$.

Proposition 1. *Every element of G^* is the value of a reduced word of G^*. Moreover, if w is a word of G^* of value 1, the identity element of G^*, then w is not reduced.*

Proof. See [1]. □

Now we generalize the above proposition as follows.

Proposition 2. *For every element g of G^* there exists a reduced word w of G^* such that $[w] = g$. Moreover, if w is a reduced word of G^* then for any word w' of G^* such that $[w] = [w'] = g$, we have $|w| = |w'|$ if and only if w' is a reduced word of G^*.*

Proof. Since G^* is generated by the generators of G and by the elements of T, therefore g can be written as the product

$$g = g_0 y_1 g_1 y_2 g_2 \cdots y_n g_n$$

where $g_i \in G$, and $y_i \in T$. Then

$$w = g_0.y_1.g_1.y_2.g_2. \cdots .y_n.g_n$$

is a word of G^* such that $[w] = g$. If w is not reduced, then for some i, $1 \leq i \leq n-1$, we have $y_{i+1} = y_i^{-1}$ and $g_i \in G_{y_i}$. Then we replace $y_i.g_i.y_i^{-1}$ by the element $\phi_{y_i}(g_i)$ in w. We get a new word

$$g_0.y_1.g_1.y_2.g_2. \cdots y_{i-1}.g_{i-1}\phi_{y_i}(g_i).y_{i+1}. \cdots .y_n.g_n.$$

Continuing the above processes on w yields a reduced of G^* of value g. In other words, the performance of the t-reductions on w yields a reduced word of G^* of value $[w] = g$.

Now let $w = g_0.y_1.g_1. \ldots .y_n.g_n$, $n \geq 0$, and $w' = h_0.x_1.h_1. \ldots .x_m.h_m$ be two words of G^* such that w is reduced and $[w] = [w'] = g$. Assume that w' is reduced. We need to show that $n = m$. Since $[w'][w]^{-1} = 1$, the identity of G^*, therefore by Proposition 1, the word

$$w_0 = h_0.x_1.h_1. \ldots .x_m.h_m g_n^{-1}.y_n^{-1}.g_{n-1}^{-1} \ldots g_1^{-1}.y_1^{-1}.g_0^{-1}$$

is not reduced. Since w and $w'$are reduced, therefore $y_n = x_m$, and $h_m g_n^{-1} \in G_{y_n}$. We substitute $\phi_{y_n}(h_m g_n^{-1})$ for $x_m.h_m g_n^{-1}.y_n^{-1}$ in w_0. We get a new word

$$w_1 = h_0.x_1.h_1. \ldots .x_{m-1}.h_{m-1} A g_{n-1}^{-1}.y_{n-1}^{-1} \ldots g_1^{-1}.y_1^{-1}.g_0^{-1}$$

where $A = \phi_{y_n}(h_m g_n^{-1})$. Then w_1 is not reduced. Similar to above we have $x_{m-1} = y_{n-1}^{-1}$, and $h_{m-1}\ \phi_{y_n}\ (h_m g_n^{-1})\ g_{n-1}^{-1} \in G_{y_{n-1}}$.

Now continuing above processes yields $x_1 = y_1$. This implies that $|w| = |w'|$. Now assume that w is reduced and $|w| = |w'|$. We need to show that w' is reduced. For, if w' is not reduced then by applying the t-reductions on w' yields a reduced word $w''$of G^* such that $[w] = [w'] = [w'']$ and $|w''| < |w'|$. From above we have $|w| = |w''|$. This contradicts the assumption that $|w| = |w'|$. Hence w' is reduced. This completes the proof. □

Definition 2. For each element g of G^* define $|g|$ to be the **length** of a reduced word of G^* of value g.

In view of Propositions 1 and 2, this concept is clear.

Definition 3. An element g of G^* is called an **invertor element** of G^* if there exist $j \in J$ and $c \in C_j$ such that g is conjugate to the element $t_j c$.

Proposition 3. *Let g is an invertor element of G^*. Then there exists $j \in J$ such that g^2 is in a conjugate of C_j and g is not in any conjugate of G.*

Proof. Since g is an invertor element of G^*, therefore $g = f t_j c f^{-1}$, where $f \in G^*$, $j \in J$ and $c \in C_j$. Then

$$g^2 = f t_j c f^{-1} f t_j c f^{-1} = f \alpha_j(c) c_j c f^{-1} \in f C_j f^{-1},$$

a conjugate of C_j, because $\alpha_j(c) c_j c \in C_j$. Now we show that g is not in any conjugate of G. For, if g is in a conjugate of G, then $g \in hGh^{-1},\ h \in G^*$. Then $t_j c = g_1 g_0 g_1^{-1}$ where $g_1 = f^{-1}h$ and $g_0 \in G$. By Proposition 1, there exists a reduced word w of G^* of value g_1. Now applying the t-reductions on the word $w.g_0.w^{-1}$ yields a reduced word w_0 of G^*. Since $g_0 \in G$, therefore $|w_0| = 2m,\ m \leq |w|$. Then the words $1.t_j.c$ and w_0 are reduced words of G^* and of value $t_j c$. By Proposition 2 we have $|1.t_j.c| = |w_0|$. This contradicts the fact that $|1.t_j.c|$ is odd and $|w_0|$ is even. Hence g is not in any conjugate of G. This completes the proof. □

Definition 4. Let w be a reduced word of G^*. We say that w is a **cyclically reduced word** of G^* if w^2 is reduced.

It is clear that every word of length zero is cyclically reduced.

The proof of the following proposition is clear.

Proposition 4. *Let $w = g_0.y_1.g_1.y_2.g_2. \dots .y_n.g_n$, $n \geq 1$, be a reduced word of G^*. Then the following are equivalent:*

1. *w is cyclically reduced,*
2. *w^m is cyclically reduced for every integer m,*
3. *$g_{n-1}.y_n.g_n g_0.y_1.g_1$ is a reduced word of G^*.*

Proposition 5. *Let w_1 and w_2 be two reduced words of G^* such that $[w_1] = [w_2]$ and w_1 is cyclically reduced. Then w_2 is cyclically reduced. Moreover, if $|w_1| \geq 1$, then $[w_1]$ is not in any conjugate of G.*

Proof. By Proposition 2, $|w_1| = |w_2|$. If $|w_1| = 0$, then by definition, w_2 is cyclically reduced. Let $w_1 = g_0.y_1.g_1.y_2.g_2. \dots .y_n.g_n$, $n \geq 1$, and $w_2 = h_0.x_1.h_1.x_2.h_2. \dots .x_n.h_n$. We need to show that $x_n.h_nh_0.x_1$ is reduced. Since w_1 is cyclically reduced, therefore $g_ng_0 \notin G_{y_n}$. By Proposition 2, $h_nh_0 \notin G_{y_n}$. So $x_n.h_nh_0.x_1$ is reduced. Therefore w_2 is cyclically reduced.

Now assume that $|w_1| \geq 1$. If $[w_1]$ is in a conjugate of G, then $[w_1] = aba^{-1}$, where $a \in G^*$, and $b \in G$. By Proposition 1, there exists a reduced word w of G^* such that $a = [w]$. Then $[w_1] = [w.b.w^{-1}]$. Now applying t-reductions on the word wbw^{-1} yields a reduced word of G^* of length zero which contradicts Proposition 2, or yields a reduced word $w'$of G^* such that w' is not cyclically reduced. This contradicts the first part of the proposition. Hence $[w_1]$ is not in any conjugate of G. This completes the proof. □

Definition 5. An element g of G^* is called **cyclically reduced** if g is the value of a cyclically reduced word of G^*.

In view of Proposition 5, this concept is well defined.

Proposition 6. *Let g be an element of G^*. Then:*

(i) *If g is an inventor, then g is not conjugate to any cyclically reduced element of G^*.*

(ii) *If g is not inventor, then g is conjugate to a cyclically reduced element of G^*.*

Proof. **(i)** Let h be cyclically reduced and $f \in G^*$ such that $g = fhf^{-1}$. If $|h| = 0$, then $h \in G$ and $g \in fGf^{-1}$. This contradicts Proposition 3 that g is not in any conjugate of G. If $|h| \geq 1$, then h^2 is cyclically reduced, and $g^2 = fh^2f^{-1}$. Then $h^2 = f^{-1}g^2f$, $\left|h^2\right| \geq 2$, and by Proposition 3, h^2 is in a conjugate of G, because g^2 is in a conjugate of G. This contradicts Proposition 5. This implies that g is not conjugate to any cyclically reduced element of G^*.

(ii) Let g' be an element in the conjugacy class of G^* containing g such that g' is represented by a reduced word w of G^* of shortest length. We need to show that w is cyclically reduced. Let

$$w = g_0.y_1.g_1.y_2.g_2. \dots .y_n.g_n.$$

If $n = 0$, then $w = g_0$ and w is cyclically reduced. Let $n \geq 1$. If $g_ng_0 \in G_{y_n}$ and $y_1 = y_n^{-1}$, then

$$w_0 = g_0.y_1.g_1.y_2.g_2. \dots .y_{n-1}.g_{n-1}\phi_{y_n}(g_ng_0)$$

is a reduced word of G^* and of value conjugating g. But w_0 has length smaller than w. Contradiction. This implies that g is conjugate to a cyclically reduced element of G^*. This completes the proof. □

3 The main result

Throughout this section G^* is the quasi-HNN group

$$G^* = \langle gen(G), t_i, t_j \mid rel\ (G),\ \ t_i A_i t_i^{-1} = B_i\ ,\ \ t_j C_j t_j^{-1}\ = C_j,\ \ t_j^2\ =\ c_j \rangle,$$
$$i \in I, \quad j \in J.$$

The aim of this section is to show that the centralizer

$$C_{G^*}(H) = \{g \in G^* : gh = hg,\ h \in H\}$$

of the finite subgroup H of G^* containing no invertor elements and not in any conjugate of the associated subgroups A_i and B_i, $i \in I$, of the base G, is contained in a conjugate of G. Then we apply such result to HNN groups, and one-relater groups.

First we start with the following lemma.

Lemma. *Finite subgroups of quasi-HNN groups containing no invertor elements are contained in conjugates of the base G.*

Proof. Let H be a finite subgroup of G^*. If H contains an invertor element, then by Proposition 3, H is not contained in any conjugates of the base G. Now assume that H contains no invertor elements**.** Let h be an element of H. Then h is of finite order, and by Propositon 6 there exist two elements f and g of G^* such that g is cyclically reduced and $h = fgf^{-1}$. If $|g| = 0$, then $g \in G$, and h is in conjugates of G. If $|g| \geq 1$, then for any integer n, g^n is cyclically reduced and $|g^n| \geq |n|\,|g| \geq 1$. Proposition 1 implies that $g^n \neq 1$, the identity of G^*. Then g is of infinite order. This implies that h is of infinite order. This contradicts the assumption that h is in H. Hence H is contained in a conjugate of G. This completes the proof. $\square$

Remark. If in the quasi-HNN group G^* defined above we have $C_j = \{1\}$ then

$$G^* = \langle gen(G), t_i, t_j \mid rel\ (G),\ \ t_i A_i t_i^{-1} = B_i\ ,\ \ t_j^2\ = 1 \rangle$$

and the subgroup H of G^* generated by t_j for a fixed j in J has the presentation

$$H = \langle t_j \mid t_j^2 = 1 \rangle = \{1, t_j\}.$$

Then H is not in any conjugate of G because H contains the invertor t_j on which by Proposition 3 is not in any conjugate of G.

The main result of this section is the following theorem.

Theorem. *The centralizers of finite subgroups of quasi-HNN groups containing no invertor elements and not contained in any conjugate of the associated subgroups of the bases are contained in conjugates of the bases.*

Proof. Let G^* be the quasi-HNN group defined above and H be a finite subgroup of G^* such that H contains no invertor elements, and H is not contained in any conjugate of the associated subgroups A_i and B_i, $i \in I$ of G. We need to show that the centralizer $C_{G*}(H)$ of H in G^* is contained in a conjugate of G. For, if $C_{G*}(H)$ is not contained in a conjugate of G, then there exists an element $g \in C_{G*}(H)$ such that g is not contained in a conjugate of G. Let h be any element of H. Then $hgh^{-1}g^{-1} = 1$. Since $g \neq 1$, then by Proposition 1, there exists a reduced word

$$w = g_0.y_1.g_1. \dots .y_n.g_n, \quad n \geq 1$$

of G^* such that $[w] = g$. Then

$$w_0 = hg_0.y_1.g_1. \dots .y_n.g_n h^{-1} g_n^{-1}.y_n^{-1}.g_{n-1}^{-1}.\dots.g_1^{-1}.y_1^{-1}.g_0^{-1}g^{-1}$$

is a word of G^* of value 1. So by Proposition 1, w_0 is not reduced. Then $L_s \in G_{y_s}$, where $L_s = g_{n-s}\phi_{y_s}(L_{s-1})g_{n-s}^{-1}$, $s = 1, \dots, n$, with convention that $L_0 = g_n h^{-1} g_n^{-1}$ and $L_n = g_0 \phi_{y_{n-1}}(L_{n-1}) g_0^{-1} h^{-1} = 1$.

Then

$$h \in g_0 \phi_{y_{n-1}}(L_{n-1}) g_0^{-1} = g_0 G_{y_{n-1}} g_0^{-1}.$$

This contradicts the assumption that H is not contained in any conjugate of the associated subgroups of G. Hence $C_{G*}(H)$ is contained in a conjugate of G. This completes the proof. □

We have the following corollaries of Theorem.

Corollary 1. *The centralizers of finite subgroups of HNN groups not contained in any conjugate of the associated subgroups of the bases are contained in conjugates of the bases.*

Proof. By taking $J = \varnothing$ in the group G^* we obtain the HNN group

$$G^* = \langle gen(G), t_i \mid rel\ (G),\ \ t_i A_i t_i^{-1} = B_i\ ,\ \ i \in I \rangle.$$

Since G^* contains no invertor elements, then Theorem implies that if H is a finite subgroup of G^* such that H is not contained in any conjugate of the associated subgroups A_i and B_i, $i \in I$, of the base G, then the centralizer $C_{G*}(H)$ of H in G^* is contained in a conjugate of the base G. This completes the proof. □

Corollary 2. *The centralizers of finite subgroups of one-relator groups where the relator element is cyclically reduced and contains at least two different letters, is contained in a conjugate of a one-relator subgroup of relator element is cyclically reduced and is shorter than the relator element of the group.*

Proof. Let $G = \langle X \mid r \rangle$ be a one-relator group, where r is cyclically reduced and contains at least two different letters from X. Let H be a finite subgroup of G. By Theorem 5.1 of [2, pages 198, 294], G can be embedded in an HNN group $\langle gen(K), t \,\big|\, rel(K),\ tUt^{-1} = V \rangle$, where the base K is a one-relator group, $K \cong$

$\langle X' \mid r' \rangle$ where r' is cyclically reduced, and r' is shorter than r, and the associated subgroups U and V of the base K are isomorphic free groups.

Let H be a finite subgroup of G. Then H contains no invertor elements because G is an HNN group. Since non-trivial elements of a free group have infinite orders, therefore H is not contained in a conjugate of U or in a conjugate of V. Then Corollary 1 implies that the centralizer $C_G(H)$ of H in G is contained in a conjugate of the base K. This completes the proof. $\square$

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