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Invariant transformations of loop transversals. 1. The case of isomorphism

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Abstract. One special class of invariant transformations of loop transversals in groups is investigated. Transformations from this class correspond to arbitrary isomorphisms of transversal operations corresponding to the loop transversals mentioned above. Isomorphisms of loop transversal operations with the same unit 1 are investigated.

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1 Introduction

The notion of a transversal in a group to its own subgroup is well-known and has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some fixed groups to their own subgroups present special interest. Loop transversal may not exist in a given group G to its subgroup H (for example, if $G = S_6$, $H = St_{12}(S_6)$), but we will study such questions further. Let a group G and its proper subgroup H be set, and some loop transversal $T_0 = \{t_i\}_{i \in E}$ in G to H is given and fixed. How to describe all other loop transversals in G to H? In other words, what kind of transformations are admissible over loop transversal T_0 so that the obtained sets were loop transversals too? And how to describe the set of all such admissible transformations?

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, they are transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies and isostrophies are well studied, but such a general transformation as crossed isotopy practically was not investigated.

These investigations are necessary and very important, since there is a number of important and known problems reduced to research of the set of all loop transversals in some given group G to its subgroup H. For example, when $G = S_n$ and $H = St_1(S_n)$, we obtain the set of all loops of some fixed order n. The calculation of their quantity for given natural number n is a well-known open problem (*enumeration problem*). Other known problem – about G-loops – also can be considered in terms of loop transversals transformations. In the present work we will investigate what transformations of loop transversals correspond to the first well-known transformation of transversal operations – to an isomorphism. We will limit ourselves only to those transformations which keep property to be loop transversals.

Let us begin with some necessary definitions and preliminary statements.

2 Necessary definitions and statements

2.1 Quasigroups, loops and transversals in groups

Definition 1. A system $< E, \cdot >$ is called a **left (right) quasigroup** if the equation $(a \cdot x = b)$ (the equation $(y \cdot a = b)$) has exactly one solution in the set E for any fixed $a, b \in E$. If for some element $e \in E$ we have

$$e \cdot x = x \cdot e = x \quad \forall x \in E,$$

then a left (right) quasigroup $< E, \cdot, e >$ is called a **left (right) loop** (the element $e \in E$ is called a **unit**). A left quasigroup $< E, \cdot >$ that is simultaneously a right quasigroup is called simply a **quasigroup**. Similarly, left loop which is simultaneously a right loop is called a **loop**.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i \in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i \in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H. If a left transversal T is simultaneously a right one, it is called a **two-side transversal**.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \; h \in H,$$

and similarly for a right transversal:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = h t_z, \; h \in H.$$

Further we will do all researches only for the left transversals in G to H; for right transversals everything is similar.

Definition 3. If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then such left transversal $T = \{t_x\}_{x \in E}$ is called a **loop transversal**.

The following statements are known (see [1, 6]):

Lemma 1. *A system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a left loop with the two-sided unit 1.*

Proof. See Lemma 1 in [6]. □

Lemma 2. *The following conditions are equivalent:*

1. *The set* $T = \{t_x\}_{x \in E}$ *is a loop transversal in* G *to* H*;*
2. *The set* $T = \{t_x\}_{x \in E}$ *is a left transversal in* G *to* $\pi H \pi^{-1} \rightleftharpoons H^\pi$, $\forall \pi \in G$*;*
3. *The set* $\pi T \pi^{-1} \rightleftharpoons T^\pi$ *is a left transversal in* G *to* H, $\forall \pi \in G$.

Proof. See [1]. □

Use further the following permutation representation $\widehat{G}$ of a group G by the left cosets of its subgroup H (see [5,6]):

$$\widehat{g}(x) = y \quad \overset{def}{\Longleftrightarrow} \quad g t_x H = t_y H.$$

For simplicity we consider

$$Core_G(H) = \underset{g \in G}{\cap} g H g^{-1} = \{e\};$$

then this representation is exact (see Lemma 6 in [6]), and we have $\widehat{G} \cong G$. Notice that $\widehat{H} = St_1(\widehat{G})$.

Lemma 3 (see [6])**.** *Let* $T = \{t_x\}_{x \in E}$ *be a left transversal in* G *to* H*. Then the following statements are true:*

1. $\widehat{h}(1) = 1 \ \forall h \epsilon H$*;*
2. $\forall x, y \in E$ *:*

   $\widehat{t}_x(y) = x \overset{(T)}{\cdot} y = \widehat{L}_x(y), \qquad \widehat{t}_1(x) = \widehat{t}_x(1) = x,$

   $\widehat{t}_x^{-1}(y) = x \overset{(T)}{\diagdown} y = \widehat{L}_x^{-1}(y), \qquad \widehat{t}_x^{-1}(1) = x \overset{(T)}{\diagdown} 1, \qquad \widehat{t}_x^{-1}(x) = 1,$

   where "$\overset{(T)}{\diagdown}$" *- is a left division for the operation* $< E, \overset{(T)}{\cdot}, 1 >$ *(i.e.* $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$*).*

Proof. See Lemma 4 in [6]. □

Remark 1. The operation "$\overset{(T)}{\diagdown}$" is named a *left division* here – as an inverse operation to the *left multiplication* (multiplication at the left) "$\overset{(T)}{\cdot}$". Sometimes in the literature this operation may be named a *right division.*

Remark 2. As we can see from Lemma 3, item **2**), the elements of a left transversal T in G to H can be represented trought its transversal operation $< E, \overset{(T)}{\cdot}, 1 >$ as left translations $\{L_x\}_{x \in E}$. The similar holds for a right transversal.

At last, remind how any two left transversals T and P in a group G to its subgroup H are connected .

Lemma 4 (see [6])**.** *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$- be left transversals in G to H. Then there is a set of elements $\{h_{(x)}\}_{x\in E}$ from H such that:*

1. $p_x = t_x h_{(x)}$ $\forall x \in E$;

2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)$.

Proof. See Lemma 7 in [6]. □

This set $\{h_{(x)}\}_{x\in E}$ is called (see [8]) a **derivation set** for transversal T (and for transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

Remind also the definitions of a left multiplicative group and of a left inner permutation group of a loop.

Definition 4. Let $< E, \cdot, e >$ be a loop. Then a group

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

generated by all left translations L_a of loop $< E, \cdot, e >$, is called a **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a\cdot b} L_a L_b, : a, b \in E >$$

generated by all permutations $l_{a,b}$, is called a **left inner permutation group** of the loop $< E, \cdot, e >$.

2.2 Morphisms of quasigroups and loops

Definition 5 (see [2])**.** A mapping $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) of the operation $< E, \cdot >$ on the operation $< E, \circ >$ is called an **isotopy** if

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\alpha, \beta, id)$, then such an isotopy is called a **principal isotopy**.

Definition 6 (see [3])**.** A mapping $\Phi = (\alpha, B, \gamma)$, where α, γ are permutations on E and $B = B(x, y)$ is a right invertible operation on E ($B(x, y) = \varphi_x(y)$, φ_x is a permutation on E $\forall x \in E$), is called a **right crossed isotopy** (**RC-isotopy**) of operations $< E, \cdot >$ and $< E, \circ >$ if

$$\gamma(x \circ y) = \alpha(x) \cdot B(x, y) \quad \forall x, y \in E.$$

A **left crossed isotopy** (*LC*-**isotopy**) is defined similarly.

It is obvious that any isotopy is both *RC*-isotopy and *LC*-isotopy simultaneously.

Definition 7 (see [2]). The operations $A(x,y)$ and $B(x,y)$ on a set E are called **orthogonal**, if a system

$$\begin{cases} A(x,y) = a \\ B(x,y) = b \end{cases}$$

has an unique solution in a set $E \times E$ for any fixed pair $(a,b) \in E \times E$.

It is easy to show (see [4]) that the orthogonality of operations A and B is equivalent to the fact: the following mapping

$$\Theta = \begin{pmatrix} (1,1) & ... & (x,y) & ... \\ (A(1,1),B(1,1)) & ... & (A(x,y),B(x,y)) & ... \end{pmatrix}$$

is a permutation on the set $E \times E$. The following is true.

Lemma 5. *Let* $< E, \cdot, e >$ *be a left loop. Then RC-isotop* $< E, \circ, e^{'} >$ *of the left loop* $< E, \cdot, e >$ *(by RC-isotopy* $T = (\alpha, B, \gamma)$*) is a loop* $\Longleftrightarrow$ *the operations* $(\cdot)^{(\alpha,id,id)}$ *and* B^{-1} *are orthogonal.*

Proof. See in [3,8]. □

2.3 Communication between transformations of transversals, morphisms of transversal operations and transformations of derivation sets

Let G be some fixed group and H be its proper subgroup. Consider further the permutation representation $\widehat{G}$ of the group G (note that $\widehat{G} \cong G$, $\widehat{H} \cong St_1(\widehat{G})$).

According to Lemma 4, any two left transversals $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ in G to H are connected with the help of some *RC*-isotopy (id, B, id) of their transversal operations $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ (where $B(x,y) = \widehat{h}_{(x)}(y)$). It means that if we fix any *"good"* left transversal T_0 in G to H (for example, a *group* transversal if it exists), then we will receive all other left transversals in G to H from T_0 by the help of *RC*-isotopy. Moreover, any loop transversal P in G to H may be received from T_0 with the help of such *RC*-isotopy (id, B, id) (where $B(x,y) = \widehat{h}_{(x)}(y)$) that the operations $< E, \overset{(T_0)}{\cdot}, 1 >$ and $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$ are ortogonal (according to Lemma 5).

Remark 3. If we consider the case $G = S_n$ and $H = St_1(S_n)$, as it is described above, it is possible to express all loops of order n as the *RC*-isotopies (id, B, id) of some loop (group) $< E, \overset{(T_0)}{\cdot}, 1 >$ of order n, and the operation $< E, \overset{(T_0)}{\cdot}, 1 >$ is orthogonal to the operation $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$.

Further we will investigate only such special cases of RC-isotopy of a fixed loop transversal T_0 in G to H, which give as a result a loop transversal in G to H again. The research will be done by the following scheme:

$$< E, \overset{(T_0)}{\cdot}, 1 > \overset{\Phi}{\longleftrightarrow} < E, \overset{(P)}{\cdot}, 1 >$$

$$\updownarrow$$

$$T_0 = \{t_x\}_{x\in E} \overset{\Phi^*}{\longrightarrow} P = \{p_x\}_{x\in E}$$

$$\updownarrow$$

$$p_x = t_x h^{(\Phi)}_{(x)},\ \{h^{(\Phi)}_{(x)}\}_{x\in E} \text{ is a derivation set, corresponding to transformation } \Phi$$

$$\updownarrow$$

$$\Theta_{(\Phi)} = \begin{pmatrix} --- & (x,y) & --- \\ --- & (x \overset{(T_0)}{\cdot} y, (\widehat{h}^{(\Phi)}_{(x)})^{-1}(y)) & --- \end{pmatrix},$$

where $\Theta_{(\Phi)}$ - is a permutation on a set $E\times E$, corresponding to orthogonal operations $< E, \overset{(T_0)}{\cdot}, 1 >$ and $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$.

Let us begin our investigation from an elementary invariant transformation on a set of loop transversals in G to H - from the transformation corresponding to isomorphism of transversal operations.

3 The transformations which correspond to isomorphisms of the transversal operations of loop transversals

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a group G to its subgroup H, and $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ are its transversal operations. Fix one of transversals, for example, $T = \{t_x\}_{x\in E}$. Consider the following group:

$$M_G(T) \overset{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{G}\alpha^{-1} >,$$

it is generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{G}\alpha^{-1}.$$

Lemma 6. *The following propositions are true:*

1. $N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T) \subseteq St_1(S_E)$,

2. $M_G(T)$ *is maximal among subgroups* $M \subseteq St_1(S_E)$ *which satisfy the following property:*

$$LM(< E, \overset{(T)}{\cdot}, 1 >) = \bigcap_{\alpha\in M} (\alpha\widehat{G}\alpha^{-1}).$$

Proof. **1**. By definition $M_G(T) \subseteq St_1(S_E)$. Let $\alpha \in N_{St_1(S_E)}(\widehat{G})$, then

$$\begin{cases} \alpha \in St_1(S_E) \\ \alpha \widehat{G} \alpha^{-1} = \widehat{G} \end{cases} .$$

The following property is always fulfilled for any left transversal T in G to H,

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \widehat{G},$$

so

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \widehat{G} = \alpha \widehat{G} \alpha^{-1}.$$

Since $\alpha \in St_1(S_E)$ then $\alpha \in M_G(T)$, and

$$N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T).$$

2. It obviously follows from the definition of the group $M_G(T)$. □

Remark 4. Both bounds in the inclusion in item **1** of previous Lemma are reached:

a) Let $LM(< E, \overset{(T)}{\cdot}, 1 >) = \widehat{G}$, then

$$M_G(T) = < \alpha \mid \alpha \in St_1(S_E),\ \widehat{G} \subseteq \alpha \widehat{G} \alpha^{-1} > = N_{St_1(S_E)}(\widehat{G}).$$

b) Let $\widehat{G} = S_E$, $\widehat{H} = St_1(S_E)$, then

$$\begin{aligned} M_G(T) &= < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha S_E \alpha^{-1} > = \\ &= < \alpha \mid \alpha \in St_1(S_E) > = St_1(S_E). \end{aligned}$$

Lemma 7. *Let loops* $< E, \overset{(T)}{\cdot}, 1 >$ *and* $< E, \overset{(P)}{\cdot}, 1 >$ *be isomorphic, and* $\varphi : E \to E$ *be this isomorphism (note that* $\varphi(1) = 1$*). Then*

1. $\widehat{P} = h_0^{-1} \widehat{T} h_0$ *for some* $h_0 \in H^* = M_G(T)$*;*
2. $\varphi \equiv h_0$ *and* $LI(< E, \overset{(T)}{\cdot}, 1 >) \subseteq h_0 \widehat{H} h_0^{-1}$.

Proof. **1**. Let conditions of Lemma hold. We have:

$$\varphi(x \overset{(P)}{\cdot} y) = \varphi(x) \overset{(T)}{\cdot} \varphi(y) \forall x, y \in E.$$

According to Lemma 3,

$$\begin{aligned} \widehat{t}_x &= L_x, \quad \textit{where} \quad L_x(y) = x \overset{(T)}{\cdot} y, \\ \widehat{p}_x &= L_x, \quad \textit{where} \quad L_x(y) = x \overset{(P)}{\cdot} y. \end{aligned}$$

Since φ is a permutation on a set E and $\varphi(1) = 1$, then $\varphi \in St_1(S_E)$. Further we have

$$\begin{aligned} \varphi L_x(y) &= L_{\varphi(x)}\varphi(y) \forall x, y \in E, \\ L_x(y) &= \varphi^{-1} L_{\varphi(x)}\varphi(y) \forall x, y \in E, \\ L_x &= \varphi^{-1} L_{\varphi(x)}\varphi \forall x \in E. \end{aligned} \tag{1}$$

It means that $\widehat{P} = \varphi^{-1}\widehat{T}\varphi$ and $\varphi \in St_1(S_E)$. Therefore we receive $\widehat{P} = h_0^{-1}\widehat{T}h_0$ for some $h_0 \in St_1(S_E)$ and $\varphi \equiv h_0$.

Moreover, since

$$LM(< E, \overset{(T)}{\cdot}, 1 >) =< L_a \mid a \in E >,$$

then from (1) it follows that

$$\begin{gathered} \varphi^{-1}(LM(< E, \overset{(T)}{\cdot}, 1 >))\varphi = \varphi^{-1} < L_a \mid a \in E > \varphi = \\ =< \varphi^{-1} L_a \varphi \mid a \in E > = < L_b \mid b \in E > = \\ = LM(< E, \overset{(P)}{\cdot}, 1 >) \subseteq \widehat{G}, \end{gathered}$$

and $h_0 = \varphi \in M_G(T)$.

2. Let $\alpha \in M_G(T)$, then we have

$$\begin{cases} \alpha \in St_1(S_E), \\ L_a \in \alpha\widehat{G}\alpha^{-1} \quad \forall a \in E. \end{cases}$$

$$\begin{cases} \alpha \in St_1(S_E), \\ \alpha^{-1} L_a \alpha \risingdotseq g_{a'} \in \widehat{G} \quad \forall a \in E. \end{cases}$$

$$a^{'} = g_{a'}(1) = \alpha^{-1} L_a \alpha(1) = \alpha^{-1}(1).$$

Then $\forall a, b \in E$

$$\begin{gathered} \alpha^{-1} l^{(T)}_{a,b} \alpha = \alpha^{-1} L^{-1}_{a \overset{(T)}{\cdot} b} L_a L_b \alpha = \\ = (\alpha^{-1} L^{-1}_{a \overset{(T)}{\cdot} b} \alpha) \cdot (\alpha^{-1} L_a \alpha) \cdot (\alpha^{-1} L_b \alpha) = \\ = g^{-1}_{\alpha^{-1}(a \overset{(T)}{\cdot} b)} g_{\alpha^{-1}(a)} g_{\alpha^{-1}(b)}. \end{gathered}$$

Assuming $a = \alpha(u)$ and $b = \alpha(v)$ (i.e. $u = \alpha^{-1}(a)$ and $v = \alpha^{-1}(b)$), we obtain

$$\alpha^{-1} l^{(T)}_{\alpha(u),\alpha(v)} \alpha = g^{-1}_{\alpha^{-1}(\alpha(u) \overset{(T)}{\cdot} \alpha(v))} g_u g_v.$$

Since α is an isomorphism of operations $(\overset{(T)}{\cdot})$ and $(\overset{(P)}{\cdot})$, then

$$\alpha(u \overset{(P)}{\cdot} v) = \alpha(u) \overset{(T)}{\cdot} \alpha(v),$$

and therefore

$$\alpha^{-1} l^{(T)}_{\alpha(u),\alpha(v)} \alpha = g^{-1}_{u \overset{(P)}{\cdot} v} g_u g_v = l^{(P)}_{u,v} \in LI(< E, \overset{(P)}{\cdot}, 1 >) \subseteq \widehat{H}.$$

It means that

$$\alpha^{-1} LI(< E, \overset{(P)}{\cdot}, 1 >)\alpha \subseteq \widehat{H},$$
$$LI(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{H} \alpha^{-1}.$$

□

Lemma 8. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in G to H and $h_0 \in N_{St_1(S_E)}(H)$. Define the set of permutations:*

$$p_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then

1. *$P = \{p_{x'}\}_{x'\in E}$ is a loop transversal in G to H;*
2. *The transversal operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$ are isomorphic, and the isomorphism is set up by the mapping $\varphi(x) = h_0(x)$.*

Proof. **1**. Let the conditions of Lemma hold. At first we can see that $P = \{p_{x'}\}_{x'\in E}$ is a left transversal in G to H. It follows from Lemma 2 and the following calculation

$$x' = \widehat{p}_{x'}(1) = h_0^{-1} \widehat{t}_x h_0(1) = h_0^{-1}.$$

Any transversal conjugated with the transversal T will be conjugated with the transversal P. According to Lemma 2, the transversal $P = \{p_{x'}\}_{x'\in E}$ is a loop transversal in G to H.

2. Consider the transversal operation $< E, \overset{(P)}{\cdot}, 1 >$ which corresponds to the transversal P. We have

$$x \overset{(P)}{\cdot} y = z \iff p_x p_y = p_z h, \quad h \in H, \quad \forall x, y \in E. \tag{2}$$

Since

$$h_0^{-1} t_x h_0 = p_{x'} = p_{h_0^{-1}(x)},$$

then after replacing $x \to h_0(u)$ we have

$$p_u = h_0^{-1} t_{h_0(u)} h_0 \quad \forall u \in E.$$

From (2) we obtain

$$p_x p_y = p_z h, \quad h \in H, \quad (where \quad z = x \overset{(P)}{\cdot} y),$$

$$h_0^{-1}t_{h_0(x)}h_0 \cdot h_0^{-1}t_{h_0(y)}h_0 = h_0^{-1}t_{h_0(x \overset{(P)}{\cdot} y)}h_0 \cdot h, \quad h \in H,$$
$$t_{h_0(x)}t_{h_0(y)} = t_{h_0(x \overset{(P)}{\cdot} y)} \cdot (h_0 h h_0^{-1}).$$

Since $h_0 \in N_{St_1(S_E)}(\widehat{H})$ then $(h_0 h h_0^{-1}) = h' \in \widehat{H}$. Therefore we obtain

$$h_0(x) \overset{(T)}{\cdot} h_0(y) = h_0(x \overset{(P)}{\cdot} y) \quad \forall x, y \in E,$$

i.e. $\varphi = h_0$ is an isomorphism of the operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$. □

It means that conjugated loop transversals in G to H correspond to isomorphic loop transversal operations and vice versa.

Further according to the scheme from Section 2, we will find out the form of derivation sets $\{h_{(x)}\}_{x\in E}$ which correspond to isomorphic transformations.

Lemma 9. *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in G to H which correspond to isomorphic transversal operations. Let $p_x = t_x h_{(x)}$ and $\{h_{(x)}\}_{x\in E}$ be a derivation set. Then*

$$h_{(x)} = t_x^{-1}h_0^{-1}t_{h_0(x)}h_0, \quad \forall x \in E$$

for some $h_0 \in M_G(T)$.

Proof. Let conditions of Lemma hold. According to Lemma 7 $\forall x \in E$:

$$p_x = h_0^{-1}t_{h_0(x)}h_0,$$

for some $h_0 \in M_G(T)$. From the other hand

$$p_x = t_x h_{(x)} \quad \forall x \in E.$$

Therefore we have

$$t_x h_{(x)} = h_0^{-1}t_{h_0(x)}h_0 \quad \forall x \in E,$$
$$h_{(x)} = t_x^{-1}h_0^{-1}t_{h_0(x)}h_0 \quad \forall x \in E,$$

as it had to be shown. □

At last according to the scheme from Section 2 we will express the form of permutations Θ which correspond to isomorphic transformations of transversals.

Lemma 10. *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be loop transversals in G to H, and its transversal operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$ are isomorphic. A permutation Θ on $E \times E$ corresponds to ortogonal operations " $\overset{(T)}{\cdot}$ " and $B^{-1}(x, y)$ (see in Section 2), can be expressed in the following form (for some $h_0 \in M_G(T)$): $\forall x, y \in E$*

$$\Theta = \begin{pmatrix} \ldots & (x, y) & \ldots \\ \ldots & (x \overset{(T)}{\cdot} y, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & \ldots \end{pmatrix}.$$

Proof. According to the previous lemma we have: it is true for some $h_0 \in M_G(T)$:

$$h_{(x)} = t_x^{-1} h_0^{-1} t_{h_0(x)} h_0 \quad \forall x \in E.$$

Then

$$h_{(x)}^{-1} = h_0^{-1} t_{h_0(x)}^{-1} h_0 t_x \quad \forall x \in E.$$

According to the definition, the permutation Θ can be expressed in the following form

$$\Theta = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_{(x)}^{-1}(y)) & \dots \end{pmatrix}.$$

We have $\forall x \in E$:

$$\begin{aligned} h_{(x)}^{-1}(y) &= h_0^{-1} \widehat{t}_{h_0(x)}^{-1} h_0 \widehat{t}_x(y) = h_0^{-1} \widehat{t}_{h_0(x)}^{-1} h_0(x \overset{(T)}{\cdot} y) = \\ &= h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y)), \end{aligned}$$

and finally we obtain

$$\Theta = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & \dots \end{pmatrix}.$$

□

Remark 5. A permutation $h_0 = id$, the derivation set $\{h_{(x)}\} = id \quad \forall x \in E$ and the permutation

$$\Theta_0 = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \end{pmatrix}$$

correspond to the trivial isomorphism $\varphi = id$.

Consider the product (composition) $\Theta_0^{-1}\Theta$ as a composition of two permutations from $S_{E\times E}$. We have

$$\begin{aligned} \Theta_0^{-1}\Theta &= \begin{pmatrix} \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \\ \dots & (x,\, y) & \dots \end{pmatrix} \circ \begin{pmatrix} \dots & (x,\, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_{(x)}^{-1}(y)) & \dots \end{pmatrix} = \\ &= \begin{pmatrix} \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & \dots \end{pmatrix} = \\ &\overset{x \overset{(T)}{\cdot} y = z}{=} \begin{pmatrix} \dots & (z,\, y) & \dots \\ \dots & (z,\, h_0^{-1}(h_0(z \overset{(T)}{/} y) \overset{(T)}{\backslash} h_0(z))) & \dots \end{pmatrix} \leftrightharpoons \Theta^*. \end{aligned}$$

As a corollary we received two interesting particular cases:

$$\Theta_0^{-1}\Theta(z,\, z) = (z,\, h_0^{-1}(\underset{\overset{\shortparallel}{1}}{h_0(1)} \overset{(T)}{\backslash} h_0(z))) = (z,\, h_0^{-1}(h_0(z))) = (z,\, z) \quad \forall z \in E.$$

$$\Theta_0^{-1}\Theta(z,\,1) = (z,\,h_0^{-1}(h_0(z)\overset{(T)}{\diagdown} h_0(z))) = (z,\,h_0^{-1}(1)) = (z,\,1) \quad \forall z \in E,$$

i.e. $\Theta^* \in St_{(a,\,a),\,(a,\,1)}(S_{E\times E}) \quad \forall a \in E.$

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