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On preradicals associated to principal functors of module categories. III

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Abstract. The classes of modules and preradicals associated to the functor $Hom_R(\text{-}, U)$ are studied, continuing the investigations of parts I and II. The properties of classes of modules and of associated preradicals are shown, as well as the relations between preradicals. A similarity with the case of functor $T = U \otimes_S$ - is explained.

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Introduction

The preradicals associated to the functors $H = Hom_R(U, \text{-})$ and $T = U \otimes_S$ - are studied in parts I and II of this paper [1, 2], observing some duality between these cases. Now we will investigate the similar question for the contravariant functor $H' = H_U = Hom_R(\text{-}, U) : R\text{-}Mod \to \mathcal{A}b$, where ${}_RU \in R$-Mod. Preradicals of R-Mod defined by ${}_RU$ and H' are revealed, the properties of these preradicals and the relations between them are specified, the conditions of coincidence of some preradicals are shown. The correlation between the cases of functors T and H' is grounded, which explains the similarity of situations for these types of functors.

For Morita contexts and adjoint situations some facts are proved in [3]. For general theory of radicals and torsions the books [4–7] can be used.

1 Preradicals defined by functor H'

Let ${}_RU$ be an arbitrary left R-module. We consider the contravariant functor

$$H' = H_U = Hom_R(\text{-}, U) : R\text{-}Mod \to \mathcal{A}b.$$

Further, we denote by

$$Cog\,({}_RU) = \{M \in R\text{-}Mod \mid \exists \text{ mono } 0 \to M \xrightarrow{i} U^{(\mathfrak{A})}\}$$

the class of modules of R-Mod, cogenerated by ${}_RU$. The following statement is obvious.

Proposition 1.1. *The class of modules $Cog\,({}_RU)$ is pretorsionfree (i.e. is closed under submodules and direct products), therefore it defines a radical r_U in R-Mod such that $\mathcal{P}(r_U) \stackrel{def}{=\!=} Cog\,({}_RU)$. For every module $M \in$ R-Mod we have:*

$$r_U(M) = \cap\{Ker\, f \mid f : M \to U\}$$

(the reject of U in M). □

For the functor $H' = Hom_R(\text{-}, U)$ we denote:

$$Ker\, H' = \{M \in R\text{-}Mod \mid H'(M) = 0\}.$$

Using the operator of Hom-orthogonality [1] we have:

$$Ker\, H' = \{{}_RU\}^{\uparrow}.$$

Proposition 1.2. *$Ker H$ is a torsionfree class (i.e. it is closed under homomorphic images, direct sums and extensions), thus it defines an idempotent radical $\overline{r}_U$ such that $\mathcal{R}(\overline{r}_U) \stackrel{def}{=\!=} Ker\, H'$ and the respective torsionfree class is:*

$$\mathcal{P}(\overline{r}_U) = (Ker\, H')^{\downarrow} = \{{}_RU\}^{\uparrow\downarrow}.$$ □

Since $\mathcal{P}(\overline{r}_U) = \{{}_RU\}^{\uparrow\downarrow}$ is the least torsionfree class containing ${}_RU$ (or: containing $Cog\,({}_RU) = \mathcal{P}(r_U)$), we obtain

Proposition 1.3. *For every module ${}_RU$ we have $r_U \geq \overline{r}_U$ and $\overline{r}_U$ is the greatest idempotent radical contained in the radical r_U.* □

To establish when the relation $r_U = \overline{r}_U$ is true we need

Definition 1. The module ${}_RU$ will be called *weakly injective* if the functor $H' = Hom_R(\text{-}, U)$ preserves the short exact sequences of the form:

$$0 \to r_U(M) \xrightarrow[\subseteq]{i} M \xrightarrow[\text{nat}]{\pi} M \,/\, r_U(M) \to 0, \quad M \in R\text{-}Mod,$$

i.e. every morphism $f : r_U(M) \to U$ can be extended to a morphism $g : M \to U$ $(gi = f)$:

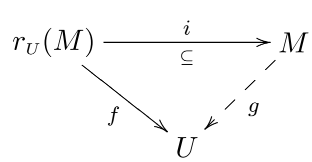

Fig. 1.

Proposition 1.4. *For the module* ${}_RU$ *the following conditions are equivalent*:

1) $r_U = \overline{r}_U$;
2) *radical* r_U *is idempotent*;
3) $Cog\,({}_RU) = (Ker\,H')^{\downarrow}\quad (= \{{}_RU\}^{\uparrow\downarrow})$;
4) ${}_RU$ *is weakly injective.*

Proof. 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3) follow from Proposition 1.3.

2) $\Rightarrow$ 4). If r_U is idempotent, then $r_U\big(r_U(M)\big) = r_U(M)$ for every $M \in R$-Mod, therefore $r_U(M) \in \mathcal{R}(r_U) = \mathcal{R}(\overline{r}_U) = Ker\,H'$. This means that $Hom_R\big(r_U(M), U\big) = 0$, thus ${}_RU$ is weakly projective ($f = 0 \Rightarrow g = 0$).

4) $\Rightarrow$ 2). Let ${}_RU$ be weakly projective module. For any $f : r_U(M) \to U$ by definition there exists such $g : M \to U$ that $g\,i = f$. Now from the definition of $r_U(M)$ it follows $r_U(M) \subseteq Ker\,g$, so $g\,i = 0$ and $f = 0$. Thus $r_U(M) \subseteq Ker\,f$ for every $f : r_U(M) \to U$, i.e. $r_U(M) \subseteq r_U\big(r_U(M)\big)$ and r_U is idempotent. □

The stronger condition on r_U is the requirement that the radical r_U is a torsion. The question when r_U is a torsion was studied earlier, see for example [6, 8]. The necessary and sufficient condition on ${}_RU$ is to be *pseudo-injective*, which is equivalent to the relation $E({}_RU) \in Cog\,({}_RU)$, where $E({}_RU)$ is the injective envelope of ${}_RU$. Now we will indicate another form of this condition.

Definition 2. Module ${}_RU$ is called *upper hereditary* if the class of modules $\{{}_RU\}^{\uparrow}$ is hereditary (i.e. from $Hom_R(M, U) = 0$ it follows $Hom_R(N, U) = 0$ for every submodule $N \subseteq M$).

From the above statements and definitions follows

Proposition 1.5. *For module* ${}_RU$ *the following conditions are equivalent*:

1) *radical* r_U *is a torsion*;
2) $r_U = \overline{r}_U$ *and the class* $Ker\,H' = \mathcal{R}(\overline{r}_U)$ *is hereditary*;
3) $r_U = \overline{r}_U$ *and the class* $Cog\,({}_RU)$ *is stable*;
4) ${}_RU$ *is weakly injective and upper hereditary.* □

If the module ${}_RU$ is injective, then it is obvious that r_U is a torsion.

2 Preradicals defined by the ideal $I = (0 : {}_RU)$ and relations with $(r_U, \overline{r}_U)$

For a fixed module ${}_RU$ we apply the radical r_U to ${}_RR$ and obtain the ideal of R:

$$I = r_U({}_RR) = \cap\{Ker\,f \mid f : {}_RR \to {}_RU\}.$$

From the isomorphism $Hom_R(M, U) \cong {}_RU$ we have that every morphism $f : {}_RR \to {}_RU$ is of the form $f_u : {}_RR \to {}_RU$, where $u \in U$ and $f_u(r) = r\,u$ for every $r \in R$. It is obvious that

$$Ker\,f_u = (0 : u) = \{r \in R \mid r\,u = 0\},$$

therefore

$$I = \bigcap\{Ker\, f \mid f : {}_RR \to {}_RU\} = \bigcap_{u\in U}(0 : u) = (0 : {}_RU),$$

i.e. I is the *annihilator* of module ${}_RU$.

As in the previous cases we consider the classes of modules and preradicals defined in R-Mod by the ideal $I \lhd R$. We denote:

${}_I\mathcal{T} = \{M \in R\text{-}Mod \mid IM = M\}$;

${}_I\mathcal{F} = \{M \in R\text{-}Mod \mid m \in M, Im = 0 \Rightarrow m = 0\}$;

$\mathcal{A}(I) = \{M \in R\text{-}Mod \mid IM = 0\}$;

r^I is the *idempotent radical* defined by ${}_I\mathcal{T} : \mathcal{R}(r^I) \overset{\text{def}}{=\!=} {}_I\mathcal{T}$;

r_I is the *torsion* defined by ${}_I\mathcal{F} : \mathcal{P}(r_I) \overset{\text{def}}{=\!=} {}_I\mathcal{F}$;

$r^{(I)}$ is the *cohereditary radical* defined by $\mathcal{A}(I) : \mathcal{P}(r^{(I)}) \overset{\text{def}}{=\!=} \mathcal{A}(I)$;

$r_{(I)}$ is the *pretorsion* defined by $\mathcal{A}(I) : \mathcal{R}(r_{(I)}) \overset{\text{def}}{=\!=} \mathcal{A}(I)$.

The relations between these classes (and respective preradicals) are indicated in part I [1]. In particular, we have:

${}_I\mathcal{T} = \mathcal{A}(I)^{\uparrow}, \quad {}_I\mathcal{F} = \mathcal{A}(I)^{\downarrow}$;

$r^I \le r^{(I)}$ and r^I is the greatest idempotent radical contained in $r^{(I)}$;

$r_I \ge r_{(I)}$ and r_I is the least idempotent radical containing $r_{(I)}$;

$r^I = r^{(I)} \Leftrightarrow r_I = r_{(I)} \Leftrightarrow I = I^2$.

Further we will study the relations between the classes of modules defined by the ideal $I \lhd R$ and classes associated to preradicals r_U and $\overline{r}_U$.

Proposition 2.1. *$Cog\,({}_RU) \subseteq \mathcal{A}(I)$ (i.e. $\mathcal{P}(r_U) \subseteq \mathcal{P}(r^{(I)})$, so $r_U \ge r^{(I)}$).*

Proof. From the definition of I we have $U \in \mathcal{A}(I)$. Class $Cog\,({}_RU)$ is the least class containing ${}_RU$ and closed under submodules and direct products. Since the class $\mathcal{A}(I)$ also possesses these properties, we have $Cog\,({}_RU) \subseteq \mathcal{A}(I)$. □

Proposition 2.2. $\{{}_RU\}^{\uparrow} = \big(Cog\,({}_RU)\big)^{\uparrow}$.

Proof. ($\supseteq$) From ${}_RU \in Cog\,({}_RU)$ it follows $\{{}_RU\}^{\uparrow} \supseteq \big(Cog\,({}_RU)\big)^{\uparrow}$.

($\subseteq$) Let $M \in \{{}_RU\}^{\uparrow}$, i.e. $Hom_R(M, U) = 0$. If $N \in Cog\,({}_RU)$, then we have a monomorphism $0 \to N \xrightarrow{\varphi} U^{\mathfrak{A}}$, and every non-zero morphism $0 \ne f : M \to N$ leads to non-zero morphism

$$M \xrightarrow{f} N \xrightarrow{\varphi} U^{\mathfrak{A}} \xrightarrow{\pi_\alpha} U_\alpha = U,$$

a contradiction. Thus $Hom_R(M, N) = 0$ for every $N \in Cog\,({}_RU)$, i.e. $M \in \big(Cog\,({}_RU)\big)^{\uparrow}$. □

Proposition 2.3. *${}_I\mathcal{T} \subseteq Ker\, H'$ (i.e. $\mathcal{R}(r^I) \subseteq \mathcal{R}(\overline{r}_U)$, so $r^I \le \overline{r}_U$).*

Proof. Since $Cog\,({}_RU) \subseteq \mathcal{A}(I)$ (Proposition 2.1), we have $\big(Cog\,({}_RU)\big)^{\uparrow} \supseteq \mathcal{A}(I)^{\uparrow}$ and by Proposition 2.2 we obtain:

$${}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} \subseteq \big(Cog\,({}_RU)\big)^{\uparrow} = \{{}_RU\}^{\uparrow} = Ker\,H'. \qquad \square$$

Totalizing we can give a review of relations between the studied classes of modules:

${}_I\mathfrak{T} \subseteq Ker\,H'$, where ${}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} = \mathcal{R}(r^I)$ and

$Ker\,H' = \{{}_RU\}^{\uparrow} = \big(Cog\,({}_RU)\big)^{\uparrow} = \mathcal{R}(\overline{r}_U)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I)$, where $Cog\,({}_RU) = \mathcal{P}(r_U)$ and $\mathcal{A}(I) = \mathcal{R}(r_{(I)}) = \mathcal{P}(r^{(I)})$;

$Cog\,({}_RU) \subseteq \big(Cog\,({}_RU)\big)^{\uparrow\downarrow} = \{{}_RU\}^{\uparrow\downarrow} = \mathcal{P}(\overline{r}_U) \subseteq {}_I\mathfrak{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow} = \mathcal{P}(r^I)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I) \subseteq {}_I\mathfrak{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow} = \mathcal{P}(r^I)$;

${}_I\mathfrak{F} = \mathcal{A}(I)^{\downarrow} = \mathcal{P}(r_I)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I) \subseteq \mathcal{A}(I)^{\downarrow\uparrow} = {}_I\mathfrak{F}^{\uparrow} = \mathcal{R}(r_I)$.

For the corresponding preradicals in particular we have the following situation:

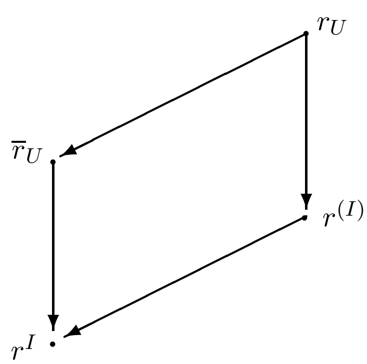

Fig. 2.

where $r_1 \leftarrow r_2$ means $r_1 \leq r_2$. The conditions when $r_U = \overline{r}_U$ or $r^I = r^{(I)}$ are mentioned above. Further we give some remarks on coincidence of other preradicals of Figure 2.

Definition 3. The module ${}_RU$ will be called *Ann-accessible* if from $Hom_R(M,U) = 0$ it follows $Hom_R\,(M,X) = 0$ for every ${}_RX$ with $IX = 0$, where $I = (0 : {}_RU)$.

If ${}_RU$ is Ann-accessible, then $\{{}_RU\}^{\uparrow} \subseteq \mathcal{A}(I)^{\uparrow}$, and the inverse inclusion is always true:

$$\mathcal{A}(I)^{\uparrow} = {}_I\mathfrak{T} \subseteq Ker\,H' = \{{}_RU\}^{\uparrow}.$$

Thus we have ${}_I\mathfrak{T} = Ker\,H'$, i.e. $\mathcal{R}(r^I) = \mathcal{R}(\overline{r}_U)$, which means that $r^I = \overline{r}_U$.

From these considerations follows

Proposition 2.4. *The following conditions are equivalent*:

1) $r^I = \overline{r}_U$;
2) $Ker\, H' = {}_I\mathfrak{T}$;
3) ${}_I\mathfrak{T}^{\downarrow} = \{{}_RU\}^{\uparrow\downarrow}$;
4) ${}_RU$ *is Ann-accessible.* □

The following particular case is worth noting.

Corollary 2.5. *Let ${}_RU$ be a faithful module: $I = (0 : {}_RU) = 0$. The relation $r^I = \overline{r}_U$ is true if and only if ${}_RU$ is a cogenerator of R-Mod.*

Proof. If $I = 0$, then $\mathcal{A}(I) = R\text{-}Mod$, so $\mathcal{A}(I)^{\uparrow} = 0$ and we have ${}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} = \mathfrak{R}(r^I) = 0$, i.e. $r^I = 0$. Thus $r^I = \overline{r}_U$ if and only if $\overline{r}_U = 0$.

($\Rightarrow$) If $r^I = \overline{r}_U$, then $Ker\, H' = {}_I\mathfrak{T} = 0$, so the relation $Hom_R(M, U) = 0$ implies $M = 0$. In particular, for every simple module $P \neq 0$ we have $Hom_R(P, U) \neq 0$. Therefore ${}_RU$ contains isomorphically every simple module, thus ${}_RU$ is a cogenerator of R-Mod.

($\Leftarrow$) If $Cog\,({}_RU) = R\text{-}Mod$, then $Cog\,({}_RU) = \mathcal{A}(I)$ and $\big(Cog\,({}_RU)\big)^{\uparrow} = \mathcal{A}(I)^{\uparrow} = 0$, i.e. $Ker\, H' = {}_I\mathfrak{T} = 0$ and this means that $r^I = \overline{r}_U = 0$. □

The relation $r^{(I)} = r_U$ is true if and only if $\mathcal{P}(r^{(I)}) = \mathcal{P}(r_U)$, i.e. $\mathcal{A}(I) = Cog\,({}_RU)$, what is reduced to the inclusion $\mathcal{A}(I) \subseteq Cog\,({}_RU)$.

The coincidence of all preradicals of Figure 2 is a strong condition can be expressed as follows.

Proposition 2.6. *The following conditions are equivalent*:

1) $r^I = r^U$ (*i.e.* $r^I = \overline{r}_U = r^{(I)} = r_U$;
2) $r^{(I)} = r^U$ *and* $I = I^2$;
3) ${}_RU$ *is Ann-accessible and weakly injective.* □

The general situation on relations between the classes of modules in the case of functor $H' = Hom_R(\text{-}, U)$ is illustrated in Figure 3.

3 Comparing the situations for functors T and H'

Analizing the cases of functors T and H' one can observe an evident resemblance of the obtained situations on classes of modules and associated preradicals. Further we give an explanation of this similarity.

Let U_S be a fixed right S-module which defines the functor

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b$$

and associated preradicals t_U and $\overline{t}_U$ with the respective classes of modules (see Part II, [2]). We will show that all classes of modules and all preradicals constructed

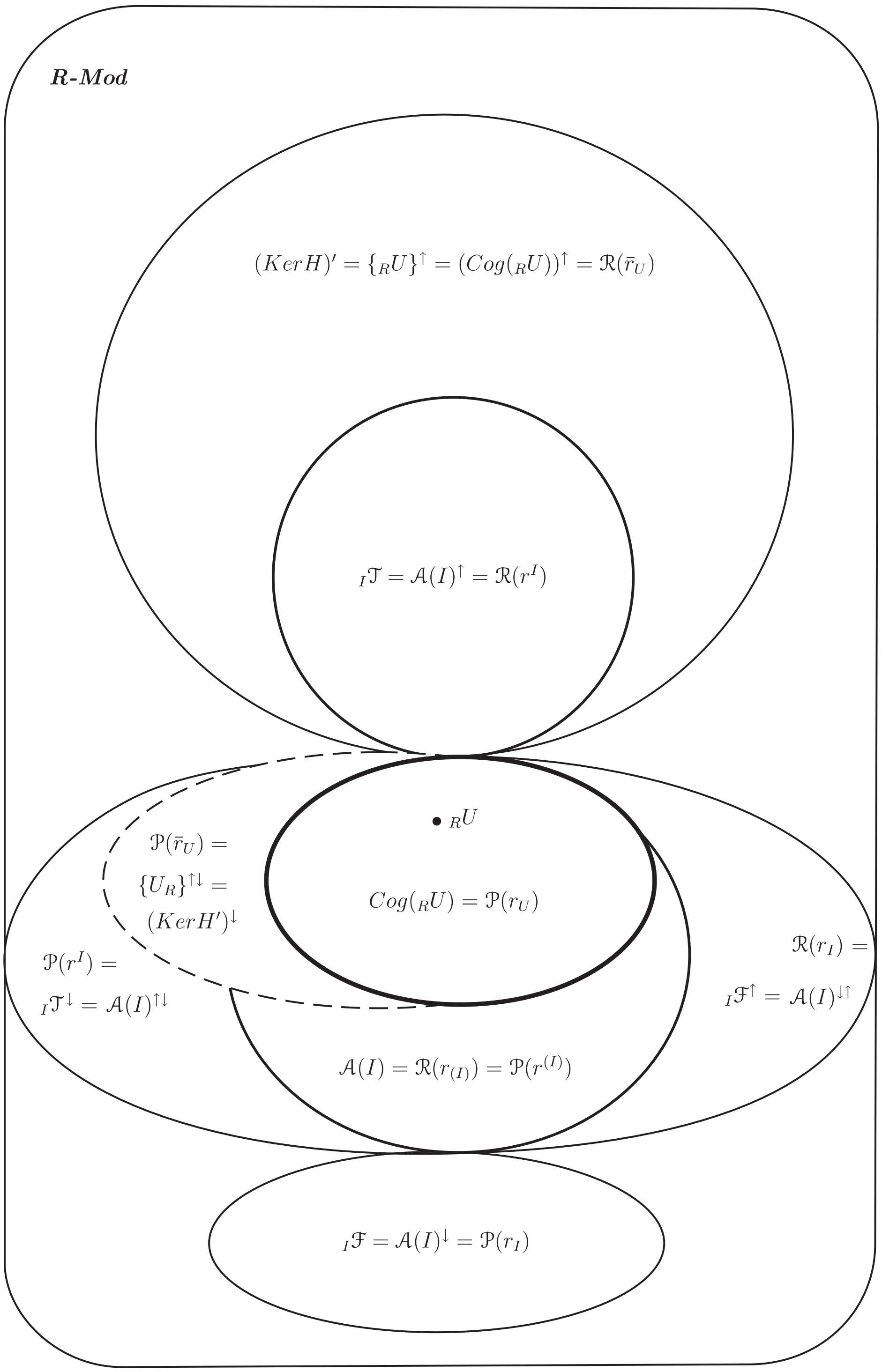
R-Mod
$(KerH)' = \{_RU\}^{\uparrow} = (Cog(_RU))^{\uparrow} = \mathcal{R}(\bar{r}_U)$
$_I\mathcal{T} = \mathcal{A}(I)^{\uparrow} = \mathcal{R}(r^I)$
$\bullet\ _RU$
$\mathcal{P}(\bar{r}_U) =$
$\{U_R\}^{\uparrow\downarrow} =$
$(KerH')^{\downarrow}$
$Cog(_RU) = \mathcal{P}(r_U)$
$\mathcal{R}(r_I) =$
$_I\mathcal{F}^{\uparrow} = \mathcal{A}(I)^{\downarrow\uparrow}$
$\mathcal{P}(r^I) =$
$_I\mathcal{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow}$
$\mathcal{A}(I) = \mathcal{R}(r_{(I)}) = \mathcal{P}(r^{(I)})$
$_I\mathcal{F} = \mathcal{A}(I)^{\downarrow} = \mathcal{P}(r_I)$

Fig. 3.

by U_S in category S-Mod can be obtained with the help of an associated module ${}_SU^*$ by the contravariant functor

$$H' = Hom_S(\text{-}, U^*) : S\text{-}Mod \to \mathcal{A}b$$

as in this part of work.

We fix an arbitrary *cogenerator* C of category $\mathcal{A}b$ of abelian groups (in particular, we can consider that $C = \mathbb{Q}/\mathbb{Z}$). We denote

$${}_SU^* = Hom_{\mathbb{Z}}\left({}_{\mathbb{Z}}U_S, C\right)$$

and consider the contravariant functor

$$H' = Hom_S\left(\text{-}, U^*\right) : S\text{-}Mod \to \mathcal{A}b.$$

The purpose of the following statements is to prove that the functors $T = U \otimes_S$- and $H' = Hom_S(\text{-}, U^*)$ define the same classes of modules, therefore they have the same associated preradicals.

For that we need some preliminary considerations. The fixed module U_S can be regarded as a bimodule ${}_{\mathbb{Z}}U_S$, so it defines the adjoint functors:

$$H = H^U = Hom_{\mathbb{Z}}(U, \text{-}) : \mathcal{A}b \to S\text{-}Mod,$$

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b,$$

(where T is the left adjoint of H), with associated natural transformations $\Phi : TH \to \mathbf{1}_{\mathcal{A}b}$ and $\Psi : \mathbf{1}_{S\text{-}Mod} \to HT$, which satisfy the relations:

$$\Phi_{T(M)} \cdot T(\Psi_M) = 1_{T(M)}, \quad H(\Phi_N) \cdot \Psi_{H(N)} = 1_{H(N)} \tag{1}$$

for every $M \in S\text{-}Mod$ and $N \in \mathcal{A}b$.

In particular, the morphism $\Psi_M : {}_SM \to Hom_{\mathbb{Z}}(U, U \otimes_S M)$ is defined by the rule:

$$[\Psi_M(m)](u) \stackrel{\text{def}}{=\!=} u \otimes_S m, \quad m \in M, \ u \in U.$$

Therefore, for every $M \in S\text{-}Mod$ we have:

$$Ker\, \Psi_M = \{m \in M \mid U \otimes_S m = 0\},$$

and Ψ_M is a monomorphism if and only if $U \otimes_S m = 0$ implies $m = 0$. From the definition of the class $\mathcal{F}(U_S)$ we have

Proposition 3.1. $\mathcal{F}(U_S) = \{M \in S\text{-}Mod \mid \Psi_M$ *is a monomorphism*$\}$. □

This permits us to prove the following essential relation.

Proposition 3.2. $\mathcal{F}(U_S) = Cog({}_SU^*)$.

Proof. ($\subseteq$) Let $M \in \mathcal{F}(U_S)$, i.e. from $U \otimes_S m = 0$ in $U \otimes_S M$ it follows $m = 0$. By Proposition 3.1 Ψ_M is a monomorphism. Since C is a cogenerator of $\mathcal{A}b$ and $U \otimes_S M \in \mathcal{A}b$, there exists a monomorphism of the form:

$$0 \to U \otimes_S M \xrightarrow{i} \prod_{\alpha \in \mathfrak{A}} C_\alpha, \quad C_\alpha = C.$$

Applying the functor $H = Hom_{\mathbb{Z}}(U, \text{-})$, which preserves monomorphisms and direct products, we obtain the exact sequence:

$$0 \to HT({}_S M) \xrightarrow{H(i)} H\Big(\prod_{\alpha \in \mathfrak{A}} C_\alpha\Big) \cong \prod_{\alpha \in \mathfrak{A}} H(C_\alpha) = \prod_{\alpha \in \mathfrak{A}} U^*_\alpha, \; U^*_\alpha = U^*.$$

Combining $H(i)$ with the monomorphism Ψ_M we obtain the monomorphism:

$$M \xrightarrow{\Psi_M} HT({}_S M) \xrightarrow{H(i)} H\Big(\prod_{\alpha \in \mathfrak{A}} C_\alpha\Big) \cong \prod_{\alpha \in \mathfrak{A}} U^*_\alpha,$$

which shows that $M \in Cog\,({}_S U^*)$.

($\supseteq$) Let $M \in Cog\,({}_S U^*)$. Then $r_{U^*}({}_S M) = \cap\{Ker\, f \mid f : M \to U^*\} = 0$. For every morphism $f : {}_S M \to {}_S U^*$ we have the following commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & U^* \overset{\text{def}}{=\!=} H(C) \\
{\scriptstyle \Psi_M}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{H(C)}} \\
HT(M) & \xrightarrow{\quad HT(f) \quad} & HTH(C)
\end{array}$$

Fig. 4.

From the relation $H(\Phi_C) \cdot \Psi_{H(C)} = 1_{H(C)}$ (see (1)) it follows that $\Psi_{H(C)}$ is a monomorphism. If $m \in Ker\, \Psi_M$ then from the diagram it is obvious that $\Psi_{H(C)}\big(f(m)\big) = 0$ and, since $\Psi_{H(C)}$ is a monomorphism, it follows that $f(m) = 0$ for all $f : M \to U^*$. Therefore $m \in \cap\{Ker\, f \mid f : M \to U^*\} = 0$ and $Ker\, \Psi_M = 0$, i.e. $M \in \mathcal{F}(U_S)$ by Proposition 3.1. □

Corollary 3.3. $t_U = r_{U^*}$ *and* $\overline{t}_U = \overline{r}_{U^*}$.

Proof. By definitions $\mathcal{F}(U_S) = \mathcal{P}(t_U)$ and $Cog\,({}_S U^*) = \mathcal{P}(r_{U^*})$, therefore by Proposition 3.2 we have $\mathcal{P}(t_U) = \mathcal{P}(r_{U^*})$, so $t_U = r_{U^*}$. But then the "nearest" idempotent radicals also coincide: $\overline{t}_U = \overline{r}_{U^*}$. □

From the above results it follows that all constructions effected in S-Mod by the module U_S and the functor $T = U \otimes_S$ - coincide with the respective constructions by the module ${}_S U^*$ and the functor $H' = Hom_S(\text{-}, U^*)$. For example, the following classes of S-Mod coincide:

$$\mathcal{F}(U_S) = Cog\,({}_S U^*), \quad Ker\, T = Ker\, H', \quad \mathcal{A}(J) = \mathcal{A}(I),$$

$$(Ker\,T)^{\downarrow} = \{_S U^*\}^{\uparrow\downarrow}, \quad _J\mathfrak{T} = {_I\mathfrak{T}}, \quad _J\mathfrak{F} = {_I\mathfrak{F}}, \quad \text{etc.}$$

These facts completely explain the similarity of the situations for the functors T and H'.

From the conditions of coincidence of "near" preradicals ($t_U = \overline{t}_U$, Part II, Proposition 1.6; $r_U = \overline{r}_U$, Part III, Proposition 1.4) now follows

Corollary 3.4. *U_S is a weakly flat module if and only if $_SU^*$ is weakly injective.* □

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