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About rings of continuous functions in the expanded field of numbers

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Abstract. In the present article the generalized rings $C_\infty(X)$ of all continuous functions on the expanded straight line are studied. The conditions under which $C_\infty(X)$ is a ring or a linear space are determined.

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Spaces of continuous maps on the expanded straight line play a leading role in the theory of topological semifields. In works [1–5] some applications of such maps have been specified. All spaces are assumed to be Tychonoff. Terminology is as in [7]. By $[A]$ or $[A]_X$ we denote the closure of a set A in a space X, $|Y|$ is the cardinality of a set Y, βX is the Stone-Čech compactification of the space X, on $N = \{0, 1, ...\}$ we consider only the discrete topology.

Let R be the field of real or complex numbers. By R_∞ we denote the one-point compactification of space R and $R_\infty = R \cup \{\infty\}$. We consider that $\infty + \infty = b + \infty = \infty$, $0 \cdot \infty = 0$, $\infty \cdot \infty = \infty$, $c \cdot \infty = \infty \cdot c = \infty$ for all $b \in R$, $c \in R \setminus \{0\}$.

Let $C_\infty(X)$ be the family of all continuous maps of space X in R_∞ in the topology of pointwise convergence and such that the set $H(f) = f^{-1}(\infty)$ is nowhere dense in X for all $f \in C_\infty(X)$. We suppose that $C(X) = \{f \in C_\infty(X) : H(f) = \emptyset\}$.

Let f, $g \in C_\infty(X)$. We say that the sum $f + g$ is defined if there exits a function $h \in C_\infty(X)$ such that $h(x) = f(x) + g(x)$ for all $x \in X \setminus (H(f) \cup H(g))$. The product $f \cdot g$ is defined if there exists a function $h \in C_\infty(X)$ such that $h(x) = f(x) \cdot g(x)$ for all $x \in X \setminus (H(f) \cup H(g))$.

For a map $\psi : C_\infty(X) \to C_\infty(Y)$ we consider the conditions:

a) if f, $g \in C_\infty(X)$, then the sum $f + g$ exists if and only if the sum $\psi(f) + \psi(g)$ exists and then $\psi(f + g) = \psi(f) + \psi(g)$;

b) $\psi(b \cdot f) = b \cdot y(f)$ for any $b \in R$, $f \in C_\infty(X)$;

c) if f, $g \in C_\infty(X)$, then the product $f \cdot g$ exists if and only if $\psi(f) \cdot \psi(g)$ exists and the product $\psi(f \cdot g) = \psi(f) \cdot \psi(g)$.

A one-to-one map $\psi : C_\infty(X) \to C_\infty(Y)$ is called

– *additive* if the condition a) is satisfied;

– *linear* if the conditions a) and b) are satisfied;

– *multiplicative* if the condition b) is satisfied;

– *an isomorphism* if the conditions a), b) and c) are satisfied.

Theorem 1. *If* $\psi : C_\infty(X) \to C_\infty(Y)$ *is a linear homeomorphism, then* $\psi(C(X)) = C(Y)$.

Proof. Let $f \in C_\infty(X)$. We put $f_n = 2^{-n} \cdot f$ and $O_X(x) = 0$ for all $x \in X$. The limit $\lim\limits_{n\to\infty} f_n = O_X$ exists in X only if $f \in C(X)$. If the limit $\lim\limits_{n\to\infty} f_n = O_X$ exists, then the limit $\lim\limits_{n\to\infty} \psi(f_n) = 2^{-n} \cdot \psi(f) = O_Y$ exists, too. Therefore, if $f \in C(X)$, then $\psi(f) \in C(Y)$ and $\psi(C(X)) \subseteq C(Y)$. Since ψ^{-1} is a linear homeomorphism, we have $\psi(C(X)) = C(Y)$. □

Theorem 2. *If* $\varphi : C_\infty C(X) \to C_\infty(Y)$ *is an additive and multiplicative homeomorphism and* R *is the field of real numbers, then* φ *is an isomorphism and* $\varphi(C(X)) = C(Y)$.

Proof. We have $f = 1_X$ only if $g \cdot f = g$ for all $g \in C_\infty(X)$. Therefore $\varphi(1_X) = 1_Y$. Let $\lambda_X = \lambda \cdot 1_X$ for any $\lambda \in R$. Then $\varphi(n_X) = n_Y$ and $\varphi((1/n)_X) = (1/n)_Y$ for all $n \in N$ and $n \geq 1$. Hence $\varphi(\lambda_X) = \lambda_Y$ for all rational numbers $\lambda \in R$. Theorem 1 complete the proof. □

Theorem 1 implies

Corollary 1. *If the homeomorphism* $\varphi : C_\infty(X) \to C_\infty(Y)$ *is an isomorphism, then the spaces* X *and* Y *are homeomorphic.*

A space X is called χ-sequential if for every nowhere dense closed set F there exist a point $x_0 \in F$ and a sequence $\{x_n \in X \backslash F : n = 1, 2, ...\}$ for which $x_0 = \lim x_n$. Each sequential space is χ-sequential.

The product of any number of metrizable compact spaces is χ-sequential. Let $X = \prod\{X_a : a \in A\}$, where $\{X_a : a \in A\}$ is a set of metrizable compact spaces. We fix nowhere dense in X set F and a point $x = \{x_a : a \in A\} \in F$. Let $Y = \{y = \{y_a : a \in A\} : |a : x_a \neq y_a\}| \leq \chi_0\}$. Then $x \in Y \cap F$ and Y is dense in X. The space Y is sequential. Therefore there is a sequence $\{x_n \in Y \setminus F\}$, converging to x.

Proposition 1. *If* $f \in C(X)$, *then* $f + g$ *exists for* $g \in C_\infty(X)$ *and the maps* $u : R_\infty \to R_\infty$ *and* $v : C(X) \times C_\infty(X) \to C_\infty(X)$, *where* $u(x, y) = x + y$ *and* $v(f, g) = f + g$, *are continuous.*

Proof. It is enough to prove the continuity of the map $u(x, y) = x + y$. If $x, y \in R$, then the function u is continuous at a point (x, y). Let $x_0 \in R$. For ∞ we consider the neighborhoods $U(n,\infty) = R_\infty \setminus \{x \in R : |x| \leq n\}$. Let $O_{x_0} = \{x \in X : |x - x_0| < 1$ and $m > n + |x_0| + 1\}$. Since $|x + y| \geq |x| - |y|$ we have $Ox_0 + U(m,\infty) \subset U(n,\infty)$. Therefore the map u is continuous. The assertion is proved. □

Proposition 2. *Let* X *be a* χ*-sequential space. Then for any function* $f \in C_\infty(X) \setminus C(X)$ *there exists such a function* $g \in C_\infty(X)$ *that the sum* $f + g$ *is not defined.*

Proof. We have $H(f) \neq \emptyset$. Then there exist a point $x_0 \in H(f)$ and a sequence $\{x_n \in X \setminus H(f) : n = 1,\ 2, ...\}$ such that $\lim x_n = x_0$, $|f(x_1)| \geq 1$ and

$|f(x_{n+1})| > |f(x_n)| + 4$. Let $U_n = \{x \in R : |x - f(x_n)| < 1\}$. Then the system $\{f^{-1}U_n : n = 1, 2, ...\}$ is open and locally finite at $x \in X \setminus H(f)$. For any $n = 1, 2, ...$ we fix a continuous function $g_n : X \to [0; 1] \subset R$, where $g_n(x_n) = 1$ and $X \setminus f^{-1}U_n \subset g_n^{-1}(0)$. Let $g = -f + \sum \{(-1)^n \cdot g_n : n = 1, 2, ...\}$. Then $H(g) = H(f)$ and $|f(x) + g(x)| \leq 1$ for any $x \in X \setminus H(f)$. By construction, $f(x_{2n}) + g(x_{2n}) = 1$ and $f(x_{2n+1}) + g(x_{2n+1}) = -1$. Therefore the limit $\lim(f(x_n) + g(x_n))$ does not exist, therefore, the sum $f + g$ does not exist. □

Corollary 2. *Let X and Y be χ-sequential spaces and $\psi : C_\infty(X) \to C_\infty(Y)$ is a one-to-one additive map. Then $\psi/(C(X)) = C(Y)$.*

Proof. By virtue of Proposition 1, $f \in C(X)$ if and only if the sum $f + g$ is defined for any $g \in C_\infty(X)$. This fact follows from Proposition 2. Therefore the conditions and $\psi(f) \in C(Y)$ are equivalent. □

Proposition 3. *Let X be a χ-sequential space. Then for each function $f \in C_\infty(X) \setminus C(X)$ there exists such a function $g \in C_\infty(X)$ that the product $f \cdot g$ is not defined.*

Proof. We have that $H(f) \neq \emptyset$. We choose a point $x_0 \in H(f)$ and a sequence $\{x_n \in X \setminus H(f) : n = 1, 2, ...\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and $|f(x_{n+1})| > |f(x_n)| + 4 > 4 + 2^{2n}$. Let $U_n = \{t \in R : |t - f(x_n)| < 1\}$. For any $n \in N$ we fix a continuous function $h_n : X \to [0; 1]$ such that $h_n(x_n) = 1$ and $X \setminus f^{-1}U_n \subset h_n^{-1}(0)$. Let $g_{2n} = 2^{-2n} \cdot h_{2n}$, and $g_{2n-1} = (f(x_{2n-1}))^{-1} \cdot h_{2n-1}$ for all $n = 1, 2, ...$ The function $g = \sum \{g_n \,|\, n = 1, 2, ...\}$ is continuous on X and $g \in C(X)$. We will prove that $f \cdot g$ does not exist. We notice that $|f(x_{2n}) \cdot g(x_{2n})| = 2^{-2n} \cdot |f(x_{2n})| > 2^{-2n} \cdot 2^{4n} = 2^{2n}$ and $|f(x_{2n-1}) \cdot g(x_{2n-1})| = |f(x_{2n-1})| \cdot |f(x_{2n-1})| = 1$. Then $\lim\limits_{n\to\infty} f(x) \cdot g(x_n)$ does not exist. The assertion is proved. □

Proposition 4. *For each space X there is a unique operator of extension $w : C_\infty(X) \to C_\infty(\beta X)$ which is linear, multiplicative and regular, i. e. $\|\omega(f)\| = \|f\|$ for all $f \in C_\infty(X)$.*

Proof. The space R_∞ is compact. Therefore for each continuous map $f : X \to R_\infty$ there exists a unique continuous map $w(f) : \beta X \to R_\infty$ such that $f = w(f)|X$. If the function is bounded, then the function $w(f)$ also is bounded and $\|\omega(f)\| = \|f\|$. Let $f, g \in C_\infty(X)$. If $\varphi = f + g$, then $w(\varphi) = w(f) + w(g)$. If $\varphi = f \cdot g$ then $w(\varphi) = w(f) \cdot w(g)$. Since $\omega(\lambda f) = \lambda \cdot \omega(f)$, the proof is complete. □

A set $H \subset X$ is functionally closed if $f^{-1}(0) = H$ for some function $f \in C(X)$. The complement to functionally closed sets are called the functionally open sets.

A space X is χ-normal if the set $[U]$ is functionally closed for any open in X set U.

A space X is extremely disconnected if the closure $[U]$ is open for any open set U.

Proposition 5. *Let X be an extremally disconnected space. Then:*

1) there exists a regular, linear and multiplicative extension operator $w : C_\infty(X) \to C_\infty(\beta X)$;

2) for any two functions $f, g \in C_\infty(X)$ the sum $f + g$ and the product $f \cdot g$ are defined;

3) $C_\infty(X)$ is a ring and a vector space.

Proof. Let $f, g \in C_\infty(X)$ and $Y = X \setminus (H(f) \cup H(g))$. Then the set Y is open in X. Let U be an open in Y set. Then the set U is open in X and the set $[U]_{\beta X}$ is open in βX. We have $\beta Y = \beta X$. Let $f_1 = f|Y,\ g_1 = g|Y$. Then $f_1,\ f_1 + g_1,\ f_1 \cdot g_1 \in C(Y)$ and by virtue of Proposition 4, there exist continuous extensions $w(f)$, $w(f + g)$, $w(f \cdot g)$ on βX. The proof is complete. □

Lemma 1. *Let U and V be open subsets of a space X, $U \cap V \neq \emptyset$, $[U] \cup [V] = X$ and $F = [U] \cap [V]$ is a non-empty functionally closed set. Then there exist such functions $f, g \in C_\infty(X)$ and $h \in C(X)$ that the sum $f + g$ and the product $f \cdot h$ do not exist.*

Proof. Clearly, $[U]$ and $[V]$ are functionally closed sets. Therefore there exist such continuous functions $\varphi_1, \varphi_2 : X \to [0; 1]$ that $\varphi_1^{-1}(0) = [U]$ and $\varphi_2^{-1}(0) = [V]$. We suppose that $\varphi = \varphi_1 + \varphi_2$ and $h = \varphi_1 - \varphi_2$. Then $\varphi^{-1}(0) = h^{-1}(0) = F$, the map $f = 1/\varphi : X \to R_\infty$ is continuous and $H(f) = F$.

The product $f \cdot h$ does not exist, since $(f \cdot g)(x) = 1$ if $x \in V$, and $(f \cdot g)(x) = -1$ if $x \in U$. The map $g : X \to R_\infty$, where $g(x) = 1 - f(x)$ if $x \in [U]$, and $g(x) = -1 - f(x)$ if $x \in V$, is continuous. The sum $f + g$ does not exist, since $(f + g)(x) = 1$ if $x \in U$, and $(f + g)(x) = -1$ if $x \in V$. The proof is complete. □

Proposition 6. *For a χ-normal space X the following statements are equivalent:*

– *the space X is extremally disconnected;*

– *for any functions $f, g \in C_\infty$ there exists the sum $f + g$;*

– *for any functions $f, g \in C_\infty$ there exists the product $f \cdot g$.*

Proof. Implications $1 \to 2$ and $1 \to 3$ follow from Proposition 5. Suppose that the space X is not extremally disconnected. Then there exists an open in X set U such that the set $[U]$ is not open. We put $V = X \setminus [U]$. We can consider that $U = X \setminus [V]$. Then $F = [U] \cap [V]$ is a nonempty functionally closed set. Therefore the implications $2 \to 1$ follow from Lemma 1. The proof is complete. □

Example 1. We consider the discrete sum $X = Y \oplus \beta N$, where Y is an infinite metrizable compact space. The space X is χ-normal and compact. However, the space X is not extremally disconnected. Therefore not for all pairs of functions $f, g \in C_\infty(X)$ the functions $f + g$ or $f \cdot g$ are defined. If $f \in C_\infty(X)$ and on $Y \subset X$ the function f is bounded, then the sum $f + g$ and the product $f \cdot g$ exist for all $g \in C_\infty(X)$. This fact follows from Proposition 5. If the function is not bounded on Y, i.e. $H(f) \cap Y \neq \emptyset$, then the sum $f + g$ and product $f \cdot \varphi$ are not defined for some $g, \varphi \in C_\infty(X)$. Therefore $C_\infty(X)$ is not a ring.

Lemma 2. *Let $g \in C(X)$ and the set $g^{-1}(0)$ is open. Then the product $g \cdot f$ exists for all $f \in C_\infty(X)$.*

Proof. The set $U = g^{-1}(0)$ is open-and-closed in X. Let $f \in C_\infty(X)$. If $H(f) = \emptyset$, then the assertion is obvious. We suppose that the set $H(f)$ is not empty. We put $h(x) = 0$ if $x \in U$ and $h(x) = g(x) \cdot f(x)$ if $x \in X \setminus U$. The function h is continuous at all points $x \in X$ for which $h(x) \neq \infty$. Let $x_0 \in X \setminus U$ and $h(x_0) = \infty$. Then $|g(x_0)| > 1/m$ for some $m \in N$. We fix $n \in N$. There exists a neighborhood Ox_0 of the point x_0 in X such that $|g(x)| > 1/m$ and $|f(x)| > nm$ for all $x \in Ox_0$. Then $|h(x)| > n$ for all $x \in Ox_0$. Therefore the function h is continuous at the point x_0 and $h = f \cdot g \in C_\infty(X)$. The proof is complete. □

Lemma 3. *Let X be a χ-normal χ-sequential space, $f \in C_\infty(X)$ and the set $f^{-1}(0)$ is not open in X. Then there exists a function $g \in C_\infty(X)$ such that the product $f \cdot g$ is not defined.*

Proof. As the set $F = f^{-1}(0)$ is not open. Then there exist a point $x_0 \in F$ and a sequence $\{x_n \in X : n = 1, 2, ...\}$ such that $\lim x_n = x_0$ and $0 < |f(x_n)| < 2^{-n}$ for all n. The set $P = F \cap [X \setminus F]$ is nowhere dense, functionally closed and $x_0 \in P$. There exists a continuous function $h \in C_\infty(X)$ such that $P = h^{-1}(0)$, $h(x_{2n}) = f(x_{2n})$ and $h(x_{2n+1}) = 2^{-1} f(x_{2n+1})$. Then $g = 1/h \in C_\infty(X)$, $g(x_{2n}) \cdot f(x_{2n}) = 1$ and $g(x_{2n+1}) \cdot f(x_{2n+1}) = 2$. The lemma is proved. □

Lemma 4. *Let $f \in C_\infty(X)$. The function $1/f$ exists if and only if the set $H(f) \cup f^{-1}(0)$ is nowhere dense.*

Proof. It is obvious. □

Theorem 3. *Let $\varphi : C_\infty(X) \to C_\infty(E)$ be a multiplicative homeomorphism with the property: if $f \in C_\infty(X)$ and $H(f) = f^{-1}(0) = \emptyset$, then $H(\varphi(f)) = \emptyset$. Then:*

1) if $f \in C(X)$ and $|f(x)| < 1$ for all $x \in X$, then $|\varphi(f)(y)| < 1$ for all $y \in Y$;

2) $\varphi(C(X)) \subseteq C(E)$.

Proof. The condition $|f(x)| < 1$ for all $x \in X$ is equivalent to $\lim f^n = 0_X$. The statement 1 of Theorem 3 is proved. Let $f \in C(X)$. We put $h(x) = 2 + |f(x)|$ and $g = 1/h$. Then $f_1 = f \cdot g \in C(X)$ and $|f(x)| < 1$ for all $x \in X$. By construction, $f = h \cdot f_1$ and $H(f_1) = f_1^{-1}(0) = \emptyset$. Considering that $\varphi(f_1), \varphi(h) \in C(Y)$ we receive $\varphi(f) = \varphi(f_1 \cdot h) = \varphi(f_1)\varphi(h) \in C(Y)$. The proof is complete. □

Corollary 3. *If $\varphi : C_\infty(X) \to C_\infty(Y)$ is a multiplicative homeomorphism and R is the field of real numbers, then:*

1) if $f \in C(X)$ and the set $f^{-1}(0)$ is open, then $g = \varphi(f) \in C(Y)$ and the set $g^{-1}(0)$ is open;

2) if $f \in C(X)$ and $f^{-1}(0) = \emptyset$, then $g = \varphi(f) \in C(Y)$ and $g^{-1}(0) = \emptyset$;

3) if $f \in C_\infty(X)$ and $g = 1/f \in C_\infty(X)$, then $\varphi(g) = 1/\varphi(f)$; $\varphi(1_X) = 1_Y$ and $\varphi(0_X) = 0_Y$;

4) if $|f(x)| = 1_X$, then $|\varphi(f)| = 1_Y$;

5) if $|f(x)| = 1_X$, then $|\varphi(f)| = 1_Y$;

6) if $f \geq 0$, then $\varphi(f) \geq 0$;

7) if $|f(x)| < 1$ for all $x \in X$, then $|\varphi(f)(y)| < 1$ for all $y \in Y$;

8) $\varphi(C(X)) = C(Y)$.

Proof. If $f \cdot g = f$ and $f \cdot h = h$ for all $f \in C_\infty(X)$, then $g = 1_X$ and $h = 0_X$. Therefore $\varphi(1_X) = 1_Y$ and $\varphi(0_X) = 0_Y$. The statement 4 of Corollary 3 is proved. The condition $|f| = 1_X$ is equivalent to the condition $f \cdot f = 1_X$. That proves the statement 5. The statement 3 is obvious. The condition $f \geq 0$ is equivalent to $f = g \cdot g$ and $g = (f)^{1/2}$. The statement 6 is proved. □

Let $f \in C_\infty(X)$. There exist such functions $g_n \in C_\infty(X)$ for which $(g_n)^n = f \cdot f$. The limit $\lim g_n$ exists in the pointwise convergensce topology if and only if $H(f) = H(g_n) = \emptyset$ and the set $f^{-1}(0) = g_n^{-1}(0)$ is open. Considering that $(\varphi(g_n))^n = \varphi(f)^2$ and $\lim \varphi(g_n) = \varphi(\lim g_n)$, we finish the proof of the statement 1. The statement 2 follows from the statements 1 and 3 and Lemma 4. The statements 7 and 8 follow from Theorem 3.

Proposition 7. *For a χ-normal χ-sequential space X the following statements are equivalent:*

1) $C(X) = C_\infty(X)$;

2) the space X is discrete.

Proof. Implication $2 \to 1$ is obvious. Assume that the space X is not discrete. Then there exists a non-isolated point $x_0 \in X$ and a sequence $\{x_n \in X \setminus \{x_0\} : n \in N\}$ such that $\lim x_n = x_0$. There exists two open in X sets U and V for which $U \cap V = \emptyset$, $\{x_{2n} : n \in N\} \subset U$ and $\{x_{2n+1} : n \in N\} \in V$. We put $F = [U]$ and $\Phi = [X \setminus F]$. Then $x_0 \in F \cap \Phi = H$, the set H is nowhere dense and there exists a continuous function $f : X \to [0;1]$ such that $H = f^{-1}(0)$. Then $g = 1/\varphi \in C_\infty(X) \setminus C(X)$. Implication $1 \to 2$ is proved. □

Example 2. Let $X = Y \cup \{b\}$ be the one-point compactification of the discrete space Y of uncountable cardinality. The neighborhoods of the point b have the form $O_b = X \setminus F$, where F is a finite subset of the set Y. The space X is χ-sequential, since X is a Frechet-Urysohn space. We will prove that $C(X) = C_\infty(X)$. Let $f \in C_\infty(X)$. Then $H(f) \subset \{b\}$. If $H(f) = \emptyset$, then $f \in C(X)$. Let $H(f) \neq \emptyset$. Then $H(f) = \{b\} = \cap\{f^{-1}((-\infty; -n) \cup (n; +\infty)) : n = 1, 2, ...\}$. This means that $H(f) = \{b\}$ is a G_δ-set. Then there exists a sequence of finite sets $F_n \subset Y : n = 1, 2, ...\}$ such that $X \setminus F_n \subset f^{-1}((-\infty; -n) \cup (n; +\infty))$, i.e. $\{b\} = \cap\{X \setminus F_n : n = 1, 2, \ldots\}$. Hence, $Y = \cup\{F_n : n = 1, 2, ...\}$, and the set Y is countable, a contradiction. Therefore $H(f) = \emptyset$.

A space X is called a P^*-space if for any monotone decreasing sequence $\{U_n : n \in N\}$ of open sets either $\cap\{U_n : n \in N\} = \emptyset$, or there exists a non-empty open set U such that $U \subset \{U_n : n \in N\}$. The space X from Example 2 is a P^*-space. The concepts of χ-normal spaces and of P^*-spaces are opposite. Only discrete spaces are simultaneously χ-normal and P^*-spaces.

Lemma 5. *If X is a P^*-space, then $C(X) = C_\infty(X)$.*

Proof. Let's suppose that there exists a function $f \in C_\infty(X)$. We put $U_n = f^{-1}([-\infty, n] \cap [n, +\infty])$ for all $n \in N$. Then $\cap\{U_n : n \in N\} = H(f)$. Let $H(f) \neq \emptyset$. There exists an open nonempty set U such that $U \subset \cap\{U_n : n \in N\} = H(f)$. Then the set $H(f)$ is not anywhere dense. Therefore $H(f) = \emptyset$ and $f \in C(X)$. The lemma is proved. $\square$

Lemma 6. *Suppose that in a space X there exists a sequence $\{U_n : n \in N\}$ of open in X sets such that $H = \cap\{U_n : n \in N\} \neq \emptyset$ and for any non-empty open in X set U we have $U \setminus H \neq \emptyset$. Then $C_\infty(X) \neq C(X)$.*

Proof. We fix $x_0 \in H$. We build such continuous functions $f_n : X \to [0;1]$ for which $f_n(x_0) = 0$ and $f_n^{-1}(0) \subset U_n$. By construction, the function $f = \sum(2^{-n} \cdot f_n : n \in N)$ is continuous, $f(x_0) = 0$ and $f^{-1}(0) \subset H$. Thus the set $f^{-1}(0)$ is nowhere dense and it is not closed. We put $g = 1/f : X \to R_\infty$. Then $H(g) = f^{-1}(0)$, $x_0 \in H(g)$ and $g \in C_\infty(X) \setminus C(X)$. The lemma is proved. $\square$

Corollary 4. *For a space X the following statements are equivalent:*
1) X is a P^*-space;
2) $C_\infty(X) = C(X)$.

Example 3. Let Y be an infinite compact space, being P^*-space, and $Z = \beta N$. Then $X = Y \oplus Z$ is a compact space, the space X is not extremally disconnented, $C_\infty(X) \neq C(X)$ and $C_\infty(X)$ is a ring.

The space X is pseudocompact if all continuous real-valued functions are bounded on X.

Theorem 4. *Let X be a P^*-space. The following statements are equivalent:*
1) X is pseudocompact;
2) βX is a P^*-space;
3) $C(\beta X) = C_\infty(\beta X)$.

Proof. Implications $2 \to 3 \to 2$ are obvious. If the space X is not pseudocompact, there exists an unbounded function $f \in C(X)$. By virtue of the proposition from [6], there exists such a continuous map $g : \beta X \to R_\infty$ for which $f = g|X$. Clearly, $H(g) \neq \emptyset$. It proves the implication $2 \to 1$. Let the space X be pseudocompact. We consider a sequence $\{U_n : n \in N\}$ of open in βX sets such that $L = \cap\{U_n : n \in N\} \neq \emptyset$. We can consider that $[U_{n+1}] \subseteq U_n$. Then the set L is functionally closed. If $L \cap X = \emptyset$, then on X there exists some unbounded continuous function and X is not pseudocompact.

Therefore there exists such an open in βX set W for which $\emptyset \neq V = W \cap X \subseteq L$. By construction, $\emptyset \neq W \subseteq L$. Implication $1 \to 2$ is proved. The proof is finished.$\square$

Example 4. Let X be not a pseudocompact P^*-space. Then the map $\varphi : C_\infty(\beta X) \to C_\infty(X) = C(X)$ satisfies the following conditions:
1) φ is a continuous isomorphism;
2) φ is not a homeomorphism;
3) $\varphi(C(\beta X)) \neq C(X)$.

Example 5. Let $X = \beta Y$, where Y be an infinite discrete space. Then there exists a function $h \in C_\infty(X) \setminus C(X)$ such that $h^{-1}(0) = \emptyset$ and the mapping $\varphi : C_\infty(X) \to C_\infty(X)$, where $\varphi(f) = f \cdot h$, satisfies the following conditions:

1) is one-to-one;
2) is linear;
3) $C(x) \cap \varphi(C(X)) = \emptyset$.

From Examples 4 and 5 it follows that the condition that φ is a homeomorphism is essential in the conditions of Theorem 1: if $\varphi : C_\infty(X) \to C_\infty(X)$ is a linear homeomorphism, then $\varphi(C(X)) = C(Y)$.

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