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About characteristics of graded algebras $S_{1,4}$ and $SI_{1,4}$

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Abstract. Hilbert series for the graded algebras of comitants $S_{1,4}$ and invariants $SI_{1,4}$ of differential system are constructed and with their help the Krull dimensions of these algebras are determined. The lower bounds for the number of the types of generators for the algebras $S_{1,4}$ and $SI_{1,4}$ are obtained

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1 Introduction

We consider differential system of the form

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \ (j, \alpha, \beta, \gamma, \delta = 1, 2), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma\delta}$ is symmetrical in lower indices in which the complete convolution holds.

In [1-7] the authors presented different methods of the study of the set of centro-affine invariants and comitants of the system (1), which later find an application for the qualitative study of these systems.

One of these methods is the method of generating functions and Hilbert series described in [5-7]. The method goes back to classical works [8-16] for invariants of binary forms, takes further investigation in works [17-28] for graded algebras of invariants of indicated forms and also for graded algebras of different abstract objects.

From [5-6] it is known that in order to construct a minimal polynomial base of invariants and comitants [3,4] of the system (1) it is enough to construct generators of algebra of unimodular comitants and invariants of indicated system.

Consider the system (1) with the group of unimodular transformations $SL(2, \mathbb{R})$, it is supposed this group acts in a natural way in $E^{16}(x, a)$, where $x = (x^1, x^2)$ is a vector of phase variables, and a is the set of coefficients of the system (1).

Let $\mathbb{R}[E^{16}(x, a)]$ be the algebra of polynomials on $E^{16}(x, a)$. The group $SL(2, \mathbb{R})$ acts also in $\mathbb{R}[E^{16}(x, a)]$.

Let $S_{1,4}$ be subalgebra of polynomials, depending only on x, a from $\mathbb{R}[E^{16}(x, a)]$, and it is formed from $SL(2, \mathbb{R})$ comitants [5-6] of the system (1).

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Following [5-6], we shall name $S_{1,4}$ the algebra of comitants, and its subalgebra $SI_{1,4}$ of polynomials depending only on x, will be called the algebra of invariants.

Let $\mathbb{R}[E^{16}(x,a)]^{(d)}$ be the set of polynomials of the type $(d) = (\delta, d_1, d_2)$, homogeneous of degree δ in variables $x = (x^1, x^2)$, of degree d_1 in coefficient tensor a^j_α, of degree d_2 in coefficient tensor $a^j_{\alpha\beta\gamma\delta}$. Let us assume

$$S^{(d)}_{1,4} = S_{1,4} \bigcap \mathbb{R}[E^{16}(x,a)]^{(d)}.$$

The algebra $S_{1,4}$ is graded through $S^{(d)}_{1,4}$ and

$$S_{1,4} = \bigoplus_{(d)} S^{(d)}_{1,4} \ (d \geq 0), \ S^{(d)}_{1,4} S^{(e)}_{1,4} \subseteq S^{(d+e)}_{1,4}$$

is considered finitely determined [5-6] for $S^{(0)}_{1,4} = \mathbb{R}$.

It is known [1-6], that $\mathbb{R}$-algebra $S_{1,4}$ is locally finite, i.e. $dim_{\mathbb{R}} S^{(d)}_{1,4} < \infty$ for all (d).

The arising here sequence is $\{dim_{\mathbb{R}} S^{(d)}_{1,4}\}$, and the corresponding generalized Hilbert series [5-6] is

$$H(S_{1,4}, u, b, e) = \sum_{(d)} dim_{\mathbb{R}} S^{(d)}_{1,4} u^{\delta} b^{d_1} e^{d_2}, \tag{2}$$

where $dim_{\mathbb{R}} S^{(0)}_{1,4} = 1$, and $(d) = (\delta, d_1, d_2)$.

The common Hilbert series is obtained from the generalized one as follows

$$H_{S_{1,4}}(u) = H(S_{1,4}, u, u, u). \tag{3}$$

Remark 1. The generalized (common) Hilbert series for the graded algebra of invariants $SI_{1,4} \subset S_{1,4}$ of the system (1) is formally obtained from (2) for $u = 0$ $(b = e = z)$.

Remark 2. Following [19], we remark that the transcendent degree over $\mathbb{R}$ of the field of quotients of algebra $S_{1,4}$ $(SI_{1,4})$ is called its dimension of Krull $\varrho(S_{1,4})$ $(\widetilde{\varrho}(SI_{1,4}))$. This dimension is equal to the maximum number of algebraically independent homogeneous elements in $S_{1,4}$ $(SI_{1,4})$, and also to the order of the pole of common Hilbert series at the unit.

2 Hilbert series and Krull dimensions for algebras $S_{1,4}$ and $SI_{1,4}$

We determine the lower bounds for the number and the totality of types of generators for algebras $S_{1,4}$ and $SI_{1,4}$. For that we construct Hilbert series for algebras $S_{1,4}$ and $SI_{1,4}$ for the system (1).

Following [5-6] we obtain that $dim_{\mathbb{R}} S_{1,4}^{(d)}$ is equal to the coefficient of $u^{\delta} b^{d_1} e^{d_2}$ in the expansion of initial function

$$\varphi_{1,4}^{(0)}(u) = \frac{1-u^{-2}}{(1-u^2b)(1-b)^2(1-u^{-2}b)} \times \\ \times \frac{1}{(1-u^5e)(1-u^3e)^2(1-ue)^2(1-u^{-1}e)^2(1-u^{-3}e)^2(1-u^{-5}e)}. \tag{4}$$

From [5-6] it is known that the generalized Hilbert series (2) is the solution of the functional Cayley equation

$$H(S_{1,4}, u, b, e) - u^{-2} H(S_{1,4}, u^{-1}, b, e) = \varphi_{1,4}^{(0)}(u),$$

where $\varphi_{1,4}^{(0)}(u)$ is from (4).

Taking into consideration the last equality takes place

Theorem 1. *The generalized Hilbert series for the graded algebra $S_{1,4}$ of the system* (1) *is a rational function of u, b, e and has the form*

$$H(S_{1,4}, u, b, e) = \frac{N_{1,4}(u, b, e)}{D_{1,4}(u, b, e)}, \tag{5}$$

where

$$D_{1,4}(u, b, e) = (1-b)(1-b^2)(1-bu^2)(1-be^2)^2(1-b^3e^2)^2(1-b^5e^2)(1-e^4)^2 \times \\ \times (1-e^2)(1-e^6)^2(1-e^8)^2(1-eu)^2(1-eu^3)^2(1-eu^5), \tag{6}$$

$$N_{1,4}(u, b, e) = \sum_{k=0}^{13} R_k(b, e) u^k, \tag{7}$$

and

$$\begin{aligned}
R_0(b,e) &= 1-e^2+4e^4+e^6+18e^8+11e^{10}+35e^{12}+13e^{14}+35e^{16}+11e^{18}+\\
&+18e^{20}+e^{22}+4e^{24}-e^{26}+e^{28}+b(e^2+5e^4+13e^6+26e^8+29e^{10}+40e^{12}+\\
&+19e^{14}+36e^{16}-5e^{18}+6e^{20}-15e^{22}+e^{24}-5e^{26}+2e^{28}-2e^{30})+b^2(e^2+\\
&+8e^4+16e^6+26e^8+27e^{10}+20e^{12}+12e^{14}-11e^{16}-29e^{18}-31e^{20}-22e^{22}-\\
&-11e^{24}-4e^{26}-2e^{28}-e^{30}+e^{32})+b^3(e^2+10e^4+10e^6+24e^8-5e^{10}+7e^{12}-\\
&-64e^{14}-49e^{16}-107e^{18}-55e^{20}-58e^{22}-10e^{24}-10e^{26}+3e^{28}+e^{30})+\\
&+b^4(e^2+6e^4+9e^6+10e^8-7e^{10}-29e^{12}-87e^{14}-75e^{16}-117e^{18}-29e^{20}-\\
&-30e^{22}+26e^{24}+2e^{26}+17e^{28}-3e^{30}+4e^{32})+b^5(5e^4+3e^6+10e^8-21e^{10}-\\
&-38e^{12}-82e^{14}-76e^{16}-72e^{18}+e^{20}+20e^{22}+44e^{24}+32e^{26}+17e^{28}+6e^{30}+\\
&+2e^{32}-2e^{34})+b^6(2e^4+3e^6-2e^8-29e^{10}-36e^{12}-84e^{14}-41e^{16}-48e^{18}+\\
&+48e^{20}+41e^{22}+84e^{24}+36e^{26}+29e^{28}+2e^{30}-3e^{32}-2e^{34})+b^7(2e^4-2e^6-\\
&-6e^8-17e^{10}-32e^{12}-44e^{14}-20e^{16}-e^{18}+72e^{20}+76e^{22}+82e^{24}+38e^{26}+\\
&+21e^{28}-10e^{30}-3e^{32}-5e^{34})+b^8(-4e^6+3e^8-17e^{10}-2e^{12}-26e^{14}+30e^{16}+\\
&+29e^{18}+117e^{20}+75e^{22}+87e^{24}+29e^{26}+7e^{28}-10e^{30}-9e^{32}-6e^{34}-e^{36})+\\
&+b^9(-e^8-3e^{10}+10e^{12}+10e^{14}+58e^{16}+55e^{18}+107e^{20}+49e^{22}+64e^{24}-7e^{26}+\\
&+5e^{28}-24e^{30}-10e^{32}-10e^{34}-e^{36})+b^{10}(-e^6+e^8+2e^{10}+4e^{12}+11e^{14}+\\
&+22e^{16}+31e^{18}+29e^{20}+11e^{22}-12e^{24}-20e^{26}-27e^{28}-26e^{30}-16e^{32}-8e^{34}-\\
&-e^{36})+b^{11}(2e^8-2e^{10}+5e^{12}-e^{14}+15e^{16}-6e^{18}+5e^{20}-36e^{22}-19e^{24}-\\
&-40e^{26}-29e^{28}-26e^{30}-13e^{32}-5e^{34}-e^{36})+b^{12}(-e^{10}+e^{12}-4e^{14}-e^{16}-\\
&-18e^{18}-11e^{20}-35e^{22}-13e^{24}-35e^{26}-11e^{28}-18e^{30}-e^{32}-4e^{34}+\\
&+e^{36}-e^{38}),
\end{aligned}$$

$$\begin{aligned}
R_1(b,e) &= -2e+4e^3+e^5+15e^7-8e^9+21e^{11}-18e^{13}+26e^{15}-27e^{17}+6e^{19}-\\
&-19e^{21}+7e^{23}-6e^{25}+2e^{27}-2e^{29}+b(e+5e^3+8e^5+5e^7+e^9+15e^{11}-8e^{13}+\\
&+16e^{15}-47e^{17}+11e^{19}-19e^{21}+18e^{23}-13e^{25}+7e^{27}-4e^{29}+4e^{31})+b^2(2e+\\
&+6e^3+2e^5+6e^7+9e^9+7e^{11}-7e^{13}-34e^{15}-24e^{17}+2e^{19}+11e^{21}+6e^{23}+\\
&+5e^{25}+5e^{27}+4e^{29}+2e^{31}-2e^{33})+b^3(e+4e^3+16e^7-11e^9+14e^{11}-77e^{13}-\\
&-e^{15}-63e^{17}+59e^{19}-6e^{21}+52e^{23}-3e^{25}+19e^{27}-3e^{29}-e^{31})+b^4(4e^3+5e^5+\\
&+3e^7-9e^9-26e^{11}-60e^{13}+4e^{15}-35e^{17}+88e^{19}-11e^{21}+51e^{23}-24e^{25}+\\
&+30e^{27}-20e^{29}+8e^{31}-8e^{33})+b^5(4e^3+7e^7-23e^9-19e^{11}-49e^{13}+e^{15}+\\
&+e^{17}+73e^{19}+12e^{21}+30e^{23}-e^{25}-9e^{27}-16e^{29}-10e^{31}-5e^{33}+4e^{35})+\\
&+b^6(e^3+e^5-3e^7-16e^9-10e^{11}-51e^{13}+26e^{15}-11e^{17}+96e^{19}-3e^{21}+\\
&+60e^{23}-45e^{25}-4e^{27}-38e^{29}-7e^{31}+e^{33}+3e^{35})+b^7(-3e^5+2e^7-6e^9-\\
&-21e^{11}-23e^{13}+18e^{15}+26e^{17}+73e^{19}+7e^{21}+11e^{23}-39e^{25}-15e^{27}-39e^{29}+\\
&+7e^{31}-4e^{33}+6e^{35})+b^8(6e^7-20e^9-24e^{13}+61e^{15}+9e^{17}+88e^{19}-49e^{21}+\\
&+20e^{23}-56e^{25}-18e^{27}-25e^{29}-e^{31}+e^{33}+6e^{35}+2e^{37})+b^9(-e^7-e^9+\\
&+7e^{11}+3e^{13}+44e^{15}+45e^{19}-53e^{21}+31e^{23}-65e^{25}+10e^{27}-47e^{29}+12e^{31}+\\
&+14e^{35}+e^{37})+b^{10}(2e^7-2e^9-e^{11}+9e^{13}+16e^{15}+7e^{17}-6e^{19}-12e^{21}-\\
&-12e^{23}-9e^{25}-21e^{27}-7e^{29}+14e^{31}+14e^{33}+8e^{35})+b^{11}(-4e^9+7e^{11}+e^{13}+\\
&+14e^{15}-23e^{17}+11e^{19}-35e^{21}+18e^{23}-34e^{25}+7e^{27}+5e^{29}+21e^{31}+8e^{33}+\\
&+3e^{35}+e^{37})+b^{12}(2e^{11}-4e^{13}-e^{15}-15e^{17}+8e^{19}-21e^{21}+18e^{23}-26e^{25}+\\
&+27e^{27}-6e^{29}+19e^{31}-7e^{33}+6e^{35}-2e^{37}+2e^{39}),
\end{aligned}$$

$$R_2(b,e) = 4e^2 - 2e^4 + 7e^6 - 5e^8 + 27e^{10} - 13e^{12} + 20e^{14} - 29e^{16} + 14e^{18} -$$
$$-17e^{20} + 5e^{22} - 14e^{24} + 3e^{26} - e^{28} + e^{30} + b(2e^2 - 3e^4 + 7e^6 + e^8 + 11e^{10} -$$
$$-53e^{12} - 14e^{14} - 73e^{16} + 9e^{18} - 45e^{20} + 8e^{22} - 17e^{24} + 21e^{26} - 5e^{28} +$$
$$+2e^{30} - 2e^{32}) + b^2(7e^6 - 12e^8 - 39e^{10} - 64e^{12} - 39e^{14} - 37e^{16} + 2e^{18} -$$
$$-16e^{20} + 16e^{22} + 17e^{24} + 16e^{26} - 3e^{28} + e^{30} - e^{32} + e^{34}) + b^3(2e^2 - e^4 +$$
$$+e^6 - 42e^8 - 30e^{10} - 85e^{12} + 8e^{14} - 44e^{16} + 67e^{18} + e^{20} + 81e^{22} + 16e^{24} +$$
$$+32e^{26} - 6e^{28} + e^{30} - e^{32}) + b^4(2e^2 - 5e^4 - 6e^6 - 30e^8 - 18e^{10} - 53e^{12} +$$
$$+65e^{14} - 3e^{16} + 159e^{18} + 44e^{20} + 143e^{22} + 9e^{24} + 40e^{26} - 47e^{28} + 5e^{30} -$$
$$-7e^{32} + 4e^{34}) + b^5(e^2 - 5e^4 - 28e^8 - 11e^{10} - 14e^{12} + 84e^{14} + 57e^{16} +$$
$$+172e^{18} + 51e^{20} + 89e^{22} - 23e^{24} - 25e^{26} - 43e^{28} + e^{30} - 4e^{32} + 2e^{34} -$$
$$-2e^{36}) + b^6(-e^4 - e^6 - 26e^8 + e^{10} - 12e^{12} + 103e^{14} + 67e^{16} + 162e^{18} +$$
$$+6e^{20} + 51e^{22} - 98e^{24} - 37e^{26} - 59e^{28} - 4e^{30} - 4e^{32} + 3e^{34}) + b^7(-e^6 -$$
$$-13e^8 + 7e^{10} + 22e^{12} + 103e^{14} + 51e^{16} + 104e^{18} - 20e^{20} - 9e^{22} - 119e^{24} -$$
$$-60e^{26} - 79e^{28} + 8e^{30} - 5e^{32} + 10e^{34} + e^{36}) + b^8(-2e^4 + 2e^6 - 11e^8 +$$
$$+26e^{10} + 23e^{12} + 87e^{14} + 79e^{18} - 92e^{20} - 24e^{22} - 141e^{24} - 68e^{26} - 61e^{28} +$$
$$+12e^{30} + 3e^{32} + 14e^{34} + 3e^{36} - e^{38}) + b^9(3e^6 - 4e^8 + 22e^{10} + 5e^{12} + 55e^{14} -$$
$$-30e^{16} + 30e^{18} - 137e^{20} - 41e^{22} - 183e^{24} - 39e^{26} - 53e^{28} + 43e^{30} + 10e^{32} +$$
$$+18e^{34} - 2e^{36} + e^{38}) + b^{10}(-e^8 + 7e^{10} - 2e^{12} - 4e^{14} - 53e^{16} - 42e^{18} -$$
$$-104e^{20} - 79e^{22} - 99e^{24} - 10e^{26} + 11e^{28} + 45e^{30} + 12e^{32} + 10e^{34} + 4e^{36} +$$
$$+3e^{38}) + b^{11}(e^6 - e^8 + 5e^{10} - 9e^{12} - 6e^{14} - 36e^{16} - 10e^{18} - 67e^{20} - 7e^{22} -$$
$$-19e^{24} + 46e^{26} + 31e^{28} + 35e^{30} + 13e^{32} + 17e^{34} + 7e^{36}) + b^{12}(-2e^8 + 2e^{10} -$$
$$-9e^{12} + 3e^{14} - 22e^{16} + 11e^{18} - 32e^{20} + 49e^{22} - e^{24} + 69e^{26} + 15e^{28} + 43e^{30} +$$
$$+8e^{32} + 19e^{34} - 2e^{36} + e^{38} - e^{40}) + b^{13}(e^{10} - e^{12} + 4e^{14} + e^{16} + 18e^{18} +$$
$$+11e^{20} + 35e^{22} + 13e^{24} + 35e^{26} + 11e^{28} + 18e^{30} + e^{32} + 4e^{34} - e^{36} + e^{38}),$$

$$R_3(b,e) = -e + e^3 + e^5 - 3e^7 - 16e^9 - 38e^{11} - 45e^{13} - 46e^{15} - 56e^{17} - 50e^{19} -$$
$$-32e^{21} - 14e^{23} - e^{27} - 2e^{29} + b(2e - e^3 - e^5 - 30e^7 - 35e^9 - 87e^{11} - 44e^{13} -$$
$$-95e^{15} - 44e^{17} - 33e^{19} + 27e^{21} + 15e^{23} + 22e^{25} - 5e^{27} + 3e^{29} + 4e^{31}) + b^2(e -$$
$$-2e^3 - 14e^5 - 32e^7 - 48e^9 - 64e^{11} - 46e^{13} - 73e^{15} + 41e^{17} + 49e^{19} + 101e^{21} +$$
$$+45e^{23} + 36e^{25} - e^{27} + 12e^{29} - 3e^{31} - 2e^{33}) + b^3(-3e^3 - 13e^5 - 31e^7 - 43e^9 -$$
$$-45e^{11} - 10e^{13} + 77e^{15} + 156e^{17} + 197e^{19} + 174e^{21} + 105e^{23} + 42e^{25} + 7e^{27} -$$
$$-5e^{29} - 5e^{31} + e^{33}) + b^4(-2e^3 - 6e^5 - 31e^7 - 23e^9 - 46e^{11} + 110e^{13} + 113e^{15} +$$
$$+271e^{17} + 189e^{19} + 148e^{21} - 16e^{23} - 19e^{25} - 61e^{27} - 10e^{29} - 6e^{31} - 7e^{33}) +$$
$$+b^5(-e^3 - 5e^5 - 20e^7 - 23e^9 + 20e^{11} + 115e^{13} + 132e^{15} + 252e^{17} + 92e^{19} +$$
$$+61e^{21} - 97e^{23} - 75e^{25} - 110e^{27} - 17e^{29} - 33e^{31} + 5e^{33} + 6e^{35}) + b^6(-4e^5 -$$
$$-23e^7 + 14e^9 + 33e^{11} + 122e^{13} + 144e^{15} + 165e^{17} + 40e^{19} - 26e^{21} - 159e^{23} -$$
$$-158e^{25} - 114e^{27} - 48e^{29} - 5e^{31} + 17e^{33} + 3e^{35} - e^{37}) + b^7(-4e^5 + e^7 + 14e^9 +$$
$$+19e^{11} + 109e^{13} + 69e^{15} + 126e^{17} - 49e^{19} - 116e^{21} - 252e^{23} - 138e^{25} - 122e^{27} +$$
$$+4e^{29} + 13e^{31} + 16e^{33} + 8e^{35}) + b^8(4e^5 - e^7 + e^9 + 31e^{11} + 52e^{13} + 31e^{15} +$$
$$+17e^{17} - 147e^{19} - 199e^{21} - 230e^{23} - 154e^{25} - 84e^{27} + 24e^{29} + 6e^{31} + 33e^{33} +$$
$$+11e^{35} + e^{37}) + b^9(-6e^7 + 21e^9 + 4e^{11} + 28e^{13} - 63e^{15} - 61e^{17} - 224e^{19} -$$
$$-145e^{21} - 207e^{23} - 58e^{25} - 18e^{27} + 35e^{29} + 35e^{31} + 38e^{33} + 16e^{35} + 2e^{37} -$$
$$-e^{39}) + b^{10}(2e^7 - 8e^{11} + e^{13} - 51e^{15} - 36e^{17} - 122e^{19} - 33e^{21} - 59e^{23} + 84e^{25} +$$
$$+23e^{27} + 97e^{29} + 44e^{31} + 48e^{33} + 11e^{35} + e^{37} - 2e^{39}) + b^{11}(-2e^7 + 3e^{11} - 8e^{13} -$$

$-24e^{15}-26e^{17}-21e^{19}+20e^{21}+58e^{23}+56e^{25}+84e^{27}+82e^{29}+50e^{31}+28e^{33}+$
$+2e^{35}-e^{37}+e^{39})+b^{12}(4e^{9}-6e^{11}-2e^{13}-15e^{15}+26e^{17}+5e^{19}+73e^{21}+$
$+27e^{23}+80e^{25}+49e^{27}+45e^{29}+11e^{31}+6e^{33}-3e^{35}+2e^{39})+b^{13}(-2e^{11}+$
$+4e^{13}+e^{15}+15e^{17}-8e^{19}+21e^{21}-18e^{23}+26e^{25}-27e^{27}+6e^{29}-19e^{31}+$
$+7e^{33}-6e^{35}+2e^{37}-2e^{39}),$

$R_4(b,e)=3e^2-2e^4-2e^6-9e^8+8e^{10}-17e^{12}+11e^{14}-43e^{16}+26e^{18}-e^{20}+$
$+25e^{22}-6e^{24}+6e^{26}-3e^{28}+4e^{30}+b(-2e^2-e^4-12e^6+7e^8-32e^{10}-6e^{12}-$
$-36e^{14}+3e^{16}+54e^{18}+14e^{20}+12e^{22}-7e^{24}+10e^{26}-2e^{28}+6e^{30}-8e^{32})+$
$+b^2(-2e^2-4e^4-3e^6-22e^8-39e^{10}-13e^{12}-11e^{14}+89e^{16}+26e^{18}+25e^{20}-$
$-20e^{22}+9e^{24}-22e^{26}-14e^{30}-3e^{32}+4e^{34})+b^3(-4e^4-16e^6-30e^8-10e^{10}-$
$-12e^{12}+92e^{14}+27e^{16}+98e^{18}-37e^{20}+10e^{22}-75e^{24}-12e^{26}-33e^{28}+$
$+2e^{32})+b^4(e^2-8e^4-22e^6-2e^8-20e^{10}+92e^{12}+44e^{14}+88e^{16}+32e^{18}-$
$-72e^{20}-49e^{22}-54e^{24}-19e^{26}-32e^{28}+17e^{30}-12e^{32}+16e^{34})+b^5(-5e^4-$
$-10e^6-16e^8+27e^{10}+61e^{12}+59e^{14}+96e^{16}-31e^{18}-71e^{20}-89e^{22}-54e^{24}-$
$-54e^{26}+46e^{28}+7e^{30}+32e^{32}+10e^{34}-8e^{36})+b^6(-2e^4-13e^6+14e^8+10e^{10}+$
$+49e^{12}+73e^{14}+35e^{16}-14e^{18}-113e^{20}-84e^{22}-105e^{24}+48e^{26}+22e^{28}+$
$+62e^{30}+23e^{32}-e^{34}-4e^{36})+b^7(-2e^4+2e^6-e^8+8e^{10}+81e^{12}+25e^{14}+$
$+32e^{16}-99e^{18}-108e^{20}-101e^{22}+18e^{24}+17e^{26}+67e^{28}+58e^{30}+4e^{32}+$
$+11e^{34}-10e^{36}-2e^{38})+b^8(-5e^6+39e^{10}+28e^{12}+20e^{14}-39e^{16}-77e^{18}-$
$-133e^{20}+10e^{22}-29e^{24}+68e^{26}+77e^{28}+33e^{30}+30e^{32}-2e^{34}-15e^{36}-$
$-5e^{38})+b^9(17e^8-8e^{10}+13e^{12}-34e^{14}-13e^{16}-90e^{18}-33e^{20}-22e^{22}+$
$+2e^{24}+88e^{26}+38e^{28}+78e^{30}+4e^{32}-10e^{34}-28e^{36}-2e^{38})+b^{10}(e^6-$
$-2e^8+6e^{12}-39e^{14}-20e^{16}-69e^{18}+22e^{20}-23e^{22}+88e^{24}+7e^{26}+92e^{28}-$
$-2e^{30}-10e^{32}-34e^{34}-14e^{36}-3e^{38})+b^{11}(2e^8+2e^{10}-14e^{12}-13e^{14}-$
$-14e^{16}+5e^{18}-14e^{20}+55e^{22}-e^{24}+73e^{26}-20e^{30}-29e^{32}-17e^{34}-10e^{36}-$
$-4e^{38}-e^{40})+b^{12}(-7e^{10}+2e^{12}-3e^{14}+16e^{16}-12e^{18}+30e^{20}+13e^{22}+$
$+20e^{24}+17e^{26}-30e^{28}-8e^{30}-22e^{32}-3e^{34}-12e^{36}+3e^{38}-4e^{40})+$
$+b^{13}(4e^{12}-2e^{14}+7e^{16}-5e^{18}+27e^{20}-13e^{22}+20e^{24}-29e^{26}+14e^{28}-$
$-17e^{30}+5e^{32}-14e^{34}+3e^{36}-e^{38}+e^{40}),$

$R_5(b,e)=-3e^3-2e^5-13e^9-38e^{11}-30e^{13}-40e^{15}-3e^{17}-26e^{19}-5e^{21}-$
$-6e^{23}+16e^{25}-e^{27}+2e^{29}-2e^{31}+b(e-3e^3-8e^7-35e^9-20e^{11}+8e^{13}+$
$+26e^{15}+64e^{17}+26e^{19}+53e^{21}+36e^{23}+19e^{25}-18e^{27}+2e^{29}-4e^{31}+4e^{33})+$
$+b^2(-4e^5-22e^7-8e^9+37e^{11}+60e^{13}+81e^{15}+78e^{17}+56e^{19}+57e^{21}+$
$+15e^{23}-29e^{25}-13e^{27}-5e^{29}-e^{31}+2e^{33}-2e^{35})+b^3(-3e^3-8e^5-3e^7+$
$+19e^9+49e^{11}+103e^{13}+101e^{15}+118e^{17}+36e^{19}+31e^{21}-60e^{23}-29e^{25}-$
$-47e^{27}-5e^{29}-3e^{31}+3e^{33})+b^4(-5e^3+4e^5+4e^7+8e^9+63e^{11}+76e^{13}+$
$+54e^{15}+29e^{17}-109e^{19}-120e^{21}-169e^{23}-115e^{25}-64e^{27}+35e^{29}+13e^{33}-$
$-6e^{35})+b^5(-e^3+e^5-6e^7+26e^9+59e^{11}+38e^{13}+3e^{15}-82e^{17}-204e^{19}-$
$-143e^{21}-159e^{23}-68e^{25}+28e^{27}+28e^{29}+16e^{31}+11e^{33}-3e^{35}+3e^{37})+$
$+b^6(-3e^5+2e^7+26e^9+42e^{11}+42e^{13}-33e^{15}-130e^{17}-192e^{19}-134e^{21}-$
$-117e^{23}+42e^{25}+40e^{27}+75e^{29}+32e^{31}+12e^{33}-6e^{35})+b^7(17e^9+30e^{11}-$
$-16e^{13}-90e^{15}-123e^{17}-165e^{19}-110e^{21}-20e^{23}+70e^{25}+120e^{27}+119e^{29}+$
$+22e^{31}+11e^{33}-13e^{35}-3e^{37})+b^8(2e^5+15e^9+2e^{11}-49e^{13}-78e^{15}-$

$-109e^{17}-145e^{19}+3e^{21}-4e^{23}+131e^{25}+143e^{27}+92e^{29}+23e^{31}+e^{33}-$
$-23e^{35}-5e^{37}+e^{39})+b^9(-2e^7+5e^9-7e^{11}-33e^{13}-67e^{15}-94e^{17}-$
$-43e^{19}+62e^{21}+92e^{23}+222e^{25}+140e^{27}+88e^{29}-14e^{31}-22e^{33}-25e^{35}+$
$+4e^{37}-4e^{39})+b^{10}(2e^9-6e^{11}-17e^{13}-7e^{15}+32e^{17}+77e^{19}+139e^{21}+$
$+187e^{23}+172e^{25}+104e^{27}+9e^{29}-34e^{31}-25e^{33}-14e^{35}-10e^{37}-5e^{39})+$
$+b^{11}(-e^7+e^9-5e^{11}-6e^{13}+12e^{15}+25e^{17}+52e^{19}+110e^{21}+61e^{23}+55e^{25}-$
$-13e^{27}-30e^{29}-37e^{31}-28e^{33}-34e^{35}-11e^{37})+b^{12}(2e^9-2e^{11}+2e^{13}+$
$+10e^{15}+19e^{17}+40e^{19}+30e^{21}-16e^{23}-7e^{25}-60e^{27}-49e^{29}-59e^{31}-36e^{33}-$
$-26e^{35}+2e^{37}-3e^{39}+2e^{41})+b^{13}(-e^{11}+e^{13}+e^{15}-3e^{17}-16e^{19}-38e^{21}-$
$-45e^{23}-46e^{25}-56e^{27}-50e^{29}-32e^{31}-14e^{33}-e^{37}-2e^{39}),$

$R_6(b,e)=2e^2-2e^4+e^6-16e^8+6e^{10}+5e^{12}+26e^{14}+3e^{16}+40e^{18}+30e^{20}+$
$+38e^{22}+13e^{24}+2e^{28}+3e^{30}+b(-3e^2-7e^6+12e^{10}+36e^{12}+9e^{14}+63e^{16}+$
$+48e^{18}+34e^{20}-e^{22}-19e^{24}-17e^{26}+8e^{28}-6e^{30}-6e^{32})+b^2(-2e^2-3e^4-$
$-e^6+6e^8+19e^{10}+19e^{12}+35e^{14}+94e^{16}-10e^{18}+3e^{20}-72e^{22}-38e^{24}-$
$-33e^{26}-6e^{28}-20e^{30}+6e^{32}+3e^{34})+b^3(-3e^4-4e^8+29e^{10}+31e^{12}+82e^{14}-$
$-15e^{16}-22e^{18}-120e^{20}-114e^{22}-110e^{24}-50e^{26}-19e^{28}+7e^{30}+8e^{32}-$
$-2e^{34})+b^4(-e^4-11e^6+4e^8+39e^{10}+70e^{12}-13e^{14}-12e^{16}-152e^{18}-154e^{20}-$
$-122e^{22}-36e^{24}+10e^{26}+50e^{28}+5e^{30}+9e^{32}+12e^{34})+b^5(-3e^4-7e^6+12e^8+$
$+50e^{10}-5e^{14}-48e^{16}-179e^{18}-67e^{20}-114e^{22}+17e^{24}+46e^{26}+91e^{28}+20e^{30}+$
$+52e^{32}-8e^{34}-8e^{36})+b^6(-3e^4+22e^8+8e^{10}+7e^{12}-19e^{14}-93e^{16}-93e^{18}-$
$-96e^{20}-65e^{22}+58e^{24}+120e^{26}+98e^{28}+69e^{30}+8e^{32}-20e^{34}-e^{36})+b^7(2e^6-$
$-e^8+14e^{10}+14e^{12}-48e^{14}-36e^{16}-144e^{18}-47e^{20}+29e^{22}+161e^{24}+112e^{26}+$
$+120e^{28}-3e^{32}-10e^{34}-11e^{36}-e^{38})+b^8(-6e^6+7e^8+24e^{10}-32e^{12}-2e^{14}-$
$-73e^{16}-65e^{18}+33e^{20}+85e^{22}+136e^{24}+159e^{26}+78e^{28}-3e^{30}+13e^{32}-40e^{34}-$
$-11e^{36}-e^{38})+b^9(e^6+11e^8-16e^{10}-14e^{12}-26e^{14}-10e^{16}+9e^{18}+98e^{20}+$
$+61e^{22}+170e^{24}+78e^{26}+23e^{28}+4e^{30}-32e^{32}-36e^{34}-16e^{36}-5e^{38}+2e^{40})+$
$+b^{10}(-2e^8+e^{10}-e^{12}-13e^{14}+17e^{16}-e^{18}+90e^{20}+35e^{22}+86e^{24}-51e^{26}+$
$+14e^{28}-82e^{30}-33e^{32}-50e^{34}-12e^{36}-e^{38}+3e^{40})+b^{11}(3e^8-e^{10}-13e^{12}+$
$+10e^{14}+e^{16}+29e^{18}+20e^{20}+10e^{22}-16e^{24}+11e^{26}-66e^{28}-68e^{30}-49e^{32}-$
$-29e^{34}+6e^{36}+3e^{38}-2e^{40})+b^{12}(-6e^{10}+4e^{12}+9e^{14}+8e^{16}-2e^{18}+5e^{20}-$
$-30e^{22}+10e^{24}-59e^{26}-56e^{28}-31e^{30}-11e^{32}+4e^{34}+7e^{36}-e^{38}-2e^{40})+$
$+b^{13}(3e^{12}-2e^{14}-2e^{16}-9e^{18}+8e^{20}-17e^{22}+11e^{24}-43e^{26}+26e^{28}-e^{30}+$
$+25e^{32}-6e^{34}+6e^{36}-3e^{38}+4e^{40}),$

$$R_{13-k}(b,e)=-b^{13}e^{45}R_k(b^{-1},e^{-1}),\ (k=\overline{0,6}). \tag{8}$$

Taking into consideration the equality (3) from Theorem 1 we obtain

Corollary 1. *The common Hilbert series for graded algebra of comitants $S_{1,4}$ of the system* (1) *has the form*

$$H_{S_{1,4}}(u)=\frac{n_{1,4}(u)}{d_{1,4}(u)}, \tag{9}$$

where

$$d_{1,4}(u)=(1-u^2)(1-u^3)(1-u^4)^3(1-u^5)^2(1-u^6)^3(1-u^7)(1-u^8)^2,$$

$$n_{1,4}(u) = 1 + u + u^2 + 5u^3 + 17u^4 + 39u^5 + 100u^6 + 218u^7 + 467u^8 +$$
$$+865u^9 + 1586u^{10} + 2685u^{11} + 4467u^{12} + 6889u^{13} + 10423u^{14} + 14934u^{15} +$$
$$+20921u^{16} + 27849u^{17} + 36293u^{18} + 45278u^{19} + 55254u^{20} + 64697u^{21} +$$
$$+74134u^{22} + 81782u^{23} + 88328u^{24} + 91866u^{25} + 93539u^{26} + 91866u^{27} +$$
$$+88328u^{28} + 81782u^{29} + 74134u^{30} + 64697u^{31} + 55254u^{32} + 45278u^{33} +$$
$$+36293u^{34} + 27849u^{35} + 20921u^{36} + 14934u^{37} + 10423u^{38} + 6889u^{39} +$$
$$+4467u^{40} + 2685u^{41} + 1586u^{42} + 865u^{43} + 467u^{44} + 218u^{45} + 100u^{46} +$$
$$+39u^{47} + 17u^{48} + 5u^{49} + u^{50} + u^{51} + u^{52}. \tag{10}$$

With the help of Remark 2 and Corollary 1 we obtain

Theorem 2. *The Krull dimension $\varrho(S_{1,4})$ for graded algebra $S_{1,4}$ is equal to 13, i.e.* $\varrho(S_{1,4}) = 13$.

According to Remark 1 from Theorem 1 follows

Corollary 2. *The generalized Hilbert series for graded algebra of invariants $SI_{1,4}$ of the system* (1) *is a rational function of b, e and has the form*

$$H(SI_{1,4}, b, e) = \frac{N_{1,4}(b,e)}{D_{1,4}(b,e)}, \tag{11}$$

where

$$\begin{gathered} D_{1,4}(b,e) = (1-b)(1-b^2)(1-e^2)(1-be^2)^2(1-b^3e^2)^2(1-b^5e^2)(1-e^4)^2\times \\ \times(1-e^6)^2(1-e^8)^2, \\ N_{1,4}(b,e) = R_0(b,e), \end{gathered} \tag{12}$$

and $R_0(b,e)$ is from (8).

With the help of Remark 2 and Corollary 2 we obtain

Corollary 3. *The common Hilbert series for graded algebras of invariants $SI_{1,4}$ for the system* (1) *has the form*

$$H_{SI_{1,4}}(z) = \frac{N_{1,4}(z)}{D_{1,4}(z)}, \tag{13}$$

where

$$\begin{gathered} D_{1,4}(z) = (1-z^3)(1-z^4)^3(1-z^5)^2(1-z^6)^2(1-z^7)(1-z^8)^2, \\ N_{1,4}(z) = 1 + z + z^2 + 3z^3 + 8z^4 + 15z^5 + 32z^6 + 67z^7 + 129z^8 + 217z^9 + \\ +355z^{10} + 546z^{11} + 812z^{12} + 1122z^{13} + 1511z^{14} + 1948z^{15} + 2447z^{16} + \\ +2923z^{17} + 3410z^{18} + 3827z^{19} + 4183z^{20} + 4375z^{21} + 4461z^{22} + 4375z^{23} + \\ +4183z^{24} + 3827z^{25} + 3410z^{26} + 2923z^{27} + 2447z^{28} + 1948z^{29} + 1511z^{30} + \\ +1122z^{31} + 812z^{32} + 546z^{33} + 355z^{34} + 217z^{35} + 129z^{36} + 67z^{37} + 32z^{38} + \\ +15z^{39} + 8z^{40} + 3z^{41} + z^{42} + z^{43} + z^{44}. \end{gathered} \tag{14}$$

With the of help Remark 2 and Corollary 3 we obtain

Theorem 3. *The Krull dimension* $\varrho(SI_{1,4})$ *for graded algebra* $SI_{1,4}$ *is equal to 11, i.e.* $\varrho(SI_{1,4}) = 11$.

Similarly [28] with the help of representative form of generating function, which is obtained from Hilbert series (5)-(8) by multiplication of the numerator and the denominator by expression $M_{1,4}(u,b,e) = (1+e^2)(1+ue)^2(1+u^3e)^2$ and taking into consideration the characteristics of algebras S_4 and SI_4 from [5-6] we have

Theorem 4. *The lower bound of the number of generators for the algebra* $S_{1,4}(SI_{1,4})$ *is not less than 311(138) irreducible comitants (invariants)[1-4], distributed in 58(20) types as follows:*

$$(0,1,0),(0,2,0),6(0,0,4),7(0,0,6),15(0,0,8),14(0,0,10),3(0,1,2),6(0,1,4),$$
$$15(0,1,6),16(0,1,8),(0,2,2),8(0,2,4),15(0,2,6),3(0,3,2),10(0,3,4),7(0,3,6),$$
$$(0,4,2),5(0,4,4),(0,5,2),3(0,5,4),2(1,0,3),11(1,0,5),20(1,0,7),2(1,0,9),$$
$$(1,1,1),8(1,1,3),20(1,1,5),2(1,2,1),9(1,2,3),4(1,2,5),(1,3,1),3(1,3,3),$$
$$3(1,4,3),(2,1,0),3(2,0,2),6(2,0,4),12(2,0,6),4(2,1,2),9(2,1,4),3(2,2,2),$$
$$2(2,3,2),(3,0,1),6(3,0,3),9(3,0,5),2(3,1,1),6(3,1,3),(3,2,1),(4,0,2),$$
$$6(4,0,4),3(4,1,2),(5,0,1),3(5,0,3),(5,1,1),2(6,0,2),2(6,0,4),(6,1,2),$$
$$(7,0,3),(9,0,3).$$

The number of comitants and invariants of the given type is indicated before brackets, the omitted number means that it is equal to one.

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