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On a Product of Classes of Algebraic Systems

Vasile I. Ursu

Abstract. This paper defines a product of classes of algebraic systems and proves that it is a universal class, a quasi-variety or variety if these classes are universal classes, quasi-varieties or varieties, respectively.

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A class of algebraic systems of signature σ is any set (possibly empty) of systems of signature σ which contains together with every its system all isomorphic to it systems. A class K of algebraic systems of signature σ is called a universal class if there exists such a set Σ of universal formulae ($\forall$-formulae) that K be formed of all systems of signature σ, with the formulae from Σ holding true within it. If all formulae from Σ are identities or quasi-identities, then the class K is called a variety or quasi-variety, respectively.

In [1] W. Taylor defined the product of two varieties of algebras of different signatures as a variety of non-indexed products of algebras from these varieties. The non-indexed product of two algebras A and B is defined as an algebra with the basic set equal to the Cartesian product of the basic sets of these algebras and with the set of operations consisting of all pairs of terms of the same number of variables. It is difficult to investigate this product, which we will name hereafter a Taylor product, due to the fact that the signature of the Taylor product of two (or more) varieties is complex, and it is difficult to investigate the operations of the algebras from this variety.

This paper presents a new concept for the product of two or more classes of algebraic systems of different signatures and shows that the product of universal classes is a universal class, the product of quasi-varieties is a quasi-variety, and the product of varieties is a variety.

1 Preliminary notions and results

For any class K of algebraic systems of signature σ we will denote by
$S(K)$ – the class of all subsystems of K-systems,
$P(K)$ – the class of all Cartesian products of K-systems,
$F(K)$ – the class of all filtered products of K-systems,
$H(K)$ – the class of all homomorphic images of K-systems.

If $S(K) = K$, $P(K) = K$, or $H(K) = K$ holds for a class K, then one class K is called hereditary, multiplicatively closed, filteredly closed, or homomorphically closed, respectively.

Next we will need the following main results from the variety and quasi-variety theory, obtained by Birkhoff G. and Mal'cev A.I.

Theorem 1 (Birkhoff [2]). *A class K of algebraic systems is a variety if and only if the class K satisfies the following conditions:*

(a) *is hereditary,*

(b) *is multiplicatively closed,*

(c) *is homomoprhically closed.*

Theorem 2 (Mal'cev [3]). *A class K of algebraic systems is a quasi-variety if and only if the class K satisfies the following conditions:*

(a) *is hereditary,*

(b) *is filteredly closed,*

(c) *contains the unitary system.*

It is worth reminding that a filter over a non-empty set I is a set D of subsets of I that satisfies the following conditions:

1) $A \in D \ \& \ B \in D \Rightarrow A \cap B \in D$,

2) $A \in D \ \& \ A \subseteq B \subseteq I \Rightarrow B \in D$,

3) $\emptyset \notin D$.

It is obvious that the set of all filters over I is a partially ordered set relative to the inclusion. A maximal filter over I is called a ultrafilter.

A filtered product of algebraic systems is defined as follows. Let A_i, $i \in I$, be a set of algebraic systems of the same signature σ and D a filter over I. We define the basic relation $\equiv$ on the Cartesian product $A = \prod_{i \in I} A_i$, putting $a \equiv b$ $(a, b \in A)$ if and only if the set of indices $\{i \in I | a(i) = b(i)\}$ belongs to the filter D. The binary relation $\equiv$ is an equivalence; moreover, it is stable relative to any operation of system A, that is, if f^A is an n operation of the system A, then

$$a_1 \equiv b_1 \& \ldots \& a_n \equiv b_n \Rightarrow f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)$$

for any elements a_i, b_i, $i = 1, \ldots, n$, from A. This means that the operations $f^{A/\equiv}$, $f \in \sigma$, and predicates r^A, $r \in \sigma$, can be naturally defined on the set $A/\equiv$ of classes of equivalences $a/\equiv$, $(a \in A)$ in such a way that:

$$f^{A/\equiv}(a_1/\equiv, \ldots, a_n/\equiv) = f^A(a_1, \ldots, a_n)/\equiv,$$

where n is the arity of the functional symbol f; in $A/\equiv$ the following relation

$$r^{A/\equiv}(a_1/\equiv, \ldots, a_m/\equiv),$$

holds, where m is the arity of predicate r if and only if the set $\{i \in I | A_i \models r^{A_i}(a_1(i), \ldots, a_m(i))\}$ belongs to the filter D. The algebraic system, built in such a way, is denoted by A/D and is called the filtered product of systems A_i, $i \in I$, and if D is an ultrafilter, it is called an ultraproduct.

2 Products of universal classes and quasi-varieties

Let A_i, $i \in I$, be a set of algebraic systems of arbitrary signatures σ_i, $i \in I$. We complete the signature of every system A_i with the functional symbols p_j, $j \in I$, that correspond to the operations of projections $p_j^{A_i}$, $j \in I$, defined on the Cartesian power A_i^I with values from $A_i : p_j^{A_i}(a) = a_j$, $j \in I$, for any element $a = a_i, i \in I) \in A_i^I$. If not all systems from this set are algebraic, then we also complete the signature of every system A_i with the predicative symbol e that corresponds to the real identical predicate e^{A_i}, defined on the Cartesian power A_i^I with real values: $A_i \models e^{A_i}(a)$ ($e^{A_i}(a)$ holds in A_i) for any $a = (a_i, i \in I) \in A_i^I$. The system we obtain in such a way will be called an *enriched algebraic system* and will be also denoted by A_i.

The enriched Cartesian product of the enriched algebraic systems A_i, $i \in I$, is an algebraic system $\otimes_i A_i$ with the basic set $A = \prod_{i \in I} A_i$, which for each family of basic n-operations $(f_i^{A_i}, i \in I)$ and each family of basic m-predicates $(r_i^{A_i}, i \in I)$ of the enriched systems $A_i, i \in I$, has a basic n-operation f^A and a basic m-predicate, defined by

$$f^A(a_1, \dots, a_m) = (f_i^{A_i}(a_1(i), \dots, a_n(i)), i \in I),$$

$$A \models r^A(a_1, \dots, a_m) \Leftrightarrow \&_{i \in I} A_i \models r_i^{A_i}(a_1(i), \dots, a_m(i)),$$

where $a_1, a_2, \dots$ are elements from A and it doesn't have any other basic operations and predicates.

We notice that if $A_i, i \in I$, are algebras, then the system $\otimes_i A_i$ is an algebra.

Let now $Q_i, i \in I$ be a set of classes of algebraic systems. The signatures of these classes may be different. We will define the *product of classes* $Q_i, i \in I$, as the class of algebraic systems, consisting of all isomorphisms of algebraic systems of the form $\otimes_i A_i$, where $A_i \in Q_i$, $i \in I$. We will denote the product of classes $Q_i, i \in I$, by $\otimes_i Q_i$ and by $Q_1 \otimes \dots \otimes Q_n$ if $I = \{1, \dots, n\}$.

Lemma 1. *The product of a finite number of filteredly closed classes is a filteredly closed class.*

Proof. Let $K_i, i = 1, \dots, n$, be a set of closed classes relative to filtered products and $K = K_1 \otimes \dots \otimes K_n$ be their product, $A^i = A_1^i \otimes \dots A_n^i, i \in I$, be a set of algebraic systems with $A_j^i \in K_j, j = 1, \dots, n$, and D be a filter over I. Then

$$\prod_{i \in I} A_1^i / D \in K_1, \dots, \prod_{i \in I} A_n^i / D \in K_n.$$

We will show that the following isomorphism holds

$$\prod_{i \in I} A^i / D \cong (\prod_{i \in I} A_1^i / D) \otimes \dots \otimes (\prod_{i \in I} A_n^i / D)$$

and then we will get

$$\prod_{i \in I} A^i / D \in K.$$

Let φ be a mapping from $\prod_{i\in I} A^i/D$ in $(\prod_{i\in I} A_1^i/D)\otimes\ldots\otimes(\prod_{i\in I} A_n^i/D)$ defined by the relation

$$\varphi(a)=\varphi((a_1^i,\ldots,a_n^i),i\in I)D=((a_1^i,i\in I)D,\ldots,(a_n^i,i\in I)D)$$

for any $a=((a_1^i,\ldots,a_n^i),i\in I)D\in\prod_{i\in I} A^i/D$, where $a_1^i\in A_1^i,\ldots,a_n^i\in A_n^i, i\in I$, is an epimorphism. We denote $A=\prod_{i\in I} A^i/D, A_1=\prod_{i\in I} A_1^i/D$, $\ldots, A_n=\prod_{i\in I} A_n^i/D$.

We consider a basic operation f^A of arity k of the algebraic system A. By the definition, we have

$$f^A(a_1,\ldots,a_k)=$$
$$=f^A(((a_{11}^i,\ldots,a_{1n}^i),i\in I)D,\ldots,((a_{k1}^i,\ldots,a_{kn}^i),i\in I)D)=$$
$$=((f^{A_1^i}(a_{11}^i,\ldots,a_{k1}^i),i\in I),\ldots,(f^{A_n^i}(a_{1n}^i,\ldots,a_{kn}^i),i\in I))D$$

for all $a_1=((a_{11}^i,\ldots,a_{1n}^i),i\in I)D,\ldots,a_k=((a_{k1}^i,\ldots,a_{kn}^i),i\in I)D$ from A. It follows from here that

$$\varphi(f^A(a_1,\ldots,a_k))=((f^{A_1^i}(a_{11}^i,\ldots,a_{k1}^i),i\in I)D,\ldots,(f^{A_n^i}(a_{1n}^i,\ldots,a_{kn}^i),i\in I)D)=$$
$$=(f^{A_1}((a_{11}^i,i\in I)D,\ldots,(a_{k1}^i,i\in I)D),\ldots,f^{A_n}((a_{1n}^i,i\in I)D,\ldots,(a_{kn}^i,i\in I)D))=$$
$$=f^{A_1\otimes\ldots\otimes A_n}(\varphi(((a_{11}^i,\ldots,a_{k1}^i),i\in I)D),\ldots,\varphi(((a_{1n}^i,\ldots,a_{kn}^i),i\in I)D))=$$
$$=f^{\varphi(A)}(\varphi(a_1),\ldots,\varphi(a_k)).$$

Let now r^A be a basic m-relation of the algebraic system A and for elements $a_1=((a_{11}^i,\ldots,a_{1n}^i),i\in I)D,\ldots,a_k=((a_{k1}^i,\ldots,a_{kn}^i),i\in I)D$ let

$$A\models r^A(((a_{11}^i,\ldots,a_{1n}^i),i\in I)D,\ldots,((a_{m1}^i,\ldots,a_{mn}^i),i\in I)D).$$

Then we have

$$I_0=\{i\in I|A_1^i\otimes\ldots\otimes A_n^i\models r^{A_1^i\otimes\ldots\otimes A_n^i}((a_{11}^i,\ldots,a_{1n}^i),\ldots,(a_{m1}^i,\ldots,a_{mn}^i))\}\in D.$$

But

$$A_1^i\otimes\ldots\otimes A_n^i\models r^{A_1^i\otimes\ldots\otimes A_n^i}((a_{11}^i,\ldots,a_{1n}^i),\ldots,(a_{m1}^i,\ldots,a_{mn}^i))$$

implies $A_j^i\models r^{A_j^i}(a_{1j}^i,\ldots,a_{mj}^i), j=1,\ldots,n$ for any $i\in I_0$. Hence

$$I_1=\{i\in I|A_1^i\models r^{A_1^i}(a_{1j}^i,\ldots,a_{mj}^i)\}\supseteq I_0,\ldots$$

$$\ldots,I_n=\{i\in I|A_n^i\models r^{A_n^i}(a_{1n}^i,\ldots,a_{mn}^i)\}\supseteq I_0,$$

therefore $I_1\in D,\ldots,I_n\in D$, thus

$$A_1\models r^{A_1}((a_{11}^i,i\in I)D,\ldots,(a_{m1}^i,i\in I)D),\ldots,A_n\models$$
$$r^{A_n}((a_{1n}^i,i\in I)D,\ldots,(a_{mn}^i,i\in I)D).$$

It follows from here that

$$A_1 \otimes \ldots \otimes A_n \models r^{A_1 \otimes \ldots \otimes A_n}(((a^i_{11}, i \in I), \ldots, (a^i_{1n}, i \in I))D, \ldots$$
$$\ldots, ((a^i_{mi}, i \in I), \ldots, (a^i_{mn}, i \in I))D)$$

or, as φ is obviously a surjective mapping, $\varphi(A) \models r^{\varphi(A)}(\varphi(a_1), \ldots, \varphi(a_m))$. Thus, φ is an epimorphism. Let us show that it is an isomorphism.

Let r^A be a basic m-relation of the algebraic system $A_1 \otimes \ldots \otimes A_n$, and for the images by φ of elements $a_1, \ldots, a_n \in A$ let $\varphi(A) \models r^{\varphi(A)}(\varphi(a_1), \ldots, \varphi(a_n))$, that is

$$A_1 \otimes \ldots \otimes A_n \models r^{A_1 \otimes \ldots \otimes A_n}(((a^i_{11}, i \in I)D, \ldots, (a^i_{1n}, i \in I)D), \ldots$$
$$\ldots, ((a^i_{m1}, i \in I)D, \ldots, (a^i_{mn}, i \in I)D)),$$

therefore

$$A_1 \models r^{A_1}((a^i_{11}, i \in I)D, \ldots, (a^i_{m1}, i \in I)D), \ldots, A_n \models$$
$$r^{A_n}((a^i_{1n}, i \in I)D, \ldots, (a^i_{mn}, i \in I)D).$$

Hence, we will get

$$I_1 = \{i \in I | A^i_1 \models r^{A_1}(a^i_{11}, \ldots, a^i_{m1})\} \in D, \ldots$$
$$\ldots, I_n = \{i \in I | A^i_n \models r^{A_1}(a^i_{1n}, \ldots, a^i_{mn})\} \in D.$$

As $I_1 \cap \ldots, \cap I_n \in D$ and

$$I_0 \supseteq I_1 \cap \ldots, \cap I_n = \{i \in I | A^i_1 \otimes \ldots \otimes A^i_n \models$$
$$r^{A^i_1 \otimes \ldots \otimes A^i_n}((a^i_{11}, \ldots, a^i_{1n}), \ldots, (a^i_{m1}, \ldots, a^i_{mn})),$$

it follows that $I_0 \in D$ and we get $A \models r^A(((a^i_{11}, \ldots, a^i_{1n}), i \in I)D, \ldots, ((a^i_{m1}, \ldots, a^i_{mn}), i \in I)D)$, that is, $A \models r^A(a_1, \ldots, a_n)$. □

Corollary. *The product of a finite number of multiplicatively closed classes is a multiplicatively closed class.*

Indeed, it follows from the proof of Lemma 1 that

$$\prod_{i \in I} A^i/D \cong (\prod_{i \in I} A^i_1/D) \otimes \ldots \otimes (\prod_{i \in I} A^i_n/D).$$

In particular, if filter D over I is a maximal filter, consisting only of the set I, then we have

$$\prod_{i \in I} A^i \cong \prod_{i \in I} A^i/D \cong (\prod_{i \in I} A^i_1/D) \otimes \ldots (\prod_{i \in I} A^i_n/D) \cong (\prod_{i \in I} A^i_1) \otimes \ldots \otimes (\prod_{i \in I} A^i_n).$$

Thus $\prod_{i \in I}(A^i_1 \otimes \ldots \otimes A^i_n) \cong \prod_{i \in I} A^i_1 \otimes \ldots \otimes \prod_{i \in I} A^i_n$ and therefore $\prod_{i \in I}(A^i_1 \otimes \ldots \ldots \otimes A^i_n) \in K$.

Lemma 2. *The product of a finite number of hereditary classes is a hereditary class.*

Proof. Let $K_i, i \in I$ be a set of hereditary classes and $K = K_1 \otimes \ldots \otimes K_n$ be their product, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system from Q with $A_i \in K_i, i = 1, \ldots n$, and B be a subsystem of system A. Then $B \subseteq \prod_{i=1}^n A_i$. Let $\pi_i : B \to A_i, i = 1, \ldots, n$, be the projective mappings from B on its components. We denote $B_i = \pi_i(B), i = 1, \ldots, n$. Then we have $B \subseteq \prod_{i=1}^n B_i$. Conversely, let $(b_1, \ldots, b_n) \in \prod_{i=1}^n B_i$, then for any $i = 1, \ldots, n$, there exists such an element $b'_i = (b_{i1}, \ldots, b_{in})$ in B that $b_{ii} = b_i$. We consider the projection operations $p_i^{A_i}, i = 1, \ldots, n$, defined by the following relations

$$p_i^{A_i}(a_1, \ldots, a_n) = a_i, \quad i = 1, \ldots, n.$$

Then the operation $p^A = (p_i^{A_i}, i = 1, \ldots, n)$, defined on the algebraic system A, corresponds to the family of operations $(p_i^{A_i}, i = 1, \ldots, n)$. As

$$p^A(b'_1, \ldots, b'_n) = (p_i^{A_i}(b_{i1}, \ldots, b_{in}), i = 1, \ldots, n) =$$

$$(b_{11}, \ldots, b_{nn}) = (b_1, \ldots, b_n) \in B,$$

finally we get $B = \prod_{i=1}^n B_i$ with $B_i \in K_i, i \in I$, and therefore $B \in K$. □

By Tarski-Los Theorem (see [4], a class K is a universal class if and only if the class K satisfies the following conditions:

(1) is hereditary,

(2) is ultrafilteredly closed.

Thus, by Lemmas 1 and 2 we get

Theorem 3. *The product of a finite number of universal classes is a universal class.*

Theorem 4. *If the classes of algebraic systems $K_1, \ldots, K_n$ are quasi-varieties, then their product $K_1 \otimes \ldots \otimes K_n$ is a quasi-variety.*

Proof. By Lemma 1

$$F(K_1 \otimes \ldots \otimes K_n) = (F(K_1), \ldots, (F(K_n)) = K_1 \otimes \ldots \otimes K_n,$$

and by Lemma 2

$$S(K_1 \otimes \ldots \otimes K_n) = (S(K_1), \ldots, (S(K_n)) = K_1 \otimes \ldots \otimes K_n.$$

As any quasi-variety K_i contains a unitary algebraic system E_i $(i = 1, \ldots, n)$, we get that the class $K_1 \otimes \ldots, \otimes K_n$ contains also the unitary algebraic system $E_1 \otimes \ldots, \otimes E_n$. Then, by Theorem 2, the product of quasi-varieties $K_1 \otimes \ldots, \otimes K_n$ is a quasi-variety. □

3 Product of varieties

Let $A = (A, \sigma)$ be an algebraic system of signature σ. It is worth reminding that the signature σ consists of a set of functional symbols σ^F, a set of predicative symbols σ^P and a function $\nu : \sigma^F Y \sigma^P \to \omega = \{0, 1, 2, \ldots\}$ that defines the arity of these symbols.

A subset $\theta \subseteq \prod_{r\in\sigma^P} A^{\nu(r)}$ is called a *congruence* on the algebraic system A (see [5]) if it satisfies the following properties:

(i) $\theta(\approx)$ is congruent on algebra (A, σ^F);

(ii) $A \models r(a_1, \ldots, a_{\nu(r)}) \Rightarrow (a_1, \ldots, a_{\nu(r)}) \in \theta(r)$;

(iii) $(a_1, \ldots, a_{\nu(r)}) \in \theta(r) \& (a_1, b_1) \in \theta(\approx) \& \ldots$
$\ldots \& (a_{\nu(r)}, b_{\nu(r)}) \in \theta(\approx) \Rightarrow (b_1, \ldots, b_{\nu(r)}) \in \theta(r), \forall r \in \sigma^P$.

The component $\theta(\approx)$ will be called an algebraic congruence on system A. The set of all congruences on algebraic system A will be denoted by $Con(A)$, and for a certain predicative symbol $r \in \sigma$ by $Con(A)(r)$ we will denote the set, consisting only of the components $\theta(r), \theta \in Con(A)$.

Relative to the inclusion $\subseteq$ the set $Con(A)$ is partially ordered and has the greatest element $\nabla = (\nabla(r) = A^{\nu(r)} | r \in \sigma^P)$. The intersection of any non-empty set $\{\theta_i, i \in I\}$ of congruences θ_i of system A is also a congruence of this system. Therefore, $Con(A)$ is a complete lattice, wherein the congruence $\triangle = (\triangle(r) = A^{\nu(r)} | r \in \sigma^P)$ is the lowest element, where $\triangle(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | A \models r^A(a_1, \ldots, a_{\nu(r)})\}$, and for any two elements α and β from $Con(A)$, the operations $\wedge, \vee$ are defined as follows:

$$\alpha \wedge \beta = (\alpha \cap \beta)(r) = (\alpha(r) \cap \beta(r) | r \in \sigma^P),$$

$$\alpha \vee \beta = \cap\{\gamma \in Con(A) | \alpha \subseteq \beta, \beta \subseteq \gamma\}.$$

We notice that the set $\theta \subseteq \prod_{r\in\sigma^P} A^{\nu(r)}$, with the components defined via the formula

$$\theta(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | (\exists b_1, \ldots, b_{\nu(r)} \in A)((a_1, b_1) \in (a \vee \beta)(\approx) \&$$

$$\ldots \& (a_{\nu(r)}, b_{\nu(r)}) \in (\alpha \vee) \beta)(\approx) \& (b_1, \ldots, b_{\nu(r)}) \in \alpha(r) \cup \beta(r)$$

is a congruence on system A. At the same time $\alpha \subseteq \theta$, $\beta \subseteq \theta$ and any other arbitrary congruence γ of system A, which contains α and β, also contains θ. Thus, $\theta = \alpha \vee \beta$ and therefore for any $r \in \sigma^P, r \neq \approx$ we have

$$(\alpha \vee \beta)(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | (\exists b_1, \ldots, b_{\nu(r)} \in A)((a_1, b_1) \in$$

$$(a \vee \beta)(\approx) \& (a_{\nu(r)}, b_{\nu(r)}) \in (\alpha \vee \beta)(\approx) \& (b_1, \ldots, b_{\nu(r)}) \in \alpha(r) \cup \beta(r)\}.$$

Lemma 3. *Let $K = K_1 \otimes \ldots \otimes K_n$ be a product of classes, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system of signature σ from K with $A_i \in K_i, i = 1, \ldots, n$. If $\theta_i \in Con(A_i)$, $i = 1, \ldots, n$, and we denote by θ a subset from $\prod_{r\in\sigma}(A_1 \otimes \ldots \otimes A_n)^{\nu(r)}$, whose components are defined as follows $\theta(r) = \{((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in (A_1 \otimes \ldots \otimes A_n)^m | (a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r)\}$, where r is any predicative m-symbol of signature σ of the algebraic system $A = A_1 \otimes \ldots \otimes A_n$, then θ is a congruence on system A.*

Proof. Indeed, the relation $\theta(\approx)$ is reflexive, symmetrical and transitive. Let now $((a_{11}, \ldots, a_{1n}), (b_{11}, \ldots, b_{1n}) \in \theta(\approx), \ldots, ((a_{m1}, \ldots, a_{mn}), (b_{m1}, \ldots, b_{mn})) \in \theta(\approx)$ and $f^A = (f^{A_i}, i = 1, \ldots, n)$ be an m-operation of the algebraic system A. As $(a_{1i}, b_{1i}) \in \theta(\approx), \ldots, (a_{mi}, b_{mi}) \in \theta(\approx), i = 1, \ldots, n$, we get

$$(f^{A_i}(a_{1i}, \ldots, a_{mi}), f^{A_i}(b_{1i}, \ldots, b_{mi})) \in \theta(\approx), i = 1, \ldots, n.$$

Then we have

$$((f^{A_1}(a_{11}, \ldots, a_{m1}), \ldots, f^{A_n}(a_{1n}, \ldots, a_{mn})), (f^{A_1}(b_{11}, \ldots, b_{m1}), \ldots$$
$$\ldots, f^{A_n}(b_{1n}, \ldots, b_{mn}))) = (f^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})),$$
$$f^A((b_{11}, \ldots, b_{1n}), \ldots, (b_{m1}, \ldots, b_{mn}))) \in \theta(\approx)$$

and, thus, $\theta(\approx)$ is closed relative to the operations of the system A. Hence, property (i) from the definition of congruence holds for θ. Let now the relation $r^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mm}))$ hold in the algebraic system A, then we have

$$A_1 \models r^{A_1}(a_{11}, \ldots, a_{m1}), \ldots, A_n \models r^{A_n}(a_{1n}, \ldots, a_{mn}),$$

which implies $((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r)$. It follows from here that $((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r)$, therefore the property (ii) from the definition of congruence holds for θ. Finally, let

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r),$$

$$((a_{11}, \ldots, a_{1n}), (b_{11}, \ldots, b_{1n})) \in \theta(\approx), \ldots, ((a_{n1}, \ldots, a_{nm}), (b_{n1}, \ldots, b_{nm})) \in \theta(\approx).$$

Then we will have

$$(a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r),$$

$$a_{11} \equiv b_{11} mod(\theta_1), \ldots, a_{1n} \equiv b_{1n} mod(\theta_n), \ldots$$

$$\ldots, a_{n1} \equiv b_{n1} mod(\theta_1), \ldots, a_{nm} \equiv b_{nm} mod(\theta_n),$$

which implies $(b_{11}, \ldots, b_{m1}) \in \theta_1(r), \ldots, (b_{1n}, \ldots, b_{mn}) \in \theta_n(r)$. It follows from here that

$$((b_{11}, \ldots, b_{1n}), \ldots, (b_{n1}, \ldots, b_{nm}) \in \theta(r),$$

hence, property (iii) from the definition of congruence holds for θ. □

Lemma 4. *Let $K = K_1 \otimes \ldots \otimes K_n$ be a product of classes, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system from K with $A_i \subseteq K_i, i = 1, \ldots, n$. Then the following isomorphism holds*

$$Con(A_1 \otimes \ldots \otimes A_n) \cong Con(A_1) \times \ldots \times Con(A_n).$$

Proof. According to Lemma 3, an element θ of the set $Con(A)$ corresponds to every element $(\theta_1, \ldots, \theta_n)$ of the set $Con(A_1) \times \ldots \times Con(A_n)$ and we will denote it by $\theta = \theta_1 \times \ldots \times \theta_n$. Conversely, let $\theta \in Con(A)$. Then we will show that such congruences $\theta_i \in Con(A_i), i = 1, \ldots, n$, can be found that $\theta = \theta_1 \times \ldots \times \theta_n$, it means that for any predicative m-symbol r of the signature of system A we will have

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta(r) \Leftrightarrow \&_{i=1}^{n}(a_{1i}, \ldots, a_{mi}) \in \theta_i(r) \Leftrightarrow$$

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta_1(r) \times \ldots, \times\theta_n(r).$$

To prove this we define $\theta_1(r), \ldots, \theta_n(r)$, so that

$$\theta_1(r) = \{(a_{11}, \ldots, a_{m1}) \in A_i^m | (\exists b_{12}^1, \ldots, b_{m2}^1 \in A_2, \ldots, \exists b_{1n}^1, \ldots, b_{mn}^1 \in A_n)$$

$$((a_{11}, b_{12}^1, \ldots, b_{1n}^1), \ldots, (a_{m1}, b_{m2}^1, \ldots, b_{mn}^1)) \in \theta(r)\},$$

$$\theta_2(r) = \{(a_{12}, \ldots, a_{m2}) \in A_2^m | (\exists b_{11}^2, \ldots, b_{m1}^2 \in A_1,$$

$$\exists b_{13}^2, \ldots, b_{m3}^2 \in A_3, \ldots, \exists b_{1n}^2, \ldots, b_{mn}^2 \in A_n)$$

$$((b_{11}^2, a_{12}, b_{13}^2, \ldots, b_{1n}^2), \ldots, (b_{m1}^2, a_{m2}, b_{m3}^2, \ldots, b_{mn}^2)) \in \theta(r)\}, \ldots,$$

$$\theta_n(r) = \{(a_{1n}, \ldots, a_{mn}) \in A_n^m | (\exists b_{11}^n, \ldots,$$

$$b_{m1}^n \in A_1, \ldots, \exists b_{1n-1}^n, \ldots, b_{mn-1}^n \in A_n)$$

$$((b_{11}^n, \ldots, b_{1n-1}^n, a_{1n}), \ldots, (b_{m1}^n, \ldots, b_{mn-1}^n, a_{mn})) \in \theta(r)\}.$$

First, if r coincides with the binary predicative symbol $\approx$, we notice that the relation $\theta_1(\approx)$ is reflexive and symmetrical. Let us show that it is also transitive. Let $(a, a') \in \theta_1(\approx)$ and $(a', a'') \in \theta_1(\approx)$. Then for any $i = 1, \ldots, n$, in A there exist such elements $b_i^1, b_i^2, c_i^1, c_i^2$ that $((a, b_2^1, \ldots, b_n^1),\ (a', b_2^2, \ldots, b_n^2)) \in \theta_1(\approx)$ and $((a', c_2^1, \ldots, c_n^1), (a'', c_2^2, \ldots, c_n^2)) \in \theta_1(\approx)$. As $\theta_1(\approx)$ is stable relative to the operation $p = (p_1^{2A}, p_2^{2A})$, where $p_i^{2A} = (p_i^{2A_1}, \ldots, p_i^{2A_n}),\ i = 1, 2$, we will have

$$p(((a, b_2^1, \ldots, b_n^1), (a', b_2^2, \ldots, b_n^2)), (a', c_2^1, \ldots, c_n^1), (a'', c_2^2, \ldots, c_n^2))) =$$

$$= (p_1^{2A}((a, b_2^1, \ldots, b_n^1), (a', c_2^1, \ldots, c_n^1)), p_2^{2A}((a', b_2^2, \ldots, b_n^2), (a'', c_2^2, \ldots, c_n^2))) =$$

$$= ((p_1^{2A_1}(a, a'), p_1^{2A_2}(b_2^1, c_2^1), \ldots, p_1^{2A_n}(b_n^1, c_n^1)), (p_2^{2A_1}(a', a''), p_2^{2A_2}(b_2^2, c_2^2), \ldots$$

$$\ldots, p_2^{2A_n}(b_n^2, c_n^2))) = ((a, b_2^1, \ldots, b_n^1), (a'', c_2^2, \ldots, c_n^2)) \in \theta(\approx),$$

resulting that $(a, a') \in \theta(\approx)$. If f^{A_1} is a certain m-operation of the system A_1 and $(a_1, a_1'') \in \theta_1(\approx), \dots, (a_m, a_m'') \in \theta_1(\approx)$, then we will take the basic operation $f^A = (f^{A_1}, p_1^{mA_2}, \dots, p_1^{mA_n})$ of the algebraic system A. As for certain $b_2^1, \dots, b_2^m, b_2'^1, \dots, b_2'^m \in A_2, \dots, b_n^1, \dots \dots, b_n^m, b_n'^1, \dots, b_n'^m \in A_n$

$$(a_1, b_2^1, \dots, b_n^1) \equiv (a_1', b_2'^1, \dots, b_n'^1) \mathrm{mod} \theta(\approx), \dots$$

$$\dots, (a_m, b_2^m, \dots, b_n^m) \equiv (a_1', b_2'^m, \dots, b_n'^m) \mathrm{mod} \theta(\approx),$$

we will have

$$f^A((a_1, b_2^1, \dots, b_n^1), \dots, (a_m, b_2^m, \dots, b_n^m)) \equiv$$
$$f^A((a_1', b_2'^1, \dots, b_n'^1), \dots, (a_m', b_2'^m, \dots, b_n'^m)) \mathrm{mod} \theta(\approx),$$

that is

$$(f^{A_1}(a_1, \dots, a_m), b_2^1, \dots, b_n^1) \equiv (f^{A_1}(a_1', \dots, a_m'), b_2'^1, \dots, b_n'^1) \mathrm{mod} \theta(\approx),$$

resulting that $(f^{A_1}(a_1, \dots, a_m), f^{A_1}(a_1', \dots, a_m') \in \theta_1(\approx)$. Hence $\theta_1(\approx)$ is stable relative to the operations of system A_1, that is, $\theta_1(\approx)$ is a congruence on algebra A.

Let now r be a predicative symbol of arity m of the signature of system A_1 and $x_{11}, \dots, x_{m1}$ be such elements from A_1 that the real relation $A_1 \models r^{A_1}(x_{11}, \dots, x_{m1})$ or $(x_{11}, \dots, x_{m1}) \in \theta_1(\approx)$ holds, and $(x_{11}, y_1) \in \theta_1(\approx), \dots, (x_{m1}, y_m) \in \theta_1(\approx)$ holds for certain elements $y_1, \dots, y_m \in A_1$. We consider the basic predicate $r^A = (r^{A_1}, r^{A_2}, \dots, r^{A_n})$ of the algebraic system A, where $r^{A_2}, \dots, r^{A_n}$ are real basic predicates on the systems $A_2, \dots, A_n$. Then for any elements $x_{12}, \dots, x_{m2} \in A_2, \dots, x_{1n}, \dots, x_{mn} \in A_n$ we have $A_2 \models r^{A_2}(x_{12}, \dots, x_{m2}) \Rightarrow (x_{12}, \dots, x_{m2}) \in \theta_1(r), \dots, A_n \models r^{A_n}(x_{1n}, \dots, x_{mn}) \Rightarrow (x_{1n}, \dots, x_{mn}) \in \theta_n(r)$, and if we have $A_1 \models r^{A_1}(x_{11}, \dots, x_{m1})$ then we get $A \models r^A((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}))$, which implies that $((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})) \in \theta(\approx)$; if we have $(x_{11}, \dots, x_{m1}) \in \theta_1(r)$ then obviously we will also get $((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}) \in \theta(r)$, for certain elements $x_{12}, \dots, x_{m2} \in A_2, \dots, x_{1n}, \dots, x_{mn} \in A_n$. As $(x_{11}, y_1) \in \theta_1(\approx), \dots, (x_{m1}, y_m) \in \theta_1(\approx)$ we have $((x_{11}, x_{12} \dots, x_{1n}), (y_1, x_{12} \dots, x_{1n})) \in \theta(\approx), \dots, ((x_{m1}, x_{m2}, \dots, x_{mn}), (y_m, x_{m2}, \dots, x_{mn})) \in \theta(\approx)$, then from $((x_{11}, \dots \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})) \in \theta(r)$, follows $((y_1, x_{12}, \dots, x_{1n}), \dots, (y_m, x_{m2}, \dots \dots, x_{mn})) \in \theta(r)$, and from here $(y_1, \dots, y_m) \in \theta_1(r)$. Therefore, θ_1 is a congruence on the algebraic system A_1. It can be shown by analogy that $\theta_2, \dots, \theta_n$ are congruences on the systems $A_2, \dots, A_n$, respectively. At the same time we notice that $\theta \subseteq \theta_1 \times \dots \times \theta_n$, that is $\theta(r) \subseteq \theta_1(r) \times \dots \times \theta_n(r)$ for any predicative symbol r of the signature of system A.

The inverse inclusion also takes place. We will also show that $(\theta_1 \times \dots \dots \times \theta_n)(\approx) \subseteq \theta(\approx)$. Indeed, let

$$((a_{11}, \dots, a_{1n}), (a_{21}, \dots, a_{2n})) \in (\theta_1 \times \dots \times \theta_n)(\approx).$$

Then $(a_{11}, a_{21}) \in \theta_1(\approx), \dots, (a_{1n}, a_{2n}) \in \theta_n(\approx)$. It follows from here that according to the constructions of $\theta_1(\approx), \dots, \theta_n(\approx)$, we will have

$$((a_{11}, b_{12}^1, \dots, b_{1n}^1), (a_{21}, b_{22}^1, \dots, b_{2n}^1)) \in \theta(\approx),$$

$$((b_{11}^2, a_{12}, b_{13}^2, \dots, b_{1n}^2), (b_{m1}^2, a_{22}, b_{m3}^2, \dots, b_{mn}^2)) \in \theta(\approx), \dots$$
$$\dots, ((b_{11}^n, \dots, b_{1n-1}^n, a_{1n}), (b_{21}^n, \dots, b_{2n-1}^n, a_{2n})) \in \theta(\approx),$$

for certain elements

$$b_{12}^1, b_{22}^1 \in A_2, \dots, b_{1n}^1, b_{2n}^1 \in A_n, b_{11}^2, b_{21}^2 \in A_1, b_{13}^2, b_{23}^2 \in A_3, \dots$$
$$\dots, b_{1n}^2, b_{2n}^2 \in A_n, \dots, b_{11}^n, b_{21}^n \in A_1, \dots, b_{1n-1}^n, b_{2n-1}^n \in A_{n-1}.$$

Applying the projection operation $p = p_1^{nA}, p_2^{nA}, \dots, p_n^{nA})$ for the last elements from $\theta(\approx)$ we get $((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, \dots, a_{2n})) \in \theta(\approx)$. Hence, $\theta(\approx) \supseteq (\theta_1 \times \dots \dots \times \theta_n)(\approx)$ and thus $\theta(\approx) = (\theta_1 \times \dots \times \theta_n)(\approx)$.

Let now $r \neq \approx$ be a predicative m-symbol of the signature of the algebraic system A and let

$$((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})) \in (\theta_1 \times \dots \times \theta_n)(r),$$

thus

$$(x_{11}, \dots, x_{m1}) \in \theta_1(r), \dots, (x_{1n}, \dots, x_{mn}) \in \theta_n(r).$$

By the constructions of the components $\theta_1(r), \dots, \theta_n(r)$, there exist such elements $b_{12}^1, \dots, b_{m2}^1 \in A_2, \dots, b_{1n}^1, \dots, b_{m2}^1 \in A_n$ that

$$((a_{11}, b_{12}^1, \dots, b_{1n}^1), (a_{11}, \dots, a_{1n})) \in \theta(\approx), \dots$$
$$\dots, ((a_{m1}, b_{m2}^1, \dots, b_{mn}^1), (a_{m1}, \dots, a_{mn})) \in \theta(\approx)$$

and $((a_{11}, b_{12}^1, \dots, b_{1n}^1), \dots, (a_{m1}, b_{m2}^1, \dots, b_{mn}^1)) \in \theta(r)$, at the same time. From here, by the definition of congruence we get $((a_{11}, \dots, a_{1n}), \dots, (a_{m1}, \dots, a_{mn})) \in \theta(r)$. Hence, $\theta_1 \times \dots \times \theta_n \subseteq \theta$ and we get the equality $\theta_1 \times \dots \times \theta_n = \theta$.

Now it is easy to see that the bijective correspondence, defined by $\theta = \theta_1 \times \dots \dots \times \theta_n \to (\theta_1 \dots \theta_n)$, defines the isomorphism we are looking for, meaning that it satisfies the following properties:

$$(\theta \wedge \theta')(r) \to ((\theta_1 \wedge \theta_1')(r), \dots, (\theta_n \wedge \theta_n')(r)),$$
$$(\theta \vee \theta')(r) \to ((\theta_1 \vee \theta_1')(r), \dots, (\theta_n \vee \theta_n')(r))$$

for any congruences $\theta = \theta_1 \times \dots \times \theta_n$ and $\theta = \theta_1' \times \dots \times \theta_n'$ from $Con(A)$. □

Lemma 5. *The product of a finite number of homomorphically closed classes is a homomorphically closed class.*

Proof. Let $K_i, i = 1, \dots, n$, be a family of homomorphically closed classes and $K = K_1 \otimes \dots \otimes K_n$ be their product, $A = A_1 \otimes \dots \otimes A_n$ – an algebraic system with $A_1 \subseteq K_i, i = 1, \dots, n$, and C – an algebraic system from Q. We will show that an epimorphism $\varphi : A_1 \otimes \dots \otimes A_n \to C$ exists if and only if there exist epimorphisms $\varphi_i : A_i \to C_i, i = 1, \dots, n$ and the isomorphism $C \cong C_1 \otimes \dots \otimes C_n$ takes place. From here we will get that $H(K) = H(K_1) \otimes \dots \otimes H(K_n)$ and hence $H(K) = H$.

Let epimorphism $\varphi : A_1 \otimes \ldots \otimes A_n \to C$ exist and let $\theta = \ker(\varphi)$ be its kernel. By Lemma 4

$$\theta = \theta_1 \otimes \ldots \otimes \theta_n,$$

where $\theta_i \in Con(A_i), i = 1, \ldots, n$. We consider the canonical isomorphisms $\varphi_i : A_i \to A_i/\theta_i, i = 1, \ldots, n$. The mapping

$$\varphi^*(a_1, \ldots, a_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))(a_i \in A_i, i = 1, \ldots, n)$$

is a homomorphism from $A_1 \otimes \ldots \otimes A_n$ to $(A_1/\theta_1) \otimes \ldots \otimes (A_n/\theta_n)$. Let $\theta^* = \ker(\varphi^*)$. We show that $\theta = \theta^*$ and then we will get

$$C \cong (A_1 \otimes \ldots \otimes A_n)/\theta \cong (A_1 \otimes \ldots \otimes A_n)/\theta^* \cong (A_1/\theta_1) \otimes \ldots \otimes (A_n/\theta_n).$$

If r is an arbitrary predicative m-symbol of the signature of system A and

$$r^{\varphi^*(A_1 \otimes \ldots \otimes A_n)}(\varphi^*(a_{11}, \ldots, a_{1n}), \ldots, \varphi^*(a_{m1}, \ldots, a_{mn})) =$$

$$= r^{\varphi_1(A_1) \otimes \ldots \otimes \varphi_n(A_n)}((\varphi_1(a_{11}), \ldots, \varphi_n(a_{1n})), \ldots, (\varphi_1(a_{m1}), \ldots, \varphi_n(a_{mn})) \Rightarrow$$

$$\Rightarrow r^{\varphi_1(A_1)}(\varphi_1(a_{11}), \ldots, \varphi_1(a_{1n}))\& \ldots \& r^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \ldots, \varphi_n(a_{mn})),$$

then $(a_{11}, \ldots, a_{1n}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r)$, resulting that

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in (\theta_1 \times \ldots \times \theta_n)(r) = \theta(r).$$

Conversely, if

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta(r),$$

then

$$(a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r),$$

resulting that

$$\varphi_1(A_1) \models r^{\varphi_1(A_1)}(\varphi_1(a_{11}), \ldots, \varphi_1(a_{m1})), \ldots$$
$$\ldots, \varphi_n(A_n) \models r^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \ldots, \varphi_n(a_{mn})),$$

therefore

$$\varphi^*(A) \models r^{\varphi^*(A)}(\varphi^*(a_{11}, \ldots, a_{1n}), \ldots, \varphi^*(a_{1n}, \ldots, a_{mn})).$$

It follows from here that $\theta(r) \subseteq \theta^*(r)$. Hence, $\theta = \theta^*$.

Conversely, let us have the r-epimorphisms $\varphi_i : A_i \to C_i, i = 1, \ldots, n$. We show that the mapping

$$\varphi(a_1, \ldots, a_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))(a_1 \in A_1, \ldots, a_n \in A_n)$$

is homomorphism from $A_1 \otimes \ldots \otimes A_n$ on $C_1 \otimes \ldots \otimes C_n$.

We consider a basic operation f^A of arity k of the algebraic system $A = A_1 \otimes \ldots \otimes A_n$. By its definition, we have

$$f^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{k1}, \ldots, a_{kn})) =$$

$$= (f^{A_1}(a_{11}, \dots, a_{1n}), \dots, f^{A_n}(a_{k1}, \dots, a_{kn}))$$

for all $a_{1i}, \dots, a_{ki}$ from A_i $(i = 1, \dots, n)$. It follows from here that

$$\varphi(f^A((a_{11}, \dots, a_{1n}), \dots, (a_{k1}, \dots, a_{kn}))) =$$

$$= (\varphi_1(f^{A_1}(a_{11}, \dots, a_{1n})), \dots, \varphi_n(f^{A_n}(a_{k1}, \dots, a_{kn}))) =$$

$$= (f^{\varphi_1(A_1)}(\varphi_1(a_{11}), \dots, \varphi_1(a_{k1})), \dots, f^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \dots, \varphi_n(a_{kn})) =$$

$$= f^{\varphi(A)}(\varphi(a_{11}, \dots, a_{n1}), \dots, \varphi(a_{k1}, \dots, a_{kn})).$$

Let now $r^A = (r^{A_1}, \dots, r^{A_n})$ be a basic m-relation of the algebraic system A and let

$$A \models r^A((a_{11}, \dots, a_{1n}), \dots, (a_{m1}, \dots, a_{mn})).$$

Then we have

$$A_i \models r^{A_i}(a_{1i}, \dots, a_{mi}),\ i = 1, \dots, n.$$

Hence, we will get

$$\varphi_i(A_i) \models r^{\varphi_i(A_i)}(\varphi_i(a_{1i}), \dots, \varphi_i(a_{mi})),\ i = 1, \dots, n.$$

Therefore

$$\varphi(A) \models r^{\varphi(A)}(\varphi(a_{1i}, \dots, a_{1n}), \dots, \varphi(a_{m1}, \dots, a_{mn})).$$ □

Theorem 5. *If the classes of algebraic systems $K_1, \dots, K_n$ are varieties, then their product $K_1 \otimes \dots \otimes K_n$ is a variety.*

Proof. By Gorollary of Lemma 1

$$P(K_1 \otimes \dots \otimes K_n) = (P(K_1), \dots, P(K_n)) = K_1 \otimes \dots \otimes K_n.$$

By Lemma 2

$$S(K_1 \otimes \dots \otimes K_n) = (S(K_1), \dots, S(K_n)) = K_1 \otimes \dots \otimes K_n,$$

and by Lemma 5

$$H(K_1 \otimes \dots \otimes K_n) = (H(K_1), \dots, H(K_n)) = K_1 \otimes \dots \otimes K_n.$$

Hence, by Theorem 2, the product of varieties $K_1 \otimes \dots \otimes K_n$ is a variety. □

4 Observations

a) We will say that the algebraic system A with signature σ is *isomorphically embedded* in the algebraic system A' with the signature σ' if there exists a mapping $\varphi : A \to A'$ and an injective mapping $\alpha : \sigma \to \sigma'$ that makes a single function n-symbol $f \in \sigma$ and a single predicative m-symbol $r \in \sigma$ *to correspond to each functional n-symbol* $f' \in \sigma'$ and each predicative m-symbol $r' \in \sigma'$, so that

$$\varphi(f^A(a_1, \dots, a_n)) = f'^{A'}(\varphi(a_1), \dots, \varphi(a_n))$$

and

$$r^A(a_1, \dots, a_m) = r'^{A'}(\varphi(a_1), \dots, \varphi(a_m))$$

for any elements $a_1, a_2, \dots$ from A.

If K and K' are two classes of algebraic systems of signatures σ and σ' respectively, then we will say that the class K is isomorphically embedded in the class K' if any algebraic system from the class K is isomorphically embedded in one of the systems of class K'.

It is easy to realize that: *any product of classes of algebras is isomorphically embedded in the Taylor product of the same classes of algebras.*

It is also easy to show that: *if any class $Q_i (i \in I)$ of algebraic systems contains a unitary algebraic system E_i, then any class Q_i is isomorphically embedded in the product of classes $Q = \otimes_{i \in I} Q_i$. In particular, if all classes $Q_i, i \in I$, are quasi-varieties (or varieties), then any quasi-variety Q_i is isomorphically embedded in the product of quasi-varieties $Q = \otimes_{i \in I} Q_i$.*

Indeed, if A_i is an arbitrary algebraic system from class Q_i $(i \in I)$, then obviously the system A_i is isomorphically embedded in the algebraic system $\otimes_{j \in J} A_j \in Q$, where $A_j = E_j$ for any $j \in I \backslash \{i\}$.

b) The product of classes of algebraic systems can be also extended for the case of an infinite number of classes, obtaining the same results as for the finite number of classes, which are proved analogously.

c) Let $K_i, i \in I = \{1, 2, \dots\}$ be an infinite totality of such classes of algebras that each class $K_i (i \in I)$ contains an algebra L_i that strictly contains a unitary subalgebra $E_i = \{e_i\}$. Then the algebra $L = \otimes_{i \in I} L_i$ belongs to the class $\otimes_{i \in I} K_i$. We consider the set M, consisting of such elements $a = (a(i), o \in I)$ from algebra L for which the sets $\{i \in I | a(i) \neq e_i\}$ are finite. It is obvious that M is closed relative to any operation of finite arity of algebra L, hence it is isomorphically embedded in L. But M is not a subalgebra of algebra L, because any subalgebra from L has the form $M_1 \otimes M_2 \dots$, where M_i is a subalgebra of L_i. Therefore, the product of an infinite number of classes of algebraic systems cannot be defined as a class of algebraic systems only with finite operations and predicates.

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Vasile I. Ursu
Institute of Mathematics "Simion Stoilov"
of the Romanian Academy
E-mail: vursu@imar.ro

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