BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(62), 2010, Pages 100–105
ISSN 1024–7696

On the action of differentiation operator in some classes of Nevanlinna-Djrbashian type in the unit disk and polydisk

Romi Shamoyan, Haiying Li

Abstract. We introduce new area Nevanlinna type spaces in the unit disk and polydisk and study the action of classical operator of differentiation on them. We substantially supplement the list of previously known assertions of this type.

Mathematics subject classification: 30D45.
Keywords and phrases: Holomorphic functions, differentiation operator, Nevanlinna-Djrbashian type analytic classes.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$, $\mathrm{T} = \{|z| = 1\}$ be the unit circle, $I^n = (0,1]^n, \mathrm{T}^n = \mathrm{T}\cdots\mathrm{T}$, $\mathbb{D}^n = \{z = (z_1, z_2, \cdots, z_n) : |z_j| < 1, j = 1, 2, \cdots, n\}$ be the unit polydisk, $H(\mathbb{D})$ be the space of all holomorphic functions in the unit disk, and let $H(\mathbb{D}^n)$ be the space of all holomorphic functions in the polydisk. Let $T(f,\tau)$ be the Nevanlinna characteristic of f, $f \in H(\mathbb{D})$ [1]. Let below always w be a function from the set of all positive slowly growing functions, $w \in L^1(0,1)$ such that there are two numbers $m_w > 0, M_w > 0$ and a number $q_w \in (0,1)$ such that $m_w \le \dfrac{w(\lambda\tau)}{w(\tau)} < M_w$, $\tau \in (0,1)$, $\lambda \in [q_w, 1]$ (see [7]). We define several subspaces of $H(\mathbb{D})$ for fixed function $w \in L^1(0,1]$, $w > 0$.

$$N^1_{p,w,\beta}=\left\{ f \in H(\mathbb{D}) : \sup_{0<R\le 1} \int_0^R (T(f,\tau))^p w(1-\tau)d\tau(1-R)^\beta < +\infty \right\},$$

$$N^2_{p,w,\alpha} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\sup_{\tau\in(0,R]} (T(f,\tau))^p w(1-\tau) \right] (1-R)^\alpha dR < +\infty \right\},$$

$$N^3_{p,q,w,\alpha} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left(\int_0^R (T(f,\tau))^p w(1-\tau)d\tau \right)^{\frac{q}{p}} (1-R)^\alpha dR < +\infty \right\},$$

$$N^4_{p,q,w} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left(\int_{-\pi}^{\pi} \ln^+ |f(\tau\xi)|^p d\xi \right)^{\frac{q}{p}} w(1-\tau)d\tau < +\infty \right\},$$

$$N^5_{p,q,w} = \left\{ f \in H(\mathbb{D}) : \int_{-\pi}^{\pi} \left(\int_0^1 \ln^+ |f(\tau\xi)|^p w(1-\tau) d\tau \right)^{\frac{q}{p}} d\xi < +\infty \right\},$$
$$N^p = \left\{ f \in H(\mathbb{D}) : \sup_{\tau<1} \int_{-\pi}^{\pi} (\ln^+ |f(\tau\xi)|)^p d\xi < \infty \right\},$$

where $0 < p, q < \infty$, $\alpha > -1$, $\beta \geq 0$.

Note that these are complete metric spaces which can be checked without difficulties.

It is obvious that for $q = \infty$, $w = 1$ the $N^4_{p,q,w}$ coincides with the well-known N^p spaces of holomorphic functions with bounded characteristic [5].

In recent papers [4, 5] it was noted that the following assertions concerning the action of differentiation $\mathcal{D}(f)(z) = f'(z)$ and integration $I(f)(z) = \int_0^z f(t)dt$ are valid in mentioned analytic classes. $N^4_{q,q,\alpha}$ is closed under differentiation and integration operator (if $w(|z|) = (1-|z|)^\alpha$ we denote $N^4_{p,q,w}$ by $N^4_{p,q,\alpha}$), $N^4_{q,q,w}$ and $N^4_{1,q,w}$ are closed under differentiation operator $\mathcal{D}(f)$ if and only if $\int_0^1 w(t)(\ln \frac{1}{t})^p dt < +\infty$. The study $I(f), \mathcal{D}(f)$ in Smirnov N^+ class were studied also earlier (see [6] and references there).

We note that much earlier in [2] Frostman then W. K. Hayman [3] established that the N^p class is not invariant under differentiation operator, but $N^p, p > 1$ are closed under integration operator, but not N^1.

The natural question is to study differentiation operator in $N^i_{p,w,\alpha}$, $i = 1, 2, 3, 4, 5$. The goal of this paper is to provide several new sharp results in this direction. Finally we would like to indicate that all assertions of this note were obtained by modification of approaches and arguments provided recently in [4]. All our results in higher dimension were obtained for $n = 1$ in [4]. Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

2 Main results

Motivated by the mentioned above results in this section we provide new assertions concerning differentiation operator $\mathcal{D}(f)$ in new Nevanlinna-Djrbashian type spaces that were defined above. In the following assertion, we provide several sharp results on the action of the differentiation operator in Nevanlinna type analytic spaces in the unit disk complementing previously known propositions of this type obtained earlier by various authors (see, for example, [2–6] and references there).

Theorem 1. *1) $N^1_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if*

$$\sup_{R\in(0,1)} (1-R)^\alpha \int_0^R \left(\ln \frac{1}{1-\tau} \right)^p w(1-\tau) d\tau < \infty, \; 0 < p < \infty, \; \alpha \geq 0.$$

2) $N^2_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1 \sup_{R<\tau} w(1-R)\left(\ln\frac{1}{1-R}\right)^p (1-\tau)^\alpha d\tau < \infty,\ 0<p<\infty,\ \alpha>-1.$$

3) $N^3_{p,q,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1 \left(\int_0^R w(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^p d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR < \infty,\ 0<p,q<\infty,\ \alpha>-1.$$

In the following theorem we provide sharp assertions concerning the operator of Differentiation in $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$.

Theorem 2. *$\mathcal{D}(f)$ is acts from $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$ to $N^1_{s,s,w}$,*

$$\widetilde{w}(1-|z|) = w(1-|z|)^{\frac{q}{s}}(1-|z|)^{\frac{2q}{s}-\frac{q}{p}-1},\ \frac{2}{s}-\frac{1}{p}>0,\ s\geq 1,\ s\geq \max\{q,p\}$$

if and only if

$$\int_0^1 \left(\ln\frac{1}{t}\right)^s w(t)dt < \infty.$$

Now we formulate some new sharp results in higher dimensions. Let always below for any function $f\in H(\mathbb{D}^n)$,

$$\mathcal{D}f(z) = \frac{\partial f(z_1,z_2,\dots,z_n)}{\partial z_1,\dots,\partial z_n}.$$

Note that Nevanlinna type classes in higher dimension were studied also earlier see for example [10] and references there.

Theorem 3. *Let $0<p<\infty$, $\int_0^1 w_j(t)dt < +\infty$, $j=1,2,\dots,n$. Then*

$$\int_{I^n}\left(\int_{\mathrm{T}^n} \ln^+ \left|\mathcal{D}f(\tau_1\xi_1,\dots,\tau_n\xi_n)\right| d\xi_1\dots d\xi_n\right)^p \Pi_{j=1}^n w_j(1-\tau_i)d\tau_1\dots d\tau_n \leq$$
$$\leq C\int_{I^n}\left(\int_{\mathrm{T}^n} \ln^+ |f(\tau_1\xi_1,\dots,\tau_n\xi_n)| d\xi_1\dots d\xi_n\right)^p \Pi_{j=1}^n w_j(1-\tau_i)d\tau_1\dots d\tau_n,$$
$$\overrightarrow{\tau} = (\tau_1,\dots,\tau_n), \tau_i\in(0,1).$$

if and only if

$$\int_0^1 w_j(t)\left(\ln\frac{1}{t}\right)^p dt < +\infty,\ j=1,2,\dots,n.$$

Theorem 4. *Let* $s \geq 1, s \geq \max\{q,p\}, w = \Pi_{j=1}^{n} w_j$. *Let*

$$\frac{2}{s} - \frac{1}{p} > 0,\ \widetilde{w_j}(1-|z_j|) = w_j(1-|z_j|)^{\frac{q}{s}}(1-|z_j|)^{\frac{2q}{s}-\frac{q}{p}-1}.$$

Then $\mathcal{D}f$ *is acts from* $N^4_{p,q,\widetilde{w}}(N^5_{p,q,\widetilde{w}})$ *to* $N^1_{s,s,w}$ *if and only if*

$$\int_0^1 w_j(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^s d\tau_1\dots d\tau_n < +\infty,\ j=1,2,\dots,n,$$

where

$$N^4_{p,q,w}(\mathbb{D}^n) = \left\{ f\in H(\mathbb{D}^n) : \int_{\mathrm{T}^n}\left(\int_{I^n}\ln^+|f(\tau\xi)|^p \Pi_{k=1}^n w(1-\tau_k)d\tau\right)^{\frac{q}{p}} d\xi < +\infty\right\},$$

$$N^5_{p,q,w}(\mathbb{D}^n) = \left\{ f\in H(\mathbb{D}^n) : \int_{I^n}\left(\int_{\mathrm{T}^n}\ln^+|f(\tau\xi)|^p d\xi\right)^{\frac{q}{p}} \times\right.$$

$$\left.\times\Pi_{k=1}^n w(1-\tau_k)d\tau_1\dots d\tau_n < +\infty\right\}.$$

Let us mention some lemmas that are needed for the proofs.

Lemma 1. *The following estimates are true.*

1)

$$\int_{\mathrm{T}^n}\ln^+|\mathcal{D}f(\tau_1\varphi_1,\dots,\tau_n\varphi_n)|d\varphi_1\dots d\varphi_n \leq$$

$$\leq C\left(\left(\sum_{j=1}^n \ln\frac{1}{1-\tau_j}\right) + \int_{\mathrm{T}^n}\ln^+|f(\overrightarrow{\tau}\xi)|dm_n(\xi)\right),\ \overrightarrow{\tau} = \left(\frac{1+\tau_1}{2},\dots,\frac{1+\tau_n}{2}\right),$$

$$\tau_i\in(0,1), i=1,\dots,n;$$

2)

$$\ln^+ T\left(\frac{1+\tau}{2}, f\right) \leq CT\left(\frac{1+\tau}{2}, f\right), \tau\in(0,1),$$

$$T(f,R) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(R\xi)|d\xi, R\in(0,1).$$

Lemma 2. *Let* $\lambda_k = 2^{\lambda_k}$, $\lambda > 0$, $\tau_n = \exp\left(-\frac{1}{2^{n\lambda}}\right)$. *Then for* $\varphi\in[0,2\pi]$, *there exist a function* $f, f\in H(\mathbb{D})$,

$$\ln^+|f'(\tau_n e^{i\varphi})| \geq C\ln\frac{1}{1-\tau_n},\ f(z) = \sum_{k=0}^{\infty}\lambda_k^{\alpha-1}z^{\lambda_k}, 0<\alpha<1, \lambda>0.$$

Lemma 3. *1) Let* $R_{m_j} = \exp\left(-\frac{1}{2^{\lambda m_j}}\right) \in (0,1]$, $t \in (0,+\infty)$, $\lambda > 0$, $j = 1,2,\dots,n$. *Then there exists a function* f, $f \in H(\mathbb{D}^n)$,

$$\left(\ln^+ |\mathcal{D}f(R_{m_1}e^{i\varphi_1},\dots,R_{m_n}e^{i\varphi_n})|\right)^t \geq C\sum_{j=0}^{n}\left(\ln\frac{1}{1-R_{m_j}}\right)^t,\ \varphi_i \in (0,2\pi].$$

2)

$$\int_{\mathrm{T}^n}\left(\ln^+ |\mathcal{D}f(\tau_1\xi_1,\dots,\tau_n\xi_n)|\right)^s d\xi_1\dots d\xi_n$$

is growing as a function of $\tau_1,\dots,\tau_n$ *for every* $s \geq 1$, $f \in H(\mathbb{D}^n)$.

Remark 1. The statements of Theorem 2 for $q = p = s$ were established in [4].

Remark 2. As W. Hayman shows in the unit disk there is a function so that $T(\tau, I(f)) > C\ln\frac{1}{1-\tau}$, $T(\tau,f) < C$, $\tau \in (0,1)$. Let X be any normed class $X \subset H(\mathbb{D})$ so that $\|f\|_{X(w)} \leq C\sup_\tau T(\tau,f)$. If for $f \in X(w)$, $I(f) \in X(w)$, then $\|\ln\frac{1}{1-\tau}\|_{X(w)} < +\infty$. As $X(w)$ we can obviously take any space $N^i_{p,q,w}, i = 1,2,3,4,5$ under some natural additional assumption on w.

Remark 3. It is not difficult to see that the statements of Theorem 1 and Theorem 2 remain true if we replace $\mathcal{D}$ operator by $\bigwedge(f)(z) = \sum_{k=0}^{n} f_k(z)\mathcal{D}^k(f)(z)$, where f_k are functions from $N^i_{p,q,w}$, $i = 1,2,3,4,5$. The same statement is true for $\widetilde{I}^k$.

Note that with the help of so-called slice functions technique in [4,9], some results of this paper can be even expanded to the unit ball.

References

[1] Nevanlinna R. *Analytic functions GITTL*. Moscow, 1991.

[2] Frostman O. *Sur les produits de Blaschke*. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar, 1942, **12(15)**, 169–182.

[3] Hayman W. K. *On the characteristic of functions meromorphic in the unit disk and of their integrals*. Acta Math., 1964, **112**, 181–214.

[4] Shamoyan F. A., Kursina I. N. *On the invariance of some classes of holomorphic functions under integral and differential operators, Investigations on linear operators and function theory*. Part 26, Zap. Nauchn. Sem. POMI, 255, POMI, St. Petersburg, 1998, 184–197.

[5] Choe B. R., Koo H., Smith W. *Carleson measures for the area Nevanlinna spaces and applications*. J. Anal. Math., 2008, **104**, 207–233.

[6] Yanagihara N., Kawase S. *On the characteristic of some classes of functions and of their integrals*. Proc. London Math. Soc., 1972, **25(3)**, 577–585.

[7] Seneta E. Regularly varying functions, Nauka, Moscow, 1985.

[8] Golʹdberg A. A. Ostrovskiĭ I. V. *The distribution of values of meromorphic functions*, Nauka, Moscow, 1970.

[9] Zhu K. *Spaces of holomorphic functions in the unit ball.* Graduate Texts in Mathematics, 226, Springer-Verlag, New York, 2005.

[10] Rudin W. *Function theory in polydisks*, W. A. Benjamin, New York, 1969.

Romi Shamoyan
Department of Mathematics
Bryansk State University
Bryansk 241050, Russia
E-mail: *rsham@mail.ru*

Received August 9, 2009

Haiying Li
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, P.R.China
E-mail: *tslhy2001@yahoo.com.cn*