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Free Moufang loops and alternative algebras

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Abstract. It is proved that any free Moufang loop can be embedded in to a loop of invertible elements of some alternative algebra. Using this embedding it is quite simple to prove the well-known result: if three elements of Moufang loop are bound by the associative law, then they generate an associative subloop. It is also proved that the intersection of the terms of the lower central series of a free Moufang loop is the identity and that a finitely generated free Moufang loop is Hopfian.

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This work offers another way of examining Moufang loops, and namely, with the help of alternative algebras. It is well known that for an alternative algebra A with unit the set $U(A)$ of all invertible elements of A forms a Moufang loop with respect to multiplication. It is known also that if L is a Moufang loop, then its loop algebra FL is not always alternative, i.e. the Moufang laws are not always true in FL [1]. These themes are stated in survey [2] and [3] in details.

However, let L be a free Moufang loop. It is shown that if we factor the loop algebra FL by some ideal I, then FL/I will be an alternative algebra and the loop L will be embedded in to the loop of invertible elements of algebra FL/I. This is a positive answer to the question raised in [4]: is it true that any Moufang loop can be imbedded into a homomorphic image of a loop of type $U(A)$ for a suitable unital alternative algebra A? The equivalent version of this question is: whether the variety generated by the loops of type $U(A)$ is a proper subvariety of the variety of all Moufang loops?

The findings of this paper also give a partial positive answer to a more general question (see, for example, [3]): is it true that any Moufang loop can be embedded into a loop of type $U(A)$ for a suitable unital alternative algebra A? A positive answer to this question was announced in [5]. Here, in fact, the answer to this question is negative: in [4] the author constructed a Moufang loop, which is not embedded into a loop of invertible elements of any alternative algebra.

Using this embedding it is quite simple to prove the well-known Moufang Theorem: if three elements of Moufang loop are bound by the associative law, then they generate an associative subloop. The Magnus Theorem for groups, stating that the intersection of the terms of the lower central series of a free group is the identity, is well known. This paper proves an analogous result for free Moufang loops. It also proves that a finitely generated free Moufang loop is Hopfian.

1 Preliminaries

A loop $(L, \cdot) \equiv L$ is called *IP-loop* if the laws ${}^{-1}x \cdot xy = yx \cdot x^{-1} = y$ are true in it, where ${}^{-1}xx = xx^{-1} = 1$. In IP- loops ${}^{-1}x = x^{-1}$ and $(xy)^{-1} = y^{-1}x^{-1}$. A loop is *Moufang* if it satisfies the law

$$x(y \cdot zy) = (xy \cdot z)y. \tag{1}$$

Every Moufang loop is an IP-loop. A subloop H of a loop L is called *normal* in L if

$$xH = Hx, \quad x \cdot yH = xy \cdot H, \quad H \cdot xy = Hx \cdot y \tag{2}$$

for every $x, y \in L$.

For elements x, y, z of a loop, the *commutator* (x, y) and the *associator* (x, y, z) are defined by

$$xy = (yx)(x, y), \quad (xy)z = (x(yz))(x, y, z). \tag{3}$$

The set of all elements z of a loop L which commute and associate with all elements of L, so that for all a, b in L, $(a, z) = 1$, $(z, a, b) = 1$, $(a, z, b) = 1$, $(a, b, z) = 1$ is a normal subloop $Z(L)$ of L, called its *center*.

If $Z_1(L) = Z(L)$, then the normal subloops $Z_{i+1}(L) : Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$ are inductively determined. A loop L is called *centrally nilpotent of class* n if its *upper central series* has the form $\{1\} \subset Z_1(L) \subset \ldots \subset Z_{n-1}(L) \subset Z_n(L) = L$.

If H is a normal subloop of a loop L, there is a unique smallest normal subloop M of L such that H/M is a part of the center of L/M, and we write $M = [H, L]$. From here it follows that M is the normal subloop of L generated by the set $\{(x, z), (z, x, y), (x, z, y), (x, y, z) | \forall z \in H, \forall x, y \in L\}$. The *lower central series* of L is defined by $L_1 = L$, $L_{i+1} = [L_i, L]$ $(i \geq 1)$ [2]. Consequently, L_{n+1} *is the normal subloop of* L *generated by the set* $\{(g, x), (g, x, y), (x, g, y), (x, y, g) | \forall g \in L_n, \forall x, y \in L\}$.

Let F be a field and L be a loop. Let us examine the *loop algebra* FL. This is a free F-module with the basis $\{q | q \in L\}$ and the product of the elements of this basis is determined as their product in loop L. Let H be a normal subloop of loop L. We denote the ideal of algebra FL, generated by the elements $1 - h$ $(h \in H)$ by ωH. If $H = L$, then ωL is called the *augmentation ideal* of algebra FL [2].

2 Embedding of free Moufang loops in to alternative algebras

Let us determine the homomorphism of F-algebras φ: $FL \to F(L/H)$ by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. Takes place

Lemma 1. *Let H, H_1, H_2 be normal subloops of loop L. Then*

1) $Ker\varphi = \omega H$;

2) $1 - h \in \omega H$ if and only if $h \in H$;

3) if the elements h_i generate the subloop H, then the elements $1 - h_i$ generate the ideal ωH; if $H_1 \neq H_2$, then $\omega H_1 \neq \omega H_2$; if $H_1 \subset H_2$, then $\omega H_1 \subset \omega H_2$; if $H = \{H_1, H_2\}$, then $\omega H = \omega H_1 + \omega H_2$;

4) $\omega L = \{\sum_{q\in L}\lambda_q q | \sum_{q\in L}\lambda_q = 0\}$*;*
5) $FL/\omega H \cong F(L/H), \quad \omega L/\omega H \cong \omega(L/H)$*;*
6) the augmentation ideal is generated as F*-module by the elements of the form* $1-q$ *(*$q\in L$*).*

Proof. 1) As the mapping φ is F-linear, then by (2) for $h\in H, q\in L$ we have $\varphi((1-h)q) = Hq - H(hq) = Hq - Hq = 0$, i.e. $\omega H \subseteq \mathrm{Ker}\varphi$. Let now $K = \{k_j | j\in J\}$ be a complete system of representatives of cosets of loop L modulo the normal subloop H and let $\varphi(r) = 0$. We present r as $r = u_1k_1 + \ldots + r_tk_t$, where $u_i = \sum_{h\in H}\lambda_h^{(i)}h, k_i\in K$. Then $0 = \varphi(r) = \varphi(u_1)\varphi(k_1) + \ldots + \varphi(u_t)\varphi(k_t) = (\sum_{h\in H}\lambda_h^{(1)})\varphi(k_1) + \ldots + (\sum_{h\in H}\lambda_h^{(t)})\varphi(k_t)$. As $\varphi(k_1),\ldots,\varphi(k_t)$ are pairwise distinct, then for all i $\sum_{h\in H}\lambda_h^{(i)} = 0$. Hence $-u_i = \sum_{h\in H}\lambda_h^{(i)}(1-h) - \sum_{h\in H}\lambda_h^{(i)} = \sum_{h\in H}\lambda_h^{(i)}(1-h)$ is an element from ωH. Consequently, $\mathrm{Ker}\varphi \subseteq \omega H$, and then $\ker\varphi = \omega H$.

2) If $q\notin H$, then $Hq \neq H$. Then $\varphi(1-q) = H - Hq \neq 0$, i.e. by 1) $1-q\notin \mathrm{Ker}\varphi = \omega H$.

3) Let elements $\{h_i\}$ generate subloop H and I be an ideal, generated by the elements $\{1-h_i\}$. Obviously $I\subseteq \omega H$. Conversely, let $g\in H$ and $g = g_1g_2$, where g_1, g_2 are words from h_i. We suppose that $1-g_1, 1-g_2\in I$. Then $1-g = (1-g_1)g_2 + 1 - g_2 \in I$, i.e. $\omega H\subseteq I$. Hence $I = \omega H$. Let $H_1\neq H_2$ (respect. $H_1\subset H_2$) and $g\in H_1, g\notin H_2$. Then by 1) $1-g\in\omega H_1$, but $1-g\notin\omega H_2$. Hence $\omega H_1 \neq \omega H_2$ (respect. $\omega H_1\subset\omega H_2$). If $H = \{H_1, H_2\}$, then by the first statement of 3) $\omega H = \omega H_1 + \omega H_2$.

4) We denote $R = \{\sum_{q\in L}\lambda_q q | \sum_{q\in L}\lambda_q = 0\}$. Obviously, $\omega L\subseteq R$. Conversely, if $r\in R$ and $r = \sum_{q\in L}\lambda_q q$, then $-r = -\sum_{q\in L}\lambda_q q = (\sum_{q\in L}\lambda_q)1 - \sum_{q\in L}\lambda_q q = \sum_{q\in L}\lambda_q(1-q)\in\omega L$, i.e. $R\subseteq\omega Q$. Hence $\omega L = R$.

5) Mapping $\varphi: FL\to F(L/H)$ is a homomorphism of loop algebras and as by 1) $\mathrm{Ker}\varphi = \omega H$, then $FL/\omega H\cong F(L/H)$. Now from 4) it follows that $\omega L/\omega H\cong \omega(L/H)$.

6) As $(1-q)q' = (1-qq') - (1-q')$, then the augmentation ideal ωL is generated by elements of the form $1-q$, where $q\in L$. This completes the proof of Lemma 1. $\square$

Lemma 2. *Let* $(L,\cdot)$ *be an* IP*-loop and let* φ *be a homomorphism of the algebra* $(FL,+,\cdot)$*. Then the* A*-homomorphism image at* φ *of the loop* $(L,\cdot)$ *will be a loop.*

Proof. We denote the A-homomorphism at image φ of the loop $(L,\cdot)$ by $(\overline{L},\star)$. It follows from the IP-loop identity $x^{-1}\cdot xy = y$ that $\varphi(x^{-1}) = (\varphi x)^{-1}$ and $(\varphi x^{-1})\star(\varphi x\star\varphi y) = \varphi y$, $(\varphi x)^{-1}\star(\varphi x\star\varphi y) = \varphi y$, $\overline{x}^{-1}\star(\overline{x}\star\overline{y}) = \overline{y}$. Let $\overline{a},\overline{b}\in\overline{L}$. It is obvious that the equation $\overline{a}\star x = \overline{b}$ is always solvable and as $\overline{a}^{-1}\star(\overline{a}\star x) = \overline{a}^{-1}\star\overline{b}$, $x = \overline{a}^{-1}\star\overline{b}$, then it is uniquely solvable. It can be shown by analogy that the equation $y\star\overline{a} = \overline{b}$ is also uniquely solvable. Therefore, $(\overline{L},\star)$ is a loop, as required.

Now, before we pass to the presentation of the basic results, we give the construction of *free* IP*-loop* with the set of free generators $X = \{x_1, x_2, \ldots\}$, using ideas from [2]. To the set X we add the disjoint set $\{x_1^{-1}, x_2^{-1}, \ldots\}$. Let us examine all groupoid words $L(X)$ from the set $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots\}$ relative to the multiplication $(\cdot)$ and let e denote the empty word. For the words from $L(X)$ we define the *inverse words*: 1) for $x_i \in X$ the inverse will be x_i^{-1}, and for x_i^{-1} the inverse will be x_i, i.e. $(x_i^{-1})^{-1} = x_i$; 2) if $u \cdot v \in L(X)$, then $(u \cdot v)^{-1} = v^{-1} \cdot u^{-1}$. Further, we define two words u, v in $F(X)$ to be *Moufang-equivalent*, $u \approx v$ if one can be obtained from other by a sequence of substitutions, each of which replaces a subword $(rs \cdot r)t$ by $r(s \cdot rt)$ and vice-versa, where r, s, t are any words in $F(X)$. By a *contraction* μ of a word in $F(X)$ we mean the substitution at a subword of the form $u^{-1}(vw)$ or $(wv)u^{-1}$, where $u \approx v$, by w. The action ν, opposite to contraction μ we call the *expansion*.

For words w, w' in $F(X)$ we define the (μ, ν)-equivalence $w \cong w'$ if one word can be obtained from the other one by a finite sequence of substitutions, each of which is either a contraction μ or an expansion ν or a single use of the Moufang law (1). The relation $\cong$ will be, obviously, a relation of equivalence on $L(X)$. Moreover, it will be congruence, as if a word $(u_1 u_2 \ldots u_n)_\alpha$ is given when α is some parentheses distributions, obtained from words $u_1, u_2, \ldots, u_n$, then the replacement of the word u_i, $i = 1, 2, \ldots, n$, with words or equivalence can be realized applying to the given word a finite number of transformations of the above described form.

With the multiplication $\{u\} \cdot \{v\} = \{uv\}$ and the inverse $\{u\}^{-1} = \{u^{-1}\}$ of congruence classes we obtain a loop with the unity $\{e\}$, as the quotient loop $L(X)/\cong$ satisfies the laws $x^{-1} \cdot xy = y$, $yx \cdot x^{-1} = y$. Moreover, $L(X)/\cong$ will be a free Moufang loop on $\{x_i\}$, $i = 1, 2, 3, \ldots$, the set of free generators of X. We identify $\{x_i\}$ with x_i and we denote $L(X)/\cong$ by $L_X(\mathfrak{M})$.

Similarly to $F(X)$, we introduce the Moufang-equivalence, transformations μ, ν and (μ, ν)-equivalence for words in $L_X(\mathfrak{M})$. We define a word in $L(\mathfrak{M})$ to be a *reduced word* if no reductions of type μ of it are possible. If $w \in L(\mathfrak{M})$, then the number $l(w)$ of the variables in X, contained in w, will be called the *length* of the word w. Now let us show that if $w \to w_1$ and $w \to w_2$ are any reductions of type μ of a word w, then there is a word w_3 obtained from each of w_1, w_2 by a sequence of reductions of type μ. We use induction on the length of w. If $l(w) = 1$, w is already a reduced word. If $l(w) = n$ and $w = u \cdot v$ where u, v are subwords of w, then $l(u) < n$, $l(v) < n$. If both reductions $w \to w_1$ and $w \to w_2$ take place in the same subword, say u, then induction on length applied to u yields the result. If the two reductions take place in separate subwords, then applying both gives the w_3 needed. This leaves the last case where at least one of the reductions $w \to w_1$ and $w \to w_2$ involves both subwords u, v of w. Then w has, for example, the form $w = u^{-1}(uv)$. Therefore $w = v$ and thus $l(w) < n$, then by inductive hypothesis the statement is true.

Using this statement, one may prove by induction on length that any word w has reduced words regarding the reductions μ and all such reduced words belong to unique class of Moufang-equivalence. Then, induction on the number of reductions and expansions connecting a pair of congruent words shows that congruent words have the same reduced words.

Any word in $L_X(\mathfrak{M})$ has a reduced words. A *normal form* of a word u in $L_X(\mathfrak{M})$ is a reduced word of the least length. Clear by every word in $L_X(\mathfrak{M})$ has a normal form. Let $u(x_1, x_2, \dots, x_k)$, $u(y_1, y_2, \dots, y_n)$, where $x_i, y_j \in X \cup X^{-1}$, be two words of normal form of u of length $l(u)$. $L_X(\mathfrak{M})$ is a free loop. Assume, for example, $y_1 \notin \{x_1, x_2, \dots, x_k\}$, then $u(x_1, x_2, \dots, x_k) = u(1, y_2, \dots, y_n)$. The length of $u(1, y_2, \dots, y_n)$ is strict by less than $l(u)$. But this contradicts the minimum condition for $l(u)$. Consequently, all words of normal form of the same word in $L_X(\mathfrak{M})$ have the same free generators in their structure. This completes the proof of the following statement. $\square$

Lemma 3. *Any word in $L_X(\mathfrak{M})$ has a reduced word that belongs to the unique class of Moufang - equivalence, two words are (μ, ν)-equivalent if and only if they have the same reduced words and all words of normal form of the same word in $L_X(\mathfrak{M})$ have the same free generators in their structure.*

Now we consider a loop algebra FM of free Moufang loop $(M, \cdot) \equiv M$ over an arbitrary field F. Let $\overline{M} = \{\overline{u} = 1 - u | u \in M\}$ and we define the *circle composition* $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{u} \cdot \overline{v}$. Then $(M, \circ)$ is a loop, denoted sometimes as $\overline{M}$. The identity $\overline{1}$ of $\overline{M}$ is the zero of FM, $\overline{1} = 1 - 1$, and the inverse of $\overline{u}$ is $\overline{u}^{-1} = 1 - u^{-1}$ as $\overline{u} \circ \overline{1} = 1 - u + 0 - (1 - u)0 = 1 - u = \overline{u}$, $\overline{1} \circ \overline{u} = \overline{u}$, $\overline{u} \circ \overline{u}^{-1} = \overline{u} + \overline{u}^{-1} - \overline{u}\overline{u}^{-1} = 1 - u + 1 - u^{-1} - (1 - u)(1 - u^{-1}) = 0$, $\overline{u}^{-1} \circ \overline{u} = 0$. Let $\overline{u}, \overline{v} \in \overline{M}$. Then $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{u}\overline{v} = 1 - u + 1 - v - (1 - u)(1 - v) = 1 - uv = 1 - \overline{uv}$. Hence $\overline{M}$ is closed under the composition $(\circ)$ and

$$\overline{u} \circ \overline{v} = 1 - uy. \tag{4}$$

Further, by (4) $\overline{u}^{-1} \circ (\overline{u} \circ \overline{v}) = 1 - u^{-1}(uv) = 1 - v = \overline{v}$ and $(\overline{v} \circ \overline{u}) \circ \overline{u}^{-1} = \overline{v}$. From here it follows that $(\overline{M}, \circ)$ is a loop. We call it the *circle loop* corresponding to the loop $(M, \cdot)$.

We define the one-to-one mapping $\overline{\varphi} : M \to \overline{M}$ by $\overline{\varphi}(a) = \overline{a}$. For $a, b \in M$ by (4) we have $\overline{\varphi}(ab) = 1 - ab = \overline{a} \circ \overline{b} = \varphi(a) \circ \varphi(b)$. Hence $\overline{\varphi}$ is an isomorphism of the loop M upon the loop $\overline{M}$. Then, by Lemma 2, it follows that $\overline{\varphi}$ induces the isomorphism φ of the loop algebra FM upon the loop algebra $F\overline{M}$ by the rule $\varphi(\Sigma_{u \in M} \alpha_u u) = \Sigma_{u \in M} \alpha_u (\overline{\varphi}(u)) = \Sigma_{u \in M} \alpha_u \overline{u}$.

Clear by if the loop M is generated by free generators $x_1, x_2, \dots$, then the loop $\overline{M}$ is generated by free generators $\overline{x}_1, \overline{x}_2, \dots$, the isomorphism $\varphi : FM \to F\overline{M}$ is defined by mappings $x_i \to \overline{x}_i$ and a word u in M has a normal form if and only if the corresponding word $\overline{u}$ also has a normal form. This completes the proof of the following lemma.

Lemma 4. *Let FM be a loop algebra of a free Moufang loop $(M, \cdot)$ with free generators $x_1, x_2, \dots$ and let $\overline{M} = \{\overline{u} = 1 - u | u \in M\}$ be the corresponding loop under the circle composition $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{uv}$. Then the mappings $x_i \to \overline{x}_i$ define an isomorphism φ of the loop algebra FM upon the loop algebra $F\overline{M}$ by the rule $\varphi(\Sigma \alpha_u u) = \Sigma \alpha_u (\overline{\varphi}(u)) = \Sigma \alpha_u \overline{u}$, $\alpha_u \in F$, $u \in M$, and a word in the loop $(M, \cdot)$ has a normal form if and only if the word φu has a normal form in the loop $(\overline{M}, \circ)$.*

From now on, according to Lemma 4 for the algebra FM we will consider only monomials of normal form. Let $u \in FM$ and let φ be the isomorphism defined in Lemma 4. We denote $\varphi(u) = \overline{u}$. If $u = \Sigma\alpha_i u_i$, $\alpha_i \in F$, $u_i \in M$, is a polynomial in FM then we denote $\mathfrak{c}(u) = \Sigma\alpha_i$. Clear by $\mathfrak{c}(u) = \mathfrak{c}(\overline{u})$, where $\overline{u} = \Sigma\alpha_i\overline{u}_i$.

Let $(a, b, c) = ab \cdot c - a \cdot bc$ be the associator in algebra. If the free Moufang loop M is non-associative, then from the definition of loop algebra it follows that the equalities

$$(a, b, c) + (b, a, c) = 0, \quad (a, b, c) + (a, c, b) = 0 \quad \forall a, b, c \in L, \tag{5}$$

do not always hold in algebra FM. Let $I(M)$ denote the ideal of algebra FM, generated by all the elements of the left part of equalities (5). It follows from the definition of loop algebra and di-associativity of Moufang loops that $FM/I(M)$ will be an alternative algebra. We remind that an algebra A is called *alternative* if the identities $(x, x, y) = (y, x, x) = 0$ hold in it. Hence we proved

Lemma 5. *Let FM and $F\overline{M}$ be the loop algebras of a free Moufang loop $(M, \cdot)$ and its corresponding circle loop $(\overline{M}, \circ)$ and let $I(M, \cdot)$, $I(\overline{M}, \circ)$ be the ideals of FM and $F\overline{M}$ respectively, defined above. Then $I(M) = I(\overline{M})$ and for any $\overline{u} \in I(\overline{M})$ and $\mathfrak{c}(\overline{u}) = 0$.*

Proof. We denote $v_1 = v_1(u_{11}, u_{12}, u_{13}) = (u_{11}, u_{12}, u_{13}) + (u_{12}, u_{11}, u_{13})$, $v_2 = v_2(u_{21}, u_{22}, u_{23}) = (u_{21}, u_{22}, u_{23}) + (u_{21}, u_{23}, u_{22})$, where $u_{ij} \in M$, $i = 1, 2$, $j = 1, 2, 3$. Then, as an F-module, the ideal $I(M)$ is generated by elements of the form

$$w(d_1, \dots, d_k, v_i, d_{k+1}, \dots, d_m),$$

where $i = 1, 2$ and $d_1, \dots, d_m$ are monomials from FM.

Let $w = w(d_1, \dots, d_k, v_1, d_{k+1}, \dots, d_m)$. Then by (4)

$$\begin{gathered}
w = w(d_1, \dots, d_k, (u_{11}, u_{12}, u_{13}) + (u_{12}, u_{11}, u_{13}), d_{k+1}, \dots, d_m) = \\
w(d_1, \dots, d_k, u_{11}u_{12} \cdot u_{13}, d_{k+1}, \dots, d_m) - \\
w(d_1, \dots, d_k, u_{11} \cdot u_{12}u_{13}, d_{k+1}, \dots, d_m) + \\
w(d_1, \dots, d_k, u_{12}u_{11} \cdot u_{13}, d_{k+1}, \dots, d_m) - \\
w(d_1, \dots, d_k, u_{12} \cdot u_{11}u_{13}, d_{k+1}, \dots, d_m) = \\
-(1 - w(d_1, \dots, d_k, u_{11}u_{12} \cdot u_{13}, d_{k+1}, \dots, d_m)) + \\
(1 - w(d_1, \dots, d_k, u_{11} \cdot u_{12}u_{13}, d_{k+1}, \dots, d_m)) - \\
(1 - w(d_1, \dots, d_k, u_{12}u_{11} \cdot u_{13}, d_{k+1}, \dots, d_m)) + \\
(1 - w(d_1, \dots, d_k, u_{12} \cdot u_{11}u_{13}, d_{k+1}, \dots, d_m)) = \\
\overline{w}(\overline{d}_1, \dots, \overline{d}_k, (\overline{u}_{11} \circ \overline{u}_{12}) \circ \overline{u}_{13}, \overline{d}_{k+1}, \dots, \overline{d}_m) - \\
\overline{w}(\overline{d}_1, \dots, \overline{d}_k, \overline{u}_{11} \circ (\overline{u}_{12} \circ \overline{u}_{13}), \overline{d}_{k+1}, \dots, \overline{d}_m) +
\end{gathered}$$

$$\overline{w}(\overline{d}_1,\dots,\overline{d}_k,(\overline{u}_{12}\circ\overline{u}_{11})\circ\overline{u}_{13},\overline{d}_{k+1},\dots,\overline{d}_m)-$$
$$\overline{w}(\overline{d}_1,\dots,\overline{d}_k,\overline{u}_{12}\circ(\overline{u}_{11}\circ\overline{u}_{13}),\overline{d}_{k+1},\dots,\overline{d}_m)=$$
$$\overline{w}(\overline{d}_1,\dots,\overline{d}_k,\overline{v}_2,\overline{d}_{k+1},\dots,\overline{d}_m).$$

Similarly, $w(d_1,\dots,d_k,v_2,d_{k+1},\dots,d_m) = \overline{w}(\overline{d}_1,\dots,\overline{d}_k,\overline{v}_2,\overline{d}_{k+1},\dots,\overline{d}_m)$. Hence $I(M)\subseteq I(\overline{M})$.

Conversely, we consider a polynomial in $f\overline{M}$ of the form $\overline{w}(\overline{d}_1,\dots,\overline{d}_k,\overline{v}_i,\overline{d}_{k+1},\dots,\overline{d}_m)$. It is clear that $\overline{w}\in I(\overline{M})$ and any element $\overline{z}\in I(\overline{M})$ will be represented as the sum of a finite number of polynomials of such a form. We have $\mathfrak{c}(\overline{v}_i)=0$, then $\mathfrak{c}(\overline{w})=0$ and, consequently, $\mathfrak{c}(\overline{z})=0$. Now, let for example $\overline{v}_i=\overline{v}_1$. By (4) we get $\overline{v}_1=(\overline{u}_{11}\circ\overline{u}_{12})\circ\overline{u}_{13}=\overline{u}_{11}\circ(\overline{u}_{12}\circ\overline{u}_{13})=1-u_{11}u_{12}\cdot u_{13}-(1-u_{11}\cdot u_{12}u_{13})=-u_{11}u_{12}\cdot u_{13}+u_{11}\cdot u_{12}u_{13}=-(u_{11},u_{12},u_{13})=-v_1$. Further, by the relation $\overline{x}\circ\overline{y}=1-xy$ in an expression $\overline{w}$ we pass from the operation $(\circ)$ to the operation $(\cdot)$. Then $\overline{w}$ can be written as the sum of a finite number of monomials, each of them containing the associators v_i in its structure. Then $\overline{w}\in I(M)$, and hence $\overline{z}\in I(M)$, $I(\overline{M})\subseteq I(M)$. Consequently, $I(\overline{M})=I(M)$. This completes the proof of Lemma 5. □

Theorem 1. *Let $(M,\cdot)$ be a free Moufang loop, let F be an arbitrary field and let $\varphi: FM\to FM/I(M)$ be the natural homomorphism of the algebra FM upon the alternative algebra $FM/I(M)$. Then the image $\varphi(M,\cdot)=(\overline{M},\star)$ of the loop $(M,\cdot)$ will be an isomorphism of these loops.*

Proof. Any Moufang loop is an IP-loop, so by Lemma 2 the image of the loop $(M,\cdot)$ under the A-homomorphism $\varphi: FM\to FM/I(M)$ will be a loop $(\overline{M},\star)$. Let H be a normal subloop of loop $(M,\cdot)$ that corresponds to φ. Then $1-H\subseteq I(M)$. We suppose that $H\neq\{1\}$ and let $1\neq u(x_1,\dots,x_k)\in H$ be a word in free generators $x_1,\dots,x_k$ of the normal form. Then the length $l(u)>0$. By (4) we write $1-u(x_1,\dots,x_k)$ in generators $\overline{x}_1,\dots,\overline{x}_k$ with respect to the circle composition $(\circ)$, $1-u(x_1,\dots,x_k)=\overline{u}(\overline{x}_1,\dots,\overline{x}_k)$. As $1-u(x_1,\dots,x_k)\in I(M)$ then by Lemma 5 $\overline{u}(\overline{x}_1,\dots,\overline{x}_k)\in I(\overline{M})$ and $\overline{u}(\overline{x}_1,\dots,\overline{x}_k)=\overline{u}$ has a normal form. Hence $l(\overline{u})>0$ and, consequently, $\mathfrak{c}(\overline{u})=1$. But by Lemma 1 $\mathfrak{c}(\overline{u})=0$. We get a contradiction with $\mathfrak{c}(\overline{u})=1$. Hence our supposition that $H\neq\{1\}$ is false. This completes the proof of Theorem 1. □

Remark. The proof of Lemma 3 has a constructive character for free Moufang loops. But Lemma 3 holds for algebras of Ω-words (see, for example, [6]). Any relatively free Moufang loop is an algebra of Ω-words. From here it follows that Lemma 3 is true for any relatively free Moufang loop. Then it is easy to see that the main result of this paper (Theorem 1) holds for every relatively free Moufang loop.

Further we identify the loop $(\overline{M},\star)$ with $(M,\cdot)$. Then every element in $FM/I(M)$ has the form $\sum_{q\in M}\lambda_q q$, $\lambda_q\in F$. Further for the alternative algebra $FM/I(M)$ we use the notation FM and we call them *"loop algebra"* (in quote marks). Let H be a normal subloop of M. We denote the ideal of "loop algebra"

FM, generated by the elements $1-h$ $(h \in H)$ by ωH. If $H = M$, then ωM will be called the *"augmentation ideal"* (in quote marks) of "loop algebra" FM. Let us determine the homomorphism φ of F-algebra FM by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. Similarly to Lemma 1 we proved

Proposition 1. *Let H be a normal subloops of a free Moufang loop M and let FM and ωM be, respectively, the "loop algebra" and the "augmentation ideal" of M. Then*

1) $\omega H \subseteq Ker\varphi$;

2) $1-h \in Ker\varphi$ if and only if $h \in H$;

3) $\omega M = \{\sum_{q\in M} \lambda_q q | \sum_{q\in M} \lambda_q = 0\}$;

4) the "augmentation ideal" ωM is generated as F-module by elements of the form $1-q$ $(q \in M)$.

Let $\overline{\omega M}$ denote the augmentation ideal (without quote marks) of $\overline{FM}$. Then from 4) of Lemma 1 and 3) of Proposition 1 it follows that

$$\omega M = \overline{\omega M}/I(M). \tag{6}$$

Any Moufang loop L has a representation $L = L/H$, where L is a free Moufang loop. As we have noted above, in [4] Moufang loops L are constructed that are not embedded into a loop of invertible elements of any alternative algebras. Then for such normal subloop H of L $\mathrm{Ker}\varphi = FL$ and by 2) of Proposition 7 the inclusion $\omega H \subset \mathrm{Ker}\varphi$ is strict.

We mention that Proposition 1 holds also for Moufang loops for which Theorem 1 is true.

3 Some corollaries

Now we consider Moufang loops with the help of alternative algebras, using the embedding of Moufang loops in alternative algebras from Theorem 6. It is obvious that from the identities $(x, y, z) = -(y, x, z)$, $(x, y, z) = -(x, z, y)$, which hold in any alternative algebra, follows

Lemma 6. *Let Q be a loop, let FQ be its "loop algebra" and let $a, b, c \in Q$. Then, if $(a, b, c) = 0$, then $(a', b', c') = 0$, where a', b', c' are obtained from a, b, c with some substitution or with the change of some loop elements a, b, c for the inverse.*

In an arbitrary alternative algebra the identities

$$(x^2, y, z) = x(x, y, z) + (x, y, z)x, \tag{7}$$

$$(x, yx, z) = x(x, y, z), \tag{8}$$

$$(x, xy, z) = (x, y, z)x \quad [7] \tag{9}$$

hold true, the linearization of the last leads to the identities

$$(x, yt, z) + (t, yx, z) = x(t, y, z) + t(x, y, z), \tag{10}$$

$$(x, ty, z) + (t, xy, z) = (x, y, z)t + (t, y, z)x. \tag{11}$$

Proposition 2. *(Moufang Theorem) If three elements a, b, c of Moufang loop Q are bounded by the associative law $ab \cdot c = a \cdot bc$, then they generate an associative subloop.*

Proof. Obviously, it is sufficient to show that if there are arbitrary monomials $u_i = u_i(x_1, x_2, x_3), i = 1, 2, 3$, of the "loop algebra" $FL_X(\mathfrak{M})$ from the generators $x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}$ of the free Moufang loop $L_X(\mathfrak{M})$, then the equality $(\overline{u}_1, \overline{u}_2, \overline{u}_3) = 0$ holds true for $a, b, c \in Q$ in the "loop algebra" FQ where $\overline{u}_i = u_i(a, b, c)$. We prove the proposition by induction on the number $n = l(u_1) + l(u_2) + l(u_3)$, where $l(u_i)$ is the length of word u_i of loop $L_X(\mathfrak{M})$. If $n = 3$, then the statement follows from Lemma 6. Let now $n > 3$ and the equality $(\overline{v}_1, \overline{v}_2, \overline{v}_3) = 0$ holds true for the words $v_1(x_1, x_2, x_3), v_2(x_1, x_2, x_3), v_3(x_1, x_2, x_3)$ of loop $L_X(\mathfrak{M})$ such that $l(v_1) + l(v_2) + l(v_3) < n$. Then by the inductive hypothesis the associator $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ does not depend on the parentheses places in the words u_i. Let us now consider the two possible cases.

1. The words u_i have, for example, the form $u_1 = x_1^k,\ u_2 = x_2^r,\ u_3 = x_3^s$. Taking into account Lemma 6, we consider that $k > 0$. If $k = 2n$, then by (7) and by the inductive hypothesis $(a^{2n}, \overline{u}_2, \overline{u}_3) = a^n(a^n, \overline{u}_2, \overline{u}_3) + (a^n, \overline{u}_2, \overline{u}_3)a^n = 0$. Let now $k = 2n + 1$. Then by (11), by the inductive hypothesis and the previous case $(a^k, \overline{u}_2, \overline{u}_3) = (a^{2n}a, \overline{u}_2, \overline{u}_3) = (a^{2n}, \overline{u}_2 a, \overline{u}_3) - (\overline{u}_2, a, \overline{u}_3)a^{2n} - (a^{2n}, a, \overline{u}_3)\overline{u}_2 = (a^{2n}a, \overline{u}_2, \overline{u}_3) = 0$.

2. Two words from u_1, u_2, u_3 have in their structure a variable of the form x_i or x_i^{-1}. Taking into account the property of IP-loop $(xy)^{-1} = y^{-1}x^{-1}$ and Lemma 6, it is sufficient to consider the case when these words have the variable x_i in their structure. We suppose, for example, that $u_1 = v_1x_1 \cdot w_1, u_2 = v_2x_1 \cdot w_2$, where v_1, w_1, v_2, w_2 can be missing. Then by the identities (8) - (11) and by the inductive hypothesis we have $(\overline{u}_1, \overline{u}_2, \overline{u}_3) = (\overline{u}_3, \overline{u}_1, \overline{u}_2) = (\overline{u}_3, \overline{v}_1 a \cdot \overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) = -(\overline{v}_1 a, \overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) + (\overline{u}_3, \overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2)(\overline{v}_1 a) + (\overline{v}_1 a, \overline{w}_1, \overline{v}_2 a \cdot w_2)\overline{u}_3 = -(\overline{v}_1 a, \overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) = -(\overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2, \overline{v}_1 a) = (\overline{v}_2 a, \overline{u}_3\overline{w}_1 \cdot \overline{w}_2, \overline{v}_1 a) - (\overline{u}_3\overline{w}_1, \overline{w}_2, \overline{v}_1 a)(\overline{v}_2 a) - (\overline{v}_2 a, \overline{w}_2, \overline{v}_1 a)(\overline{u}_3\overline{w}_1) = (\overline{v}_2 a, \overline{u}_3\overline{w}_1 \cdot \overline{w}_2, \overline{v}_1 a) = (\overline{v}_2 a, t, \overline{v}_1 a) = (t, \overline{v}_1 a, \overline{v}_2 a) = -(a, \overline{v}_1 t, \overline{v}_2 a) + t(a, \overline{v}_1, \overline{v}_2 a) + a(t, \overline{v}_1, \overline{v}_2 a) = -(a, \overline{v}_1 t, \overline{v}_2 a) = (a, \overline{v}_2 a, \overline{v}_1 t) = a(a, \overline{v}_2, \overline{v}_1 t) = 0$. This completes the proof of Proposition 2. ☐

If we apply the Proposition 2 to the equality $a \cdot ab = aa \cdot b$, which follows from (1), we get

Corollary. *The Moufang loop is di-associative, i.e. any its two elements generate an associative subloop.*

Let $L_X(\mathfrak{M})$ be a free Moufang loop with the set of free generators X. By Lemma 3 every word in $L_X(\mathfrak{M})$ can be presented as a reduced word in different ways. As $L_X(\mathfrak{M})$ is a free loop, all reduced words of the same element in $L_X(\mathfrak{M})$ have the same free generators in their structure. Hence, their number is finite. The reduced words of element w in $L_X(\mathfrak{M})$ of the least length will be called *normal reduced words* of w. Hence every word in $L_X(\mathfrak{M})$ has normal reduced words. We will call the normal reduced words u, v in $L_X(\mathfrak{M})$ *l-homogeneous* if u, v have the same length, $l(u) = l(v)$ with respect to the variables $y \in X \cup X^{-1}$.

By the definition of the loop algebra $FL_X(\mathfrak{M})$ any element in $FL_X(\mathfrak{M})$ has the form $\sum \alpha_g g, g \in L_X(\mathfrak{M})$, and only a finite number of coefficients $\alpha_g \in F$ differ from zero. We have introduced earlier a notion of l-homogeneity for the monomials g_j. We extend it to the polynomials of algebra $FL_X(\mathfrak{M})$. It can be done, as $FL_X(\mathfrak{M})$ is a free F-module with free generators g_j. Then the algebra $FL_X(\mathfrak{M})$ decomposes into a direct sum of *l-homogeneous submodules*, consisting of l-homogeneous polynomials.

By 3) of Lemma 1 the augmentation ideal $\omega L_X(\mathfrak{M})$ of the loop algebra $FL_X(\mathfrak{M})$ is generated by the set $\overline{X} = \{1 - x_i | \forall x_i \in X \cup X^{-1}\}$. If $u(x_1, x_2, \ldots$ $\ldots, x_k\}$, where $x_i \in X \cup X^{-1}$ is a normal reduced word in $L_X(\mathfrak{M})$, then the monomial $u(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k)$ in $\omega L_X(\mathfrak{M})$ will be called *normal reduced with respect to the generating set* $\overline{X}$. We transfer the notions of length and l-homogeneity of monomials $u(x_1, x_2, \ldots, x_k\}$ to monomials $u(\overline{x}_i, \ldots, \overline{x}_k)$.

Lemma 7. *Let $FL_X(\mathfrak{M})$ be a loop algebra of the free Moufang loop $L_X(\mathfrak{M})$ with a set of free generators X and let $u(x_1, \ldots, x_k)$ be a normal reduced word in the variables $x_1, \ldots, x_k \in X \cup X^{-1}$ of length $l(u)$. Then*

1) the polynomial $1 - u(x_1, \ldots, x_k)$ of the augmentation ideal $\omega L_X(\mathfrak{M})$ is represented as a sum of normal reduced monomials of $\omega L_X(\mathfrak{M})$ in variables $\overline{x}_1, \ldots, \overline{x}_k \in \overline{X}$ whose lengths do not exceed $l(u)$, and in this representation there is only one monomial of length $l(u)$ which has the form $\pm u(\overline{x}_1, \ldots, \overline{x}_k)$;

2) $\omega L_X(\mathfrak{M})$ is generated as F-module by the normal reduced monomials from the set of generators $\overline{X}$ and decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i \in I} (\omega L_X(\mathfrak{M}))_i$.

Proof. 1) We will prove by induction on length $l(u)$. Let $x_1, x_2 \in X \cup X^{-1}$. We have $(1 - x_1)(1 - x_2) = 1 - x_1 - x_2 + x_1 x_2 = (1 - x_1) - (1 + x_2) - (1 - x_1 x_2)$, $1 - x_1 x_2 = (1 + x_1) + (1 + x_2) - (1 - x_1)(1 - x_2)$, $1 - x_1 x_2 = \overline{x}_1 + \overline{x}_2 - \overline{x}_1 \overline{x}_2$. Hence the statement of lemma for $l(u) = 2$ holds. Let us now consider the normal reduced loop word $u(x_1, \ldots, x_k)$ of length $l(u) > 2$. We expand the expression $u(1 + x_1, \ldots, 1 + x_k)$ and get

$$\begin{aligned} u(\overline{x}_1, \ldots, \overline{x}_k) = 1 + \sum x_j + \sum v_2(x_{j_1}, x_{j_2}) + \ldots \\ \ldots + \sum v_r(x_{j_1}, \ldots, x_{j_r}) + \ldots + u(x_1, \ldots, x_k), \end{aligned} \tag{12}$$

where $v_r(x_{j_1}, \ldots, x_{j_r})$ is a loop word, containing in its structure r ($r \leq l(u) - 1$) generators $x_{j_1}, \ldots, x_{j_r} \in \{x_1, \ldots, x_k\}$. We consider that the loop words $v_r(x_{j_1}, \ldots$ $\ldots, x_{j_r})$ are reduced, as in the opposite case we can bring them to this form. It is easy to see by induction on k that the right part of the equality (12) contains even number of monomials. That is why $u(\overline{x}_1, \ldots, \overline{x}_k)$ can be presented as a sum of terms of the form $1 - v_r(x_{j_1}, \ldots, x_{j_r})$ or $1 - u(x_1, \ldots, x_n)$. Then it follows from (12) that $1 - u(x_1, \ldots, x_k) = \sum \epsilon(1 - v_r(x_{j_1}, \ldots, x_{j_r})) + u(\overline{x}_1, \ldots, \overline{x}_n)$, where $\epsilon = \pm 1, r \leq l(u) - 1$. Using the inductive hypothesis for the monomials $1 - v_r(x_{j_1}, \ldots, x_{j_r})$ we obtain from here the troth of 1).

2) We have proved above that the algebra $FL_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $FL_X(\mathfrak{M}) = \bigoplus_{i\in I}(FL_X(\mathfrak{M}))_i$. From 6) of Lemma 1 and 1) of this lemma it follows that $\omega L_X(\mathfrak{M}) = \sum_{i\in I}(\omega L_X(\mathfrak{M}))_i$. Let $u(\overline{x}_1,\dots,\overline{x}_k) \in (\omega L_X(\mathfrak{M}))_i \cap (\omega L_X(\mathfrak{M}))_j$, where $i \neq j$. From the definition of a normal reduced word with respect to the set $\overline{X}$ it follows that $u(x_1,\dots,x_k) \in (\omega L_X(\mathfrak{M}))_i \cap (\omega L_X(\mathfrak{M}))_j$. From here it follows that $u(x_1,\dots,x_k) = 0$. Then $u(\overline{x}_1,\dots,\overline{x}_k) = 0$, as well. Hence $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$. This completes the proof of Lemma 7. □

Now, according to Theorem 1 we transfer the notions of length, l-homogeneity of polynomials and l-homogeneity submodules of augmentation ideal of loop algebra for polynomials of "augmentation ideal" of "loop algebra".

Lemma 8. *Let $\omega L_X(\mathfrak{M})$ be the "augmentation ideal" of "loop algebra" of free Moufang loop $L_X(\mathfrak{M})$. Then*

1) $\omega L_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$;

2) the intersection of the l-homogeneous submodules of $\omega L_X(\mathfrak{M})$ is the zero.

Proof. Expanding the expression we obtain that $(1-a, 1-b, 1-c) = -(a,b,c)$. Then from the definition (5) of ideal $I(L_X(\mathfrak{M}))$ of loop algebra $FL_X(\mathfrak{M})$ it follows that this ideal is generated by elements of the form

$$v(d_1,\dots,d_k,\overline{w}_i,d_{k+1},\dots,d_m),$$

where $i = 1,2$ and $d_1,\dots,d_m$ are normal reduced words from $FL_X(\mathfrak{M})$. Now, by the relations $yz = z-(1-y)z,\ zy = z-z(1-y),\ yz = (1+y)z-z,\ zy = z(1+y)-z$, and by 1) of Lemma 4 it is easy to see that the ideal $I(L_X(\mathfrak{M}))$ is generated as F-module by l-homogeneous polynomials of the form

$$v(\overline{b}_1,\dots,\overline{b}_r,\overline{w}_i,\overline{b}_{r+1},\dots,\overline{b}_s), \tag{13}$$

where $\overline{b}_i$ are normal reduced monomials from $\omega L_X(\mathfrak{M})$ with respect respect to the set $\overline{X}$.

By 2) of Lemma 7 the augmentation ideal $\omega L_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$. Then the ideal $I(L_X(\mathfrak{M}))$ decomposes into a direct sum of l-homogeneous submodules $I(L_X(\mathfrak{M})) = \bigoplus_{i\in I}(I(L_X(\mathfrak{M})))_i$ as well. From here it follows that the decomposition of algebra $\omega L_X(\mathfrak{M})$ into a direct sum of submodules $(\omega L_X(\mathfrak{M}))$ induces a similar decomposition also for the quotient algebra $\omega L_X(\mathfrak{M})/I(L_X(\mathfrak{M}))$: $\omega L_X(\mathfrak{M})/I(L_X(\mathfrak{M})) = \bigoplus_{i\in I}((\omega L_X(\mathfrak{M}))_i \cap I(FL_X(\mathfrak{M})))$, which by (6) is the "augmentation ideal" of the "loop algebra" $FL_X(\mathfrak{M})/FL_X(\mathfrak{M})$. This completes the proof of item 1). The item 2) follows from item 1) and item 2) of Lemma 7. □

Theorem 2. *The intersection of the terms of the lower central series of a free Moufang loop $L_X(\mathfrak{M})$ is the identity.*

Proof. We denote $L_X(\mathfrak{M}) = Q$, $\omega L_X(\mathfrak{M}) = B$. Let $Q = Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_n \supseteq \ldots$ be the lower central series of free Moufang loop Q. We have to prove that

$$\cap_{n=0}^{\infty} Q_n = 1. \tag{14}$$

Really, let $B^0 = B, B^n = \sum_{i+j=n} B^i \cdot B^j$. By 2) of Lemma 8 it is easy to see that $\cap_{n=0}^{\infty} B^n = 0$. Further, $D_n = \{g \in Q | 1 - g \in B^n\}$ is a normal subloop of loop Q, as this is the kernel of homomorphism, induced by natural homomorphism $FL_X(\mathfrak{M}) \to FL_X(\mathfrak{M})/B^n$. From the relation $\cap_{n=0}^{\infty} B^n = 0$ it follows that $\cap_{n=0}^{\infty} D_n = 1$. Now to prove (14) it is sufficient to show that $Q_n \subseteq D_n$. We will prove this by induction on n. We have $Q_0 = Q = D_0$. Let $a, b \in Q$ and we suppose that the element $g_n \in Q_n$ belongs to D_n. Then $u_n = 1 - g_n \in B^n$, $v = 1 - a \in B^0, w = 1 - b \in B^0$. Any Moufang loop is an IP-loop. Then from (3) we get $1 - (g_n, a, b) = 1 - (g_n a \cdot b)(g_n \cdot ab)^{-1} = (g_n \cdot ab - g_n a \cdot b)(g_n \cdot ab)^{-1} = ((1 - g_n)((1 - a)(1 - b)) - (((1 - g_n)(1 - a))(1 - b))(g_n \cdot ab)^{-1} = (u_n \cdot vw - u_n v \cdot w)(g_n \cdot ab)^{-1} = (u_n \cdot vw - u_n v \cdot w) - (u_n \cdot vw - u_n v \cdot w)(1 - (g_n \cdot ab)^{-1}) \in B^{n+1}$. By analogy we prove that $1 - (a, b_n, b) \in B^{n+1}$, $1 - (a, b, g_n) \in B^{n+1}$, $1 - (g_n, a) \in B^{n+1}$. Then $(g_n, a, b), (a, g_n, b), (a, b, g_n), (g_n, a) \in D_{n+1}$. But as shown at the beginning of this paper elements of the form $(g_n, a, b), (a, g_n, b), (a, b, g_n), (g_n, a)$ generate the normal subloop Q_{n+1}. Then $Q_{n+1} \subseteq D_{n+1}$. Consequently, $\cap_{n=0}^{\infty} Q_n = 1$. This completes the proof of Theorem 2. □

We remind now that a loop Q is called a *Hopfian loop* and it has a Hopfian property if it can't be isomorphic to any of its quotient loop. Obviously, any finite loop is Hopfian, but no free loop of infinite rank F_∞ can be Hopfian. Really, if $x_1, x_2, \ldots, x_i, \ldots$ is a free generators for F_∞, then the map $x_1 \to 1, x_i \to x_{i-1}$ $(x > 1)$ defines an endomorphism on with non-trivial kernel.

Proposition 3. *A finitely generated centrally nilpotent Moufang loop L is Hopfian.*

Proof. Let us consider a normal subloop $N \neq 1$ of the loop L such that $\overline{L} = L/N$ is isomorphic to L. We must come to a contradiction. For that we will prove that no element $g \neq 1$ of the loop L can be mapped into the unit of the loop $\overline{L}$. In [8] it is proved that the loop L is residually finite. Then let K be a normal subloop of L of index n, not containing g. We denote by $K^\star$ the intersection of all normal subloops of L of index $\leq n$.Then the subloop $K^\star$ also has a finite index $n^\star$ in L and also doesn't contain g. Under a homomorphic mapping of L on $\overline{L}$, the subloop $K^\star$ is mapped on subloop $K^{\star\star}$ of loop $\overline{L}$. As the index of a finite loop is not augmented by a homomorphic mapping, $K^{\star\star}$ will contain subloop $\overline{K}^\star$ of loop $\overline{L}$, which corresponds to $K^\star \subseteq L$ under an isomorphic mapping of L on $\overline{L}$. In such a way the inverse image of $\overline{K}^\star$ in L (denoted by P) should be contained in $K^\star$. On the other hand, P contains N and, consequently, g is not mapped on 1 (under a natural homomorphism of L on $\overline{L}$). □

Lemma 9. *A loop L has a Hopfian property if and only if it has a set of fully invariant normal subloops, whose quotient loop has a Hopfian property and whose intersection is trivial.*

Proof. The necessity is trivial. To prove this, it is enough to denote by φ some endomorphism on loop L and by N we denote the fully invariant normal subloop of L, whose quotient loop is Hopfian. As $\varphi N \subseteq N$ and $\varphi L = L$ then φ induces an endomorphism on of L/N. According to the supposion, it is an automorphism of loop L/N, so that ker$\varphi \subseteq N$. It means that the intersection at any set of such fully invariant subloops contains kerφ. If the intersection is trivial, then kerφ is trivial and φ is an automorphism, as required. $\square$

Combining (14), Proposition 3 and Lemma 9 we get

Theorem 3. *Any finitely generated free Moufang loop is Hopfian.*

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