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Some more results on b-θ-open sets

N. Rajesh, Z. Salleh

Abstract. In this paper, we consider the class of b-θ-open sets in topological spaces and investigate some of their properties. We also present and study some weak separation axioms by involving the notion of b-θ-open sets. We define the concepts of b-θ-kernal of sets and slightly b-θ-R_0 spaces. We apply them to investigate some properties of the graph functions.

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1 Introduction

In 1996, Andrijevic [1] initiated the study of so called b-open sets. This notion has been studied extensively in recent years by many topologists (see [2–5]). In this paper, we will continue the study of related spaces by using b-θ-open [7] sets.

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be b-open [1] if $A \subseteq \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$. The complement of b-open set is called b-closed. The intersection of all b-closed sets of X containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{cl}(A)$. A set A is b-closed if and only if $b\,\mathrm{cl}(A) = A$. The union of all b-open sets of X contained in A is called the b-interior of A and is denoted by $b\,\mathrm{int}(A)$. A set A is said to be b-regular [7] if it is b-open and b-closed. The family of all b-open (resp. b-closed, b-regular) sets of X is denoted by $BO(X)$ (resp. $BC(X)$, $BR(X)$). We set $BO(X, x) = \{V \in BO(X) | x \in V\}$ for $x \in X$.

2 Preliminaries

A point x of X is called a b-θ-cluster [7] point of $S \subseteq X$ if $b\,\mathrm{cl}(U) \cap S \neq \varnothing$ for every $U \in BO(X, x)$. The set of all b-θ-cluster points of S is called the b-θ-closure of S and is denoted by $b\,\mathrm{cl}_\theta(S)$. A subset S is said to be b-θ-closed if and only if $S = b\,\mathrm{cl}_\theta(S)$. The complement of a b-θ-closed set is said to be b-θ-open. The family of all b-θ-open subsets of X is denoted by $B_\theta O(X)$.

Theorem 1 (see [7]). *Let A be a subset of a topological space X. Then,*

(i) $A \in BO(X)$ *if and only if* $b\,\mathrm{cl}(A) \in BR(X)$.

(ii) $A \in BO(X)$ *if and only if* $b\,\mathrm{int}(A) \in BR(X)$.

Theorem 2 (see [7]). *For a subset A of a topological space X, the following properties hold:*

(i) If $A \in BO(X)$, *then* $b\,\mathrm{cl}(A) = b\,\mathrm{cl}_\theta(A)$,

(ii) $A \in BR(X)$ *if and only if A is b-θ-open and b-θ-closed.*

Definition 1. A topological space X is said to be b-regular [7] if for each $F \in BC(X)$ and each $x \notin F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 3 (see [7]). *For a topological space X, the following properties are equivalent:*

(i) X *is b-regular;*

(ii) For each $U \in BO(X)$ *and each* $x \in U$, *there exists* $V \in BO(X)$ *such that* $x \in V \subseteq b\,\mathrm{cl}(V) \subseteq U$;

(iii) For each $U \in BO(X)$ *and each* $x \in U$, *there exists* $V \in BR(X)$ *such that* $x \in V \subseteq U$.

Definition 2. A function $f : X \to Y$ is said to be b-irresolute [6] if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.

3 b-θ-open sets

Remark 1. It is easy to prove that

(i) the intersection of an arbitrary collection of b-θ-closed sets is b-θ-closed.

(ii) X and $\varnothing$ are b-θ-closed sets.

Remark 2. The following example shows that an union of any two b-θ-closed sets of X is not necessarily b-θ-closed in X.

Example 1. Let $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, \{a, b\}, X\}$. Clearly, $\{a\}, \{b\}$ are b-θ-closed sets in X, but their union $\{a, b\}$ is not b-θ-closed in X.

Lemma 1 (see [7]). *Let A be a subset of a topological space (X, τ). The following hold:*

(i) If $A \in BO(X)$, *then* $b\,\mathrm{cl}(A)$ *is b-regular and* $b\,\mathrm{cl}(A) = b\,\mathrm{cl}_\theta(A)$;

(ii) *A is b-regular if and only if A is b-θ-closed and b-θ-open;*

(iii) *A is b-regular if and only if A is* $b\,\mathrm{int}(b\,\mathrm{cl}(A))$.

Lemma 2. *For any subset A of a topological space* (X, τ), $b\,\mathrm{cl}_\theta(A)$ *is b-θ-closed.*

Definition 3. A subset S of a topological space (X, τ) is said to be θ-completment b-open (briefly θ-c-b-open) provided there exists a subset A of X for which $X - S$ $= b\,\mathrm{cl}_\theta(A)$. We call a set θ-complement b-closed if its complement is θ-c-b-open.

Remark 3. It should be mentioned that by Lemma 2, $X - S = b\,\mathrm{cl}_\theta(A)$ is b-θ-closed and S is b-θ-open. Therefore, the equivalence of θ-c-b-open and b-θ-open is obvious from the definition.

Theorem 4. *If* $A \subseteq X$ *is b-open, then* $b\,\mathrm{int}(b\,\mathrm{cl}_\theta(A))$ *is b-θ-open.*

Proof. Since $X - b\,\mathrm{int}(b\,\mathrm{cl}(A)) = b\,\mathrm{cl}(X - b\,\mathrm{cl}(A))$, then by complements $b\,\mathrm{int}(b\,\mathrm{cl}(A))$ $= (X - b\,\mathrm{cl}(X - b\,\mathrm{cl}(A)))$. Since $X - b\,\mathrm{cl}(A)$ $(=B$, say) is b-open, $b\,\mathrm{cl}(B) = b\,\mathrm{cl}_\theta(B)$ from Lemma 1. Therefore, there exists a subset $B = X - b\,\mathrm{cl}(A)$ for which $X - b\,\mathrm{int}(b\,\mathrm{cl}(A)) = b\,\mathrm{cl}_\theta(B)$. Hence $b\,\mathrm{int}(b\,\mathrm{cl}(A))$ is b-θ-open. □

Corollary 1. *If* $A \subseteq X$ *is b-regular, then A is b-θ-open.*

Proof. Obvious by Lemma 1, since A is b-regular if and only if $A = b\,\mathrm{int}(b\,\mathrm{cl}(A))$. □

Theorem 5. *b-θ-open is equivalent to b-regular if and only if* $b\,\mathrm{cl}_\theta(A)$ *is b-regular for every set* $A \subseteq X$.

Proof. Let X be a topological space. Assume b-θ-open is equivalent to b-regular and let $A \subseteq X$. Then by Lemma 2, $X - b\,\mathrm{cl}_\theta(A)$ is b-θ-open which implies that $b\,\mathrm{cl}_\theta(A)$ is b-regular. Conversely, assume $b\,\mathrm{cl}_\theta(A)$ is b-regular for every set A. Suppose U is b-θ-open and let $A \subseteq X$ such that $X - U = b\,\mathrm{cl}_\theta(A)$. That is, $U = X - b\,\mathrm{cl}_\theta(A)$. Then, $b\,\mathrm{cl}_\theta(A)$ is b-regular and U is b-regular. Therefore, b-θ-open is equivalent to b-regular. □

Theorem 6. *If* $B \subseteq X$ *is b-θ-open, then B is an union of b-regular sets.*

Proof. Let B be b-θ-open and $x \in B$. Since B is b-θ-open, then there exists a set $A \subseteq X$ such that $B = X - b\,\mathrm{cl}_\theta(A)$. Because $x \notin b\,\mathrm{cl}_\theta(A)$, there exists a b-open set W for which $x \in W$ and $b\,\mathrm{cl}(W) \cap A = \varnothing$. Hence $x \in b\,\mathrm{int}(b\,\mathrm{cl}(W)) \subseteq X - b\,\mathrm{cl}_\theta(A)$, where $b\,\mathrm{int}(b\,\mathrm{cl}(W))$ $(=V$ (say)$) \in BR(X)$, that is, $B = \bigcup\{V : V \subseteq B, V \in BR(X)\}$. □

Corollary 2. *If B is b-θ-closed, then B is the intersection of b-regular sets.*

4 On b-θ-D_i (resp. b-θ-T_i) topological spaces

Now, we study some classes of topological spaces in terms of the concept of b-θ-open sets. The relations with other notions, directly or indirectly connected with these classes are investiaged.

Definition 4. A subset A of a topological space (X, τ) is called a b-θ-D-set if there are two sets $U, V \in B_\theta O(X)$ such that $U \neq X$ and $A = U - V$.

It is true that every b-θ-open set U different from X is a b-θ-D set if $A = U$ and $V = \varnothing$.

Definition 5. A topological space (X, τ) is called b-θ-D_0 if for any distinct pair of points x and y of X, there exists a b-θ-D-set of X containing one of the points but not the other.

Definition 6. A topological space (X, τ) is called b-θ-D_1 if for any distinct pair of points x and y of X, there exists a b-θ-D-set F of X containing x but not y and a b-θ-D set G of X containing y but not x.

Definition 7. A topological space (X, τ) is called b-θ-D_2 if for any distinct pair of points x and y of X, there exists disjoint b-θ-D-sets G and E of X containing x and y respectively.

Definition 8. A topological space (X, τ) is called b-θ-T_0 if for any distinct pair of points in X, there exists a b-θ-open set containing one of the points but not the other.

Definition 9. A topological space (X, τ) is called b-θ-T_1 if for any distinct pair of points x and y in X, there exists a b-θ-open set U in X containing x but not y and a b-θ-open set V in X containing y but not x.

Definition 10. A topological space (X, τ) is called b-θ-T_2 if for any distinct pair of points x and y in X, there exist b-θ-open sets U and V in X containing x and y, respectively, such that $U \cap V = \varnothing$.

Remark 4. From Definitions 4 to 10, we obtain the following diagram:

$$\begin{array}{ccccc} b\text{–}\theta\text{–}T_2 & \Rightarrow & b\text{–}\theta\text{–}T_1 & \Rightarrow & b\text{–}\theta\text{–}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ b\text{–}\theta\text{–}D_2 & \Rightarrow & b\text{–}\theta\text{–}D_1 & \Rightarrow & b\text{–}\theta\text{–}D_0 \end{array}$$

Theorem 7. *If a topological space (X, τ) is b-θ-T_0, then it is b-θ-T_2.*

Proof. For any points $x \neq y$, let V be a b-θ-open set such that $x \in V$ and $y \notin V$. Then, there exists $U \in BO(X)$ such that $x \in U \subseteq b\,\mathrm{cl}(U) \subseteq V$. By Lemma 1, $b\,\mathrm{cl}(U) \in BR(X)$. Then $b\,\mathrm{cl}(U)$ is b-θ-open and also $X - b\,\mathrm{cl}(U)$ is a b-θ-open set containing y. Therefore, X is b-θ-T_2. $\square$

Theorem 8. *For a topological space (X, τ), the six properties in the diagram are equivalent.*

Proof. By Theorem 7, we have that b-θ-T_0 implies b-θ-T_2. Now we prove that b-θ-D_0 implies b-θ-T_0. Let (X, τ) be b-θ-D_0 so that for any distinct pair of points x and y of X, one of them belongs to a b-θ-D set A. Therefore, we choose $x \in A$ and $y \notin A$. Suppose $A = U - V$ for which $U \neq X$ and $U, V \in B_\theta O(X)$. This implies that $x \in U$. For the case that $y \notin A$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X is b-θ-T_0 since $x \in U$ but $y \notin U$. For (ii), the space X is also b-θ-T_0 since $y \in V$ but $x \notin V$. □

Let x be a point of X and V a subset of X. The set V is called a b-θ-neighbourhood of x in X if there exists a b-θ-open set A of X such that $x \in A \subseteq V$.

Definition 11. A point $x \in X$ which has only X as the b-θ-neighbourhood is called a point common to all b-θ-closed sets (briefly b-θ-*cc*).

Theorem 9. *If a topological space (X, τ) is b-θ-D_1, then (X, τ) has no b-θ-cc-point.*

Proof. Since (X, τ) is b-θ-D_1, so each point x of X is contained in a b-θ-D set $A = U - V$ and thus in U. By definition $U \neq X$ and this implies that x is not a b-θ-*cc*-point. □

Definition 12. A subset A of a topological space (X, τ) is called a quasi b-θ-closed (briefly *qbt*-closed) set if $b\,\mathrm{cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is b-θ-open in (X, τ).

Lemma 3. *[7] Let A be any subset of a topological space X. Then $x \in b\,\mathrm{cl}_\theta(A)$ if and only if $V \cap A = \varnothing$ for every $V \in BR(X, x)$.*

Theorem 10. *For a topological space (X, τ), the following properties hold:*

(i) For each pair of points x and y in X, $x \in b\,\mathrm{cl}_\theta(\{y\})$ implies $y \in b\,\mathrm{cl}_\theta(\{x\})$;

(ii) For each $x \in X$, the singleton $\{x\}$ is qbt-closed in (X, τ).

Proof. (i): Let $y \notin b\,\mathrm{cl}_\theta(\{x\})$. This implies that there exists $V \in BO(X, y)$ such that $b\,\mathrm{cl}(V) \cap \{x\} = \varnothing$ and $X - b\,\mathrm{cl}(V) \in BR(X, x)$ which means that $x \notin b\,\mathrm{cl}_\theta(\{y\})$.
(ii): Suppose that $U \in B_\theta O(X)$. This implies that there exists $V \in BO(X)$ such that $x \in V \subseteq b\,\mathrm{cl}(V) \subseteq U$. Now we have $b\,\mathrm{cl}_\theta(\{x\}) \subseteq b\,\mathrm{cl}_\theta(V) = b\,\mathrm{cl}(V) \subseteq U$. □

Definition 13. A topological space (X, τ) is said to be b-θ-$T_{1/2}$ if every *qbt*-closed set is b-θ-closed.

Theorem 11. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) is b-θ-$T_{1/2}$;

(ii) (X, τ) is b-θ-T_1.

Proof. (i)⇒(ii): For distinct points x, y of $X, \{x\}$ is *qbt*-closed by Theorem 10. By hypothesis, $X - \{x\}$ is b-θ-open and $y \in X - \{x\}$. By the same token, $x \in X - \{y\}$ and $X - \{y\}$ is b-θ-open. Therefore, (X, τ) is b-θ-T_1.

(ii)⇒(i): Suppose that A is a *qbt*-closed set which is not b-θ-closed. There exists $x \in b\,\mathrm{cl}_\theta(A) - A$. For each $a \in A$, there exists a b-θ-open set V_a such that $a \in V_a$ and $x \notin V_a$. Since $A \subseteq \bigcup_{a \in V_a} V_a$ and $\bigcup_{a \in V_a} V_a$ is b-θ-open, we have $b\,\mathrm{cl}_\theta(A) \subseteq \bigcup_{a \in V_a} V_a$. Since $x \in b\,\mathrm{cl}_\theta(A)$, there exists $a_0 \in A$ such that $x \in V_{a_0}$. But this is a condradiction. □

Recall that a topological space (X, τ) is called b-T_2 [2] if for any distinct pair of points x and y in X, there exist b-open sets U and V in X containing x and y respectively such that $U \cap V = \varnothing$.

Theorem 12. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) *is* b-θ-T_2*;*

(ii) (X, τ) *is* b-T_2*.*

Proof. (i)⇒(ii): This is obvious since every b-θ-open set is b-open.

(ii)⇒(i): Let x and y be distinct points of X. There exist b-open sets U and V such that $x \in U$, $y \in V$ and $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \varnothing$. Since $b\,\mathrm{cl}(U)$ and $b\,\mathrm{cl}(V)$ are b-regular, then they are b-θ-open and hence (X, τ) is b-θ-T_2. □

Definition 14. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weak b-irresolute [8] if for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists a $U \in BO(X, x)$ such that $f(U) \subseteq b\,\mathrm{cl}(V)$.

Remark 5. A function $f : (X, \tau) \to (Y, \sigma)$ is weak b-irresolute if and only if $f^{-1}(V)$ is b-θ-closed (resp. b-θ-open) in (X, τ) for every b-θ-closed (resp. b-θ-open) set V in (Y, σ).

Theorem 13. *If $f : (X, \tau) \to (Y, \sigma)$ is a weak b-irresolute surjective function and E is a b-θ-D set in Y, then the inverse image of E is a b-θ-D set in X.*

Proof. Let E be a b-θ-D-set in Y. Then there are b-θ-open sets U and V in Y such that $E = U - V$ and $U \neq Y$. By weak b-irresoluteness of f, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-open in X. Since $U \neq Y$, we have $f^{-1}(U) \neq X$. Hence $f^{-1}(E) = f^{-1}(U) - f^{-1}(V)$ is a b-θ-D-set in X. □

Theorem 14. *If (Y, σ) is a b-θ-D_1 space and $f : (X, \tau) \to (Y, \sigma)$ is a weak b-irresolute bijection, then (X, τ) is b-θ-D_1.*

Proof. Suppose that Y is a b-θ-D_1 space. Let x and y be any pair of distinct poins in X. Since f is injective and Y is b-θ-D_1, there exists b-θ-D sets U and V of Y containing $f(x)$ and $f(y)$, respectively such that $f(y) \notin U$ and $f(x) \notin V$. By Theorem 13, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-D sets in X containing x and y, respectively such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. This implies that X is a b-θ-D_1 space. □

Theorem 15. *For a topological space (X, τ), the following statements are equivalent:*

(i) (X, τ) *is* b-θ-D_1;

(ii) For each pair of distinct points $x, y \in X$, *there exists a weak* b*-irresolute surjective function* $f : (X, \tau) \to (Y, \sigma)$, *where* Y *is a* b-θ-D_1 *space such that* $f(x)$ *and* $f(y)$ *are distinct.*

Proof. (i)⇒(ii): For every pair of distinct points of X, it suffices to take the identity function $f : (X, \tau) \to (X, \tau)$.

(ii)⇒(i): Let x and y be any pair of distinct points in X. By hypothesis, there exists a surjective weak b-irresolute function f of the space X into a b-θ-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint b-θ-D sets U and V of Y containing $f(x)$ and $f(y)$, respectively such that $f(y) \notin U$ and $f(x) \notin V$. Since f is weak b-irresolute and surjective, by Theorem 13, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-D sets in X containing x and y, respectively such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. Hence X is a b-θ-D_1 space. □

5 Further properties

Definition 15. Let A be a subset of a topological space (X, τ). The b-θ-kernel of $A \subseteq X$ denoted by $bker_\theta(A)$, is defined to be the set
$\bigcap \{O : O \in B_\theta O(X, \tau) \text{ and } A \subseteq O\} = \{x : b\,\mathrm{cl}_\theta(\{x\}) \cap A \neq \varnothing\}$.

Definition 16. A topological space (X, τ) is said to be slightly b-θ-R_0 if $\bigcap\{b\,\mathrm{cl}_\theta(\{x\}) : x \in X\} = \varnothing$.

Theorem 16. *A topological space* (X, τ) *is slightly* b-θ-R_0 *if and only if* $bker_\theta(\{x\}) \neq X$ *for any* $x \in X$.

Proof. Necessity. Let the space (X, τ) be slightly b-θ-R_0. Assume that there is a point y in X such that $bker_\theta(\{y\}) = X$. Then $y \notin O$ which is some proper b-θ-open subset of X. This implies that $y \in \bigcap\{b\,\mathrm{cl}_\theta(\{x\}) \colon x \in X\}$. But this is a contradiction.

Sufficiency. Now assume that $bker_\theta(\{x\}) \neq X$ for any $x \in X$. If there exists a point y in X such that $y \in \bigcap\{b\,\mathrm{cl}_\theta(\{x\}) \colon x \in X\}$, then every b-θ-open set containing y must contain every point of X. This implies that the space X is the unique b-θ-open set containing y. Hence $b\,\mathrm{cl}_\theta(\{x\}) = X$ which is a contracdiction . Therefore, (X, τ) is slightly b-θ-R_0. □

Theorem 17. *If the topological space* X *is slightly* b-θ-R_0 *and* Y *is any topological space, then the product* $X \times Y$ *is slightly* b-θ-R_0.

Proof. By showing that $\bigcap\{b\,\mathrm{cl}_\theta(\{x, y\}) \colon (x, y) \in X \times Y\} = \varnothing$ we are done. We have $\bigcap\{b\,\mathrm{cl}_\theta(\{x, y\}) \colon (x, y) \in X \times Y\} \subseteq \bigcap\{b\,\mathrm{cl}_\theta(\{x\}) \times b\,\mathrm{cl}_\theta(\{y\}) \colon (x, y) \in X \times Y\} = \bigcap\{b\,\mathrm{cl}_\theta(\{x\}) \colon x \in X\} \times \bigcap\{b\,\mathrm{cl}_\theta(\{y\}) \colon y \in Y\} \subseteq \varnothing \times Y = \varnothing$. □

Definition 17. A function $f : (X, \tau) \to (Y, \sigma)$ is R-continuous (resp. θ-R-b-continuous, R-b-continuous) if for each $x \in X$ and each b-open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $\mathrm{cl}(f(U)) \subseteq V$ (resp. $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$, $b\,\mathrm{cl}(f(U)) \subseteq V$).

Definition 18. A function $f : X \to Y$ is said to be b-open [6] if $f(U)$ is b-open in Y for every open set U of X.

Remark 6. (i): Since $A \subseteq b\,\mathrm{cl}(A) \subseteq b\,\mathrm{cl}_\theta(A)$ for any set A, θ-R-b-continuity implies R-b-continuity.

(ii): Since the b-closure and b-θ-closure operate agree on b-open sets (Lemma 1) it follows that if $f : (X, \tau) \to (Y, \sigma)$ is R-b-continuous and b-open, then f is θ-R-b-continuous.

Definition 19. The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly b-θ-closed if for each point $(x, y) \in (X \times Y) - G(f)$, there exists subsets $U \in BO(X, x)$ and $V \in B_\theta O(Y, y)$ such that $(b\,\mathrm{cl}(U) \times V) \cap G(f) = \varnothing$.

Lemma 4. *The graph $G(f)$ of $f : (X, \tau) \to (Y, \sigma)$ is strongly b-θ-closed in $X \times Y$ if and only if for each point $(x, y) \in (X \times Y) - G(f)$, there exist $U \in BO(X, x)$ and $V \in B_\theta O(Y, y)$ such that $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.*

Proof. It follows immediately from Definition 19. □

Recall that a topological space (X, τ) is called b-T_1 [2] if for any distinct pair of points x and y in X, there is a b-open set U in X containing x but not y and a b-open set V in X containing y but not x.

Theorem 18. *If $f : (X, \tau) \to (Y, \sigma)$ is θ-R-b-continuous, weak b-irresolute and Y is b-T_1, then $G(f)$ is strongly b-θ-closed.*

Proof. Assume $(x, y) \in (X \times Y) - G(f)$. Since $y \neq f(x)$ and Y is b-T_1, there exists a b-open set V of Y such that $f(x) \in V$ and $y \notin V$. The θ-R-b-continuity of f implies the existence of an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore, $(x, y) \in b\,\mathrm{cl}(U) \times (Y - b\,\mathrm{cl}_\theta f(U))$ which is disjoint from $G(f)$ because if $a \in b\,\mathrm{cl}(U)$, then since f is a weak b-irresolute function, $f(a) \in f(b\,\mathrm{cl}(U)) \subseteq b\,\mathrm{cl}_\theta f(U)$. Note that $Y - b\,\mathrm{cl}_\theta(f(U))$ is b-θ-open. □

Theorem 19. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weak b-irresolute function. Then f is θ-R-b-continuous if and only if for each $x \in X$ and each b-closed subset F of Y with $f(x) \notin F$, there exists an open subset U of X containing x and a b-θ-open subset V of Y with $F \subseteq V$ such that $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.*

Proof. Necessity. Let $x \in X$ and F be a b-closed subset of Y with $f(x) \in Y - F$. Since F is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq Y - F$. Let $V = Y - b\,\mathrm{cl}_\theta(f(U))$. Then V is b-θ-open and $F \subseteq V$.

Since f is weak b-irresolute, $f(b\,\mathrm{cl}(U)) \subseteq b\,\mathrm{cl}_\theta(f(U))$. Therefore $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.

Sufficiency. Let $x \in X$ and let V be a b-open subset of Y with $f(x) \in V$. Let $F = Y - V$. Since $f(x) \notin F$ there exists an open subset U of X containing x and a b-θ-open subset W of Y with $F \subseteq W$ such that $f(b\,\mathrm{cl}(U)) \cap W = \varnothing$. Then $f(b\,\mathrm{cl}(U)) \subseteq Y - W$, thus $b\,\mathrm{cl}_\theta(f(U)) \subseteq b\,\mathrm{cl}_\theta(Y - W) = Y - W \subseteq Y - F = V$. Therefore, f is θ-R-b-continuous. □

Corollary 3. *Let X and Y be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a weak b-irresolute function. Then f is θ-R-b-continuous if and only if for each $x \in X$ and each b-open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq V$.*

Proof. Assume f is θ-R-b-continuous. Let $x \in X$ and let V be a b-open subset of Y with $f(x) \in V$. Then there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. By hypothesis of f, we have $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq b\,\mathrm{cl}_\theta(b\,\mathrm{cl}_\theta(f(U))) = b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Thus, $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq V$. The converse implication is immediate. □

Definition 20. A topological space (X, τ) is said to be b-R_1 if for $x, y \in X$ with $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$, there exist disjoint b-open sets U and V such that $b\,\mathrm{cl}(\{x\}) \subseteq U$ and $b\,\mathrm{cl}(\{y\}) \subseteq V$.

Lemma 5. *A topological space X is b-R_1 if and only if $b\,\mathrm{cl}_\theta(\{x\}) = bcl(\{x\})$ for all $x \in X$.*

Proof. Necessity. Generally we have $b\,\mathrm{cl}(\{x\}) \subseteq b\,\mathrm{cl}_\theta(\{x\})$ for all $x \in X$. Suppose that $y \notin b\,\mathrm{cl}(\{x\})$ for any $x \in X$. Then there exists $A \in BO(X, y)$ such that $A \cap \{x\} = \varnothing$. Since X is b-R_1 and $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$, there exist b-open sets U and V such that $b\,\mathrm{cl}(\{x\}) \subseteq U$, $b\,\mathrm{cl}(\{y\}) \subseteq V$ and $U \cap V = \varnothing$. Since $U \in BO(X, x)$ and $V \in BO(X, y)$, then $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \varnothing$. This implies $b\,\mathrm{cl}(\{x\}) \cap b\,\mathrm{cl}(V) = \varnothing$ and hence $\{x\} \cap b\,\mathrm{cl}(V) = \varnothing$. Therefore $y \notin bcl_\theta(\{x\})$ and thus $b\,\mathrm{cl}_\theta(\{x\}) \subseteq b\,\mathrm{cl}(\{x\})$.

Sufficiency. Let $x, y \in X$ with $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$. Then there exists a $k \in b\,\mathrm{cl}(\{x\})$ such that $k \notin b\,\mathrm{cl}(\{y\})$. Since $k \in b\,\mathrm{cl}(\{x\}) = b\,\mathrm{cl}_\theta(\{x\})$, then $U \cap \{x\} \neq \varnothing$ for every $U \in BR(X, k)$ and hence $b\,\mathrm{cl}(\{x\}) \subseteq U$. Since $k \notin b\,\mathrm{cl}(\{y\}) = b\,\mathrm{cl}_\theta(\{y\})$, there exists $U \in BR(X, k)$ such that $U \cap \{y\} = \varnothing$. Since $U \in BO(X, k)$, $U \cap b\,\mathrm{cl}(\{y\}) = \varnothing$ and hence $bcl(\{y\}) \subseteq X \backslash U$. Therefore, there exists disjoint b-open sets U and $X \backslash U$ such that $b\,\mathrm{cl}(\{x\}) \subseteq U$ and $b\,\mathrm{cl}(\{y\}) \subseteq X \backslash U$. □

Proposition 1. *A space X is b-R_1 if and only if for each b-open set A and each $x \in A$, $b\,\mathrm{cl}_\theta(\{x\}) \subseteq A$.*

Proof. Necessity. Assume X is b-R_1. Suppose that A is a b-open subset of X and $x \in A$. Let y be an arbitrary element of $X - A$. Since X is b-R_1, then $b\,\mathrm{cl}_\theta(\{y\}) = b\,\mathrm{cl}(\{y\}) \subseteq X - A$. Hence we have that $x \notin b\,\mathrm{cl}_\theta(\{y\})$ and so $y \notin b\,\mathrm{cl}_\theta(\{x\})$. It follows that $b\,\mathrm{cl}_\theta(\{x\}) \subseteq A$.

Sufficiency. Assume now that $y \in b\,\mathrm{cl}_\theta(\{x\}) - b\,\mathrm{cl}(\{x\})$ for some $x \in X$. Then there exists a b-open set A containing y such that $b\,\mathrm{cl}(A) \cap \{x\} \neq \varnothing$ but $A \cap \{x\} = \varnothing$. Then $b\,\mathrm{cl}_\theta(\{y\}) \subseteq A$ and $b\,\mathrm{cl}_\theta(\{y\}) \cap \{x\} = \varnothing$. Hence $x \notin b\,\mathrm{cl}_\theta(\{y\})$. Thus $y \notin b\,\mathrm{cl}_\theta(\{x\})$. By this contradiction, we obtain $b\,\mathrm{cl}_\theta(\{x\}) = b\,\mathrm{cl}(\{x\})$ for each $x \in X$. Thus, X is b-R_1. □

Theorem 20. *If $f : (X,\tau) \to (Y,\sigma)$ is a θ-R-b-continuous surjection, then (Y,σ) is a b-R_1 space.*

Proof. Let V be a b-open subset of Y and $y \in V$. Let $x \in X$ such that $y = f(x)$. Since f is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Then $b\,\mathrm{cl}_\theta(\{y\}) \subseteq b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore, by Proposition 1, Y is b-R_1. □

We give some basic properties of θ-R-b-continuous functions concerning composition and restriction.

Theorem 21. *If $f : (X,\tau) \to (Y,\sigma)$ is continuous and $g : (Y,\sigma) \to (Z,\gamma)$ is θ-R-b-continuous, then $g \circ f\colon (X,\tau) \to (Z,\gamma)$ is θ-R-b-continuous.*

Proof. Let $x \in X$ and W be a b-open subset of Z containing $g(f(x))$. Since g is θ-R-b-continuous, there exists an open subset V of Y containing $f(x)$ such that $b\,\mathrm{cl}_\theta(g(V)) \subseteq W$. Since f is continuous, there exists an open subset U of X containing x, $f(U) \subseteq V$; hence $b\,\mathrm{cl}_\theta(g(f(U))) \subseteq W$. Therefore $g \circ f$ is θ-R-continuous. □

Theorem 22. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be functions. If $g \circ f\colon (X,\tau) \to (Z,\gamma)$ is θ-R-b-continuous and f is fn open surjection, then g is θ-R-b-continuous.*

Proof. Let $y \in Y$ and W be a b-open subset of Z containing $g(y)$. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. Since $g \circ f$ is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(g(f(U))) \subseteq W$. Note that $f(U)$ is an open set containing y. Therefore g is θ-R-b-continuous. □

Theorem 23. *If $f : (X,\tau) \to (Y,\sigma)$ is θ-R-b-continuous and $A \subseteq X$, then $f|_A\colon A \to Y$ is θ-R-b-continuous.*

Proof. Let $x \in A$ and let V be any b-open subset of Y containing $f(x)$ ($= f|_A(x)$). Since f is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Put $O = U \cap A$, then O is an open subset of A containing x such that $b\,\mathrm{cl}_\theta(f|_A(O)) = b\,\mathrm{cl}_\theta(f(O)) \subseteq b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore $f|_A : A \to Y$ is θ-R-b-continuous. □

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N. Rajesh
Department of Mathematics
Kongu Engineering College
Perundurai, Erode-638 052
TamilNadu, India
E-mail: *nrajesh_topology@yahoo.co.in*

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Z. Salleh
Institute for Mathematical Research
University Putra Malaysia, 43400 UPM, Serdang
Selangor, Malaysia
E-mail: *bidisalleh@yahoo.com*