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# A lower bound for a quotient of roots of factorials

Cristinel Mortici

**Abstract.** With the aid of asymptotic properties of polygamma functions a new lower bound is established for the quotient $\phi(r+1)/\phi(r)$ where $\phi(r) = (r!)^{1/r}$.

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## 1 Introduction

In 1965, H. Minc and L. Sathre [12] have given one of the first estimations of the expression

$$\phi(r) = (r!)^{1/r}.$$

Inequalities involving the function $\phi(r)$ are of interest in themselves, but they also have important applications in the theory of $(0,1)$-matrices.

The permanent of an $n$-by-$n$ matrix $A = (a_{ij})$ is defined as

$$\operatorname{Per}(A) = \sum a_{1\sigma(1)} a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)},$$

where the sum goes over every permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$. Although it looks similar to the determinant of matrices, the permanent is much harder to be computed. The literature on bounds for permanents is quite extensive. It was first conjectured by H. Minc [10], then proved by L.M. Brégman [4] that for a $(0,1)$-matrix with row sums $r_1, r_2, \ldots, r_n$, the following upper bound holds:

$$\operatorname{Per}(A) \leq \prod_{i=1}^{n} \phi(r_i).$$

This kind of bounds and some others, see [5, 9, 11, 14], motivated many authors [12, 15, 16, 17] to introduce new inequalities involving $(r!)^{1/r}$, or the ratio $\phi(r+1)/\phi(r)$.

H. Minc and L. Sathre [13, Cor. 2] proved that for every positive integer $r$:

$$1 < \frac{\phi(r+1)}{\phi(r)} < 1 + \frac{1}{r}. \tag{1.1}$$

One of the main results of this paper is the following new inequality, for every $x \geq 1$,

$$\frac{\Gamma\left(x+2\right)^{1/(x+1)}}{\Gamma\left(x+1\right)^{1/x}} \geq \frac{\left(4x+4\right)^{1/(x+1)}}{\left(4x\right)^{1/x}}\left(1+\frac{1}{x}\right) > 1.$$

Since $\Gamma(r+1) = r!$ for the positive integer $r$, this improves the estimation from the left-hand side of (1.1).

## 2 The Results

In the early 18th century, famous Swiss mathematician Leonhard Euler (1707 – 1783), introduced the function

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1}e^{-t}dt\ , \quad x > 0,$$

now known as the Euler's gamma function. It is the natural extension of the factorial function to every positive real number (or more exactly to $\mathbb{C}\setminus\mathbb{Z}_-$), since $\Gamma\left(n+1\right) = n!$, for every counting number $n$. The famous Bohr-Mollerup theorem [2, 3] states that the gamma function extends uniquely the factorial function, as $f = \Gamma$ is the only solution of the functional equation

$$f\left(x+1\right) = xf\left(x\right), \quad f\left(1\right) = 1$$

in the class of log-convex functions $f : (0,1) \to (0,1)$. (Another result of this kind says, that $f = \Gamma$ also in the case where there is such a $g : (0,1) \to \mathbb{R}$ that the function $g \circ f$ is convex in an interval $(\gamma, 1)$, $\gamma > 0$, and $g(x) = a \ln x + b$, $x \to \infty$, with some $a > 0$ and $b \in \mathbb{R}$, cf. [6]). The psi or digamma function is defined as

$$\psi\left(x\right) = \frac{d}{dx}\ln\Gamma\left(x\right) = \frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)},$$

while the derivatives $\psi'$, $\psi''$, $\psi'''$, ... are called the tri-, tetra-, pentagamma functions, or simply the polygamma functions. In what follows, we use the following integral representations [1, 13, 18]

$$\psi^{(n)}\left(x\right) = \left(-1\right)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-tx}dt \tag{2.1}$$

and for every $\omega > 0$,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma\left(\omega\right)}\int_0^\infty t^{\omega-1}e^{-tx}dt. \tag{2.2}$$

Recall that a function $z$ is said to be completely monotonic on $(0,\infty)$ if it has derivatives of all orders and for every positive integer $k$ and $x \geq 0$, we have

$$\left(-1\right)^k z^{(k)}\left(x\right) \geq 0.$$

This notion was introduced in 1921 by F. Hausdorff [8], under the name 'total monoton'. J. Dubourdieu [7] proved that every non-constant, completely monotonic function satisfies $(-1)^k z^{(k)}(x) > 0$. According with the well-known Hausdorff–Bernstein–Widder theorem in [18, Theorem 12a, p. 160], a function $z$ on $(0,\infty)$ is completely monotonic if and only if there exists a non-negative measure $\mu(t)$ such that for every $x \geq 0$,

$$z(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{2.3}$$

such that the integral converges for all $x > 0$. Completely monotonic functions involving the gamma function are very useful, since they produce sharp bounds for the polygamma functions. They also play a basic role in probability theory, or asymptotic and numerical analysis and in physics.

Motivated by the right-hand inequality of (1.1), we introduce the function $h : (0,\infty) \to \mathbb{R}$, by the formula

$$h(x) = x(x+1)\ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}}.$$

**Theorem 2.1.** *The function $h'$ is completely monotonic.*

*Proof.* We have

$$h(x) = x\ln\Gamma(x+1) - (x+1)\ln\Gamma(x) - (x^2+x)\ln\left(1+\frac{1}{x}\right).$$

Then

$$h'(x) = 2 + \ln x - (2x+1)\ln\left(1+\frac{1}{x}\right) - \psi(x) \tag{2.4}$$

and

$$h''(x) = \frac{2}{x} + \frac{1}{x+1} - 2\ln\left(1+\frac{1}{x}\right) - \psi'(x) \tag{2.5}$$

and

$$h'''(x) = \frac{2}{x} - \frac{2}{x+1} - \frac{2}{x^2} - \frac{1}{(x+1)^2} - \psi''(x). \tag{2.6}$$

Using (2.1)–(2.2), we have

$$h'''(x) = \int_0^\infty 2e^{-tx}dt - \int_0^\infty 2e^{-t(x+1)}dt -$$

$$-\int_0^\infty 2te^{-tx}dt - \int_0^\infty te^{-t(x+1)}dt + \int_0^\infty \frac{t^2}{1-e^{-t}}e^{-tx}dt,$$

or

$$h'''(x) = \int_0^\infty \varphi(t)\frac{e^{-t(x+1)}}{e^t-1}dt,$$

where

$$\varphi(t) = t^2 e^{2t} - \left(e^t - 1\right)\left(2 + t - 2e^t + 2te^t\right) =$$
$$= \sum_{n=3}^{\infty} \frac{2^{n-2}\left(n^2 - 5n + 8\right) + n - 4}{n!} t^n > 0.$$

Next we use the fact that

$$\lim_{x\to\infty} \left(\psi(x) - \ln x\right) = \lim_{x\to\infty} \psi'(x) = \lim_{x\to\infty} \psi''(x) = 0,$$

as it results from the asymptotic expansions of the polygamma functions, *e.g.*, [1, p. 259 – Rel. 6.3.18; p. 260 – Rel. 6.4.12 and 6.4.13]. Thus, from (2.4)–(2.6), we have

$$\lim_{x\to\infty} h'(x) = \lim_{x\to\infty} h''(x) = \lim_{x\to\infty} h'''(x) = 0.$$

Now, from $h''' > 0$, it results that $h''$ is strictly increasing. As $\lim_{x\to\infty} h''(x) = 0$, we have $h'' < 0$. Further, $h'$ is strictly decreasing, with $\lim_{x\to\infty} h'(x) = 0$, so $h' > 0$. Finally, from (2.3) it results that $h'$ is completely monotonic. □

**Corollary 2.1.** *For every $x \geq 1$, we have:*

$$\frac{\Gamma(x+1)^{1/(x+1)}}{\Gamma(x)^{1/x}} \geq 4^{\frac{-1}{x(x+1)}} \left(1 + \frac{1}{x}\right) > 1, \tag{2.7}$$

*where the constant 4 is best possible.*

*Proof.* The function $h'$ is positive, so $h$ is strictly increasing. In consequence, for every $x \geq 1$, we have $h(1) \leq h(x)$. As $h(1) = -\ln 4$, we obtain

$$-\ln 4 \leq x(x+1) \ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}}.$$

By exponentiating, we get

$$4^{\frac{-1}{x(x+1)}} \leq \frac{x}{x+1} \cdot \frac{\Gamma(x+1)^{1/(x+1)}}{\Gamma(x)^{1/x}},$$

which is the conclusion. □

By using the recurrence $\Gamma(y+1) = y\Gamma(y)$ in (2.7), we can state the following

**Corollary 2.2.** *For every $x \geq 1$, we have:*

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \geq \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}} \left(1 + \frac{1}{x}\right) > 1,$$

*where the constant 4 is best possible.*

As a consequence, this inequality can be used as a good approximation

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \approx \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}}\left(1+\frac{1}{x}\right),$$

as we can see from numerical computations:

| $x$ | $\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}}$ | $\frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}}\left(1+\frac{1}{x}\right)$ |
|---|---|---|
| 10 | 1.084 021 393 | 1.072 979 624 |
| 50 | 1.019 047 171 | 1.018 278 181 |
| 125 | 1.007 818 486 | 1.007 666 066 |
| 350 | 1.002 829 804 | 1.002 806 159 |
| 500 | 1.001 985 892 | 1.001 973 593 |
| 2500 | 1.000 399 307 | 1.000 398 686 |

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Cristinel Mortici
Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18, 130082 Târgovişte
Romania

E-mail: *cmortici@valahia.ro*