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On commutative Moufang loops with some restrictions for subloops and subgroups of its multiplication groups

Natalia Lupashco

Abstract. It is proved that if an infinite commutative Moufang loop L has such an infinite subloop H that in L every associative subloop which has with H an infinite intersection is a normal subloop then the loop L is associative. It is also proved that if the multiplication group $\mathfrak{M}$ of infinite commutative Moufang loop L has such an infinite subgroup $\mathfrak{N}$ that in $\mathfrak{M}$ every abelian subgroup which has with $\mathfrak{N}$ an infinite intersection is a normal subgroup then the loop L is associative.

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When considering different classes of algebras (rings, groups, loops) it is very important to know whether they have subalgebras (systems of subalgebras) with prescribed features. For example, in [1] it is proved that *every infinite CML L contains an infinite associative subloop* and, *if all infinite associative subloops of L are normal in L, then L is associative* [2]. Similarly from the equivalence of statements 1), 2), 8) of Theorem 3.5 from [1] it follows that *every multiplication group $\mathfrak{M}$ of infinite CML L contains an infinite abelian subgroup* and if all infinite abelian subgroups of the multiplication group $\mathfrak{M}$ are normal in $\mathfrak{M}$ then CML L is associative [3].

In this work the restriction on infinite associative subloops and infinite abelian subgroups is reduced. We prove that if an infinite CML L (respect. multiplication group $\mathfrak{M}$ of infinite CML L) has such an infinite subloop H (respect. infinite subgroup $\mathfrak{N}$) that in L (respect. $\mathfrak{M}$) every associative subloop (respect. abelian subgroup) which has with H (respect. $\mathfrak{N}$) an infinite intersection is a normal subloop (respect. subgroup) then the CML L is associative.

We remind that the *commutative Moufang loop* (abbreviated *CML*) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$.

The *multiplication group* $\mathfrak{M}(L)$ of a CML L is the group generated by all the *translations* $R(x)$, where $R(x)y = yx$.

The subgroup $\mathfrak{I}(L)$ of the group $\mathfrak{M}(L)$, generated by all the *inner mappings* $R(x,y) = R^{-1}(xy)R(y)R(x)$ is called the *inner mapping group* of the CML L.

A subloop H of the CML L is called *normal* in L if $x \cdot yH = xy \cdot H$ for all $x, y \in L$. Equivalently, H is normal in L if $\mathfrak{I}(Q)H = H$.

The *center* $Z(L)$ of the CML L is the normal subloop $Z(L) = \{x \in L | xy \cdot z = x \cdot yz \quad \forall y, z \in L\}$ [4].

Further we will denote by $< M >$ the subloop of the loop L, generated by the set $M \subseteq L$.

Theorem. *For an infinite CML L with multiplication group* $\mathfrak{M}$ *the following statements are equivalent:*

1) the CML L is associative;

2) the CML L has such an infinite subloop H that every associative subloop which hase an infinite intersection with H is a normal subloop in L;

3) the group $\mathfrak{M}$ *is abelian;*

4) the group $\mathfrak{M}$ *has such an infinite subloop* $\mathfrak{N}$ *that every associative subloop which hase an infinite intersection with* $\mathfrak{N}$ *is a normal subgroup in* $\mathfrak{M}$*.*

Proof. The implications 1) $\Rightarrow$ 2), 3) $\Rightarrow$ 4), 1) $\Leftrightarrow$ 3) are obvious.

2) $\Rightarrow$ 1). We suppose that H is a non-periodic subloop and let $a \in H$ be an element of infinite order. By [4] the element a^3 belongs to the center $Z(L)$ of CML L. Let b be an arbitrary element in L such that $< b > \cap < a >= 1$. The subloop $< a^3 >$ is normal in L. Then by [4] the product $< b >< a^3 >$ is a subgroup. As $< a^3 > \subseteq < b >< a^3 > \cap H$ then by statement 2) $< b >< a^3 >$ is a normal subloop in L. Let φ be an inner mapping of CML L. In CML the inner mappings are its automorphisms [4]. Then $< b >< a^3 >= \varphi(< b >< a^3 >) = \varphi(< b >)\varphi(< a^3 >) = \varphi(< b >) < a^3 >$, $\varphi(< b >) < a^3 >=< b >< a^3 >$. We have $< b > \cap < a^3 >= 1$. Then and $\varphi(< b >) \cap < a^3 >= 1$.

We denote $< a^3 >= A$, $< b >= B$. Let θ, η be the restrictions of natural homomorphism $\lambda : AB \to AB/A$ onto B and φB respectively. Obviously, $\ker\theta = B \cap A$, $\ker\eta = \varphi B \cap A$. Then from equalities $B \cap A = 1$, $\varphi B \cap A = 1$ it follows that θ, η are monomorphisms.

Let $b \in B$. Then $b = ca$ for some $c \in \varphi B$, $a \in A$. Further, $\lambda b = \lambda(ca)$, $\lambda b = \lambda c \cdot \lambda a$, $\lambda b = \lambda c \cdot \lambda 1$, $\lambda b = \lambda c$. The homomorphism λ acts onto φB as η. Hence $\lambda c = \eta c$. η is a restriction of λ onto φB and is a monomorphism of φB. Then from $\lambda b = \eta c$ it follows that $b \in \varphi B$, $B \subseteq \varphi B$. Analogously, $\varphi B \subseteq B$. Hence $\varphi B = B$. Consequently, the subloop $< b >$ is normal in L.

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Next, using the normality of $< b >$ in L by analogy it is proved that the subloop $< a >$ is normal in L. We get that any element of L generates a normal subloop in L. This means that CML L is hamiltonian. But any hamiltonian CML is associative [5]. Hence the implication 2) $\Rightarrow$ 1 holds.

Now we suppose that the abelian group H is periodic. Then H decomposes into a direct product of its maximal p-subgroups H_p. Let $H = D \times H_3$. By [6] $D \subseteq Z(L)$. The subgroup D is normal in L. If D is infinite then, as in the previous case, it is proved that CML L is hamiltonian and, consequently, is associative. If D is finite then the subgroup H_3 is infinite.

We suppose that the infinite abelian group H_3 satisfies the minimum condition for its subgroups. Then $H_3 = T \times K$, where K is a finite group and T is an infinite divisible group. By [1] $T \subset Z(L)$ and, as in the previous case the CML L is associative.

To prove the implication 2 $\Rightarrow$ 1) we have only to consider the case when the abelian group H_3 does not satisfy the minimum condition for its subgroups. In this case H_3 has an infinite abelian subgroup B which decomposes into a direct product of cyclic groups of order 3. Let $b \in B$ and let $R \subseteq B$ be such a subgroup that $< b > \cap R = 1$. Let $R = R_1 \times R_2$ be a certain decomposition of group R into a direct product of two infinite subgroups. From statement 2) it follows that the subloops R_1, $< b > \times R_1$, R_2, $< b > \times R_2$ are normal in CML L. In [2] it is proved that if in a CML an element of order 3 generates a normal subloop, then this element belongs to the center of this CML. Then $b \in Z(L)$ and, consequently, $B \subseteq Z(L)$. The subgroup B is infinite, then as in the previous cases, it may be proved that the CML L is associative. Consequently, the implication 2) $\Rightarrow$ 1) holds.

4) $\Rightarrow$ 3). We suppose that $\mathfrak{N}$ is a non-periodic subgroup and let $\alpha \in \mathfrak{N}$ be an element of infinite order. Let $C(\mathfrak{M})$ denote the center of group $\mathfrak{M}$. In [4] it is proved that the quotient group $\mathfrak{M}/C(\mathfrak{M})$ is a locally finite 3-group. Then the element α^k belongs to the center $C(\mathfrak{M})$ for some integer k. Let ε be the unity of the group $\mathfrak{M}$ and let β be an arbitrary element in $\mathfrak{M}$ such that $< \beta > \cap < \alpha >= \varepsilon$. The subgroup $< \alpha^k >$ is normal in $\mathfrak{M}$. Then the product $< \beta >< \alpha^k >$ is a subgroup. As $< \alpha^k > \subseteq < \beta >< \alpha^k > \cap \mathfrak{N}$ then by statement 4) $< \beta >< \alpha^k >$ is a normal subgroup in $\mathfrak{M}$. Let φ be an inner automorphism of group $\mathfrak{M}$. Then $< \beta >< \alpha^k >= \varphi(< \beta >< \alpha^k >) = \varphi(< \beta >)\varphi(< \alpha^k >) = \varphi(< \beta >) < \alpha^k >$, $\varphi(< \beta >) < \alpha^k >=< \beta >< \alpha^k >$. We have $< \beta > \cap < \alpha^k >= \varepsilon$. Then and $\varphi(< \beta >) \cap < a^3 >= \varepsilon$.

We denote $< \alpha^k >= \mathfrak{A}$, $< \beta >= \mathfrak{B}$. Let θ, η be the restrictions of natural homomorphism $\lambda : \mathfrak{A}\mathfrak{B} \rightarrow \mathfrak{A}\mathfrak{B}/\mathfrak{A}$ onto $\mathfrak{B}$ and $\varphi\mathfrak{B}$ respectively. Obviously, $\ker\theta = \mathfrak{B} \cap \mathfrak{A}$, $\ker\eta = \varphi\mathfrak{B} \cap \mathfrak{A}$. Then from equalities $\mathfrak{B} \cap \mathfrak{A} = \varepsilon$, $\varphi\mathfrak{B} \cap \mathfrak{A} = \varepsilon$ it follows that θ, η are monomorphisms.

Let $\beta \in \mathfrak{B}$. Then $\beta = \gamma\alpha$ for some $\gamma \in \varphi\mathfrak{B}$, $\alpha \in \mathfrak{A}$. Further, $\lambda\beta = \lambda(\gamma\alpha)$, $\lambda\beta = \lambda\gamma \cdot \lambda\alpha$, $\lambda\beta = \lambda\gamma \cdot \lambda\varepsilon$, $\lambda\beta = \lambda\gamma$. The homomorphism λ acts onto $\varphi\mathfrak{B}$ as η.

Hence $\lambda\gamma = \eta\gamma$. η is a restriction of λ onto $\varphi\mathfrak{B}$ and is a monomorphism of $\varphi\mathfrak{B}$. Then from $\lambda\beta = \eta\gamma$ it follows that $\beta \in \varphi\mathfrak{B}$, $\mathfrak{B} \subseteq \varphi\mathfrak{B}$. Analogously, $\varphi\mathfrak{B} \subseteq \mathfrak{B}$. Hence $\varphi\mathfrak{B} = \mathfrak{B}$. Consequently, the subgroup $< \beta >$ is normal in $\mathfrak{M}$.

Further, using the normality of $< \beta >$ in $\mathfrak{M}$ it is proved by analogy that the subgroup $< \alpha >$ is normal in $\mathfrak{M}$. We get that any element in $\mathfrak{M}$ generates a normal subgroup in $\mathfrak{M}$. This means that the group $\mathfrak{M}$ is hamiltonian. But any hamiltonian multiplication group of CML is abelian [3]. Hence the implication 4)$\Rightarrow$ 3) holds.

Now we suppose that the abelian group $\mathfrak{N}$ is periodic. Then $\mathfrak{N}$ decomposes into a direct product of its maximal p-subgroups $\mathfrak{N}_p$. Let $\mathfrak{N} = \mathfrak{D} \times \mathfrak{N}_3$. By [2] $\mathfrak{D} \subseteq C(\mathfrak{M})$. The subgroup $\mathfrak{D}$ is normal in $\mathfrak{M}$. If $\mathfrak{D}$ is infinite then as in the previous case, we show that the group $\mathfrak{M}$ is hamiltonian and, consequently, is abelian. If $\mathfrak{D}$ is finite then the subgroup $\mathfrak{N}_3$ is infinite.

Let the infinite abelian group $\mathfrak{N}_3$ satisfy the minimum condition for its subgroups. Then $\mathfrak{N}_3 = \mathfrak{T} \times \mathfrak{K}$, where $\mathfrak{K}$ is a finite group and $\mathfrak{T}$ is an infinite divisible group. By [1] $\mathfrak{T} \subset Z(\mathfrak{M})$ and as in the previous case the CML $\mathfrak{M}$ is abelian.

To prove the implication $4 \Rightarrow 3$) we have to consider only the case when the abelian group $\mathfrak{N}_3$ does not satisfy the minimum condition for its subgroups. In this case $\mathfrak{N}_3$ has an infinite abelian subgroup $\mathfrak{B}$, which decomposes into a direct product of cyclic groups of order 3. Let $\beta \in \mathfrak{B}$ and let $\mathfrak{R} \subseteq \mathfrak{B}$ be such a subgroup that $< \beta > \cap \mathfrak{R} = \varepsilon$. Let $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$ be a certain decomposition of group $\mathfrak{R}$ into a direct product of two infinite subgroups. From statement 2) it follows that the subloops $\mathfrak{R}_1$, $< \beta > \times\mathfrak{R}_1$, $\mathfrak{R}_2$, $< \beta > \times\mathfrak{R}_2$ are normal in group $\mathfrak{M}$. In [3] it is proved that if in a multiplication group of CML an element of order 3 generates a normal subgroup, then this element belongs to the center of this multiplication group. Then $\beta \in C(\mathfrak{M})$ and, consequently, $\mathfrak{B} \subseteq C(\mathfrak{M})$. The subgroup $\mathfrak{B}$ is infinite then as in the previous cases it may be proved that the group $\mathfrak{M}$ is abelian. Consequently, the implication 4) $\Rightarrow$ 3) holds. This completes the proof of Theorem. □

We note that the construction of arbitrary groups that satisfy the equivalence of statements 3), 4) of Theorem is described in [7]. It is easy to see that the equivalence of statements 3), 4) and the equivalence of statements 1), 2) of Theorem are proved by the same schema. But if we use the results of paper [7] to prove the equivalence of statements 3), 4), then the proof doesn't get easier but, on the contrary, it gets more complicated.

References

[1] Sandu N. I. *Commutative Moufang loops with minimum condition for subloops, I.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, No. 3(43), 25–40.

[2] Sandu N. I. *Commutative Moufang loops with minimum condition for subloops, II.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2004, No. 2(45), 33–48.

[3] Lupashco N. T. *On commutative Moufang loops with some restrictions for subgroups of its multiplication groups.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2006, No. 2(51), 95–101.

[4] Bruck R. H. *A survey of binary systems.* Springer Verlag, Berlin-Heidelberg, 1958.

[5] Norton D. *Hamiltonian loops.* Proc. Amer. Math. Soc., 1952, **3**, 56–65.

[6] Sandu N. I. *Centrally nilpotent commutative Moufang loops.* Quasigroups and loops, Mat. Issled., vol. 51, 1979, 145–155 (in Russian).

[7] Semko N. N. *Some forms of non-abelian groups with given systems of invariant infinite abelian subgroups.* Ukr. mat. jurn., 1981, **33**, No. 2, 270–273 (in Russian).

Natalia Lupashco
Tiraspol State University
Departament of Mathematics
str. Iablocichin, 5, Chişinău, MD-2069
Moldova
E-mail: *nlupashco@gmail.com*

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