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Postoptimal analysis of multicriteria combinatorial center location problem

Vladimir Emelichev, Eberhard Girlich, Olga Karelkina

Abstract. A multicriteria variant of a well known combinatorial MINMAX location problem with Pareto and lexicographic optimality principles is considered. Necessary and sufficient conditions of an optimal solution stability of such problems to the initial data perturbations are formulated in terms of binary relations. Numerical examples are given.

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1 Introduction

Many problems of design, planning and management in technical and organizational systems have a pronounced multicriteria character. Multiobjective models appeared in these cases are reduced to the choice of "best" (in a certain sense) values of variable parameters from some discrete aggregate of the given quantities. Therefore recent interest of mathematicians in multicriteria discrete optimization problems keeps very high, as confirmed by the intensive publishing activity (see, e.g., bibliography [1], which contains 234 references).

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty such as lack of input data, inadequacy of mathematical models to real processes, rounding off, calculation errors, etc. Therefore widespread use of discrete optimization models in the last decades stimulated many experts to investigate various aspects of incorrect problems theory and, in particular, to the questions of stability. The most important results in this topic are concerned with postoptimal and parametric behavior analysis of the solutions of the optimization problems with respect to variation of their input data. Generally the technique of such analysis is based on using the properties of multi-valued functions. Such research methods are elaborated in detail and covered in literature about optimization problems with a continuous set of feasible solutions. Numerous articles are devoted to the analysis of conditions when problem possesses some property of invariance under the problem parameters perturbations (see, e. g., [2–5]).

The main difficulty while studying stability of discrete optimization problems is the essential complexity of discrete models. They behave unpredictable even for small changes of initial data. There are a lot of papers (see, e. g., [6–15]) devoted to the analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations. The present work continues our investigations of different stability types of such problems with various partial criteria and optimality principles (see, e. g., [16–23]). The here multicriteria variant of the well-known center location problem (p-center problem) is considered. Some necessary and sufficient conditions of lexicographic and Pareto optima stability under perturbations of initial data are obtained. Numerical examples are given.

2 Basic definitions and notations

Problems of finding the "best" location of equipment and facilities abound in practical situations. Often such problems are formulated as extreme problems in graphs and networks. In particular, if a graph represents a road network with its vertices representing communities, one may have the problem of locating optimally a hospital, fire station or any other emergency service facility. The criterion of optimality may justifiably be taken to be the minimization of the distance (traveling time or other costs) from the facility to the most remote vertex of the graph, i. e. the optimization of the worst-case. In a more general problem, a large number of such facilities may be required to be located. For instance, in the problems which involve the location of emergency facilities it is required to minimize the largest travel distance to any consumer from its nearest facility (center). If there are several costs criteria which have to be minimized, the vector variant of the center location problem arises. Let us consider this problem in the following formulation.

Let $N_m = \{1, 2, \ldots, m\}$ be the set of possible points (centers) of suppliers (equipment, storehouses, facility, etc.) location, N_n be consumers (clients) location, $A = (a_{ijk}) \in \mathbb{R}^{m\times n\times s}$ be the cost matrix a_{ijk}. The cost is connected with delivery of required quantity of products from point $i \in N_m$ to point $j \in N_n$ with criterion $k \in N_s$.

On the set T of nonempty subsets (trajectories) $T \subset 2^{N_m}$, $|T| \geq 2$, let the vector function

$$f(t, A) = (f_1(t, A), f_2(t, A), \ldots, f_s(t, A))$$

be defined with "bottle neck" (MINMAX) criteria:

$$f_k(t, A) = \max_{j\in N_n} \min_{i\in t} a_{ijk} \to \min_{t\in T}, \ k \in N_s.$$

We give the traditional definition of the set of Pareto optimal trajectories:

$$P^s(A) = \{t \in T : \forall t' \in T\backslash\{t\} \ \ (t \underset{A,P}{\not\succ} t')\},$$

where

$$t \underset{A,P}{\succ} t' \ \Leftrightarrow \ f(t, A) \geq f(t', A) \& f(t, A) \neq f(t', A),$$

and the sign $\underset{A,P}{\overline{\succ}}$ is a negation of the relation $\underset{A,P}{\succ}$. The set $P^s(A)$ is nonempty for any matrix $A \in \mathbb{R}^{m\times n\times s}$ as $1 < |T| < \infty$.

The set of lexicographically optimal trajectories is denoted by the formula:

$$L^s(A) = \{t \in T \ : \ \forall t' \in T \ \ (t \underset{A,L}{\overline{\succ}} t')\},$$

where

$$t \underset{A,L}{\succ} t' \ \Leftrightarrow \ \exists\, l \in N_s \ (f_l(t,A) > f_l(t',A) \ \& \ l = \min\{k \in N_s : f_k(t,A) \neq f_k(t',A)\}),$$

and the sign $\underset{A,L}{\overline{\succ}}$ is a negation of the relation $\underset{A,L}{\succ}$. It is easy to see that $L^s(A) \subseteq P^s(A)$ for any matrix $A \in \mathbb{R}^{m\times n\times s}$.

Thus, two multicriteria center location problems appear: with Pareto principle of optimality, i. e. the problem of finding the set $P^s(A)$, and with lexicographic principle of optimality, i. e. the problem of finding the set $L^s(A)$.

In particular in scalar case ($s = 1$) we get the well-known p-center problem [24–27], i.e. minimax location problem:

$$\max_{j\in N_n} \min_{i\in t} a_{ij} \to \min$$

$$t \in T, \ |t| = p,$$

where p is an integer number, which satisfies the inequalities $1 \leq p \leq m-1$. Thereby in this problem the situation is modeled, when it is required to locate p facilities in N_m possible points to minimize the largest travel distance to any consumer from its nearest facility.

It is known (see, e. g., [28]), that the set of lexicographically optimal trajectories $L^s(A)$ can be defined as the result of solving the sequence of s scalar problems

$$L^s_k(A) = \operatorname{Arg\,min}\{f_k(t,A) \mid t \in L^s_{k-1}(A)\}, \qquad k \in N_s, \tag{1}$$

where $L^s_0(A) = T$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all optimal trajectories of the corresponding scalar optimization problem. Hence the following inclusions

$$T \supseteq L^s_1(A) \supseteq L^s_2(A) \supseteq \ldots \supseteq L^s_s(A) = L^s(A) \tag{2}$$

are true.

Perturbations of the vector criterion $f(t,A)$ parameters are modeled by adding matrix A to the matrices of the set

$$\Omega(\varepsilon) = \{A' \in \mathbb{R}^{m\times n\times s} \ \ : \ \ ||A'|| < \varepsilon\},$$

where $\varepsilon > 0$, $||A'|| = \max\{|a'_{ijk}| \ : \ (i,j,k) \in N_m \times N_n \times N_s\}$, $A' = (a'_{ijk})$. The set $\Omega(\varepsilon)$ is called the set of perturbing matrices.

Pareto optimal trajectory $t \in P^s(A)$ is called stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (t \in P^s(A + A')).$$

Lexicographically optimal trajectory t is called stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (t \in L^s(A + A')).$$

To prove stability criteria, we consider a number of evident properties and also formulate and prove 4 lemmas.

3 Properties

Directly from definitions of the binary relations $t \underset{A,P}{\succ} t'$ and $t \underset{A,L}{\succ} t'$ follows

Property 1. *If $t \underset{A,P}{\succ} t'$, then $t \underset{A,L}{\succ} t'$.*

For any indexes $k \in N_s,\ j \in N_n$ and trajectory t put

$$N_{jk}(t, A) = \{l \in t \;:\; f_k(t, A) = g_{jk}(t, A) = a_{ljk}\},$$

$$J_k(t, A) = \{j \in N_n \;: f_k(t, A) = g_{jk}(t, A)\},$$

where

$$g_{jk}(t, A) = \min_{i \in t} a_{ijk}.$$

Next properties directly follow from these notions.

Property 2. *If $q \in J_k(t, A)$, then $f_k(t, A) = g_{qk}(t, A)$.*

Property 3. *If $q \in J_k(t, A)$ and $p \in N_{qk}(t, A)$, then $f_k(t, A) = g_{qk}(t, A) = a_{pqk}$.*

Property 4. *$N_{jk}(t, A) \neq \emptyset$ if and only if $j \in J_k(t, A)$.*

Property 5. *If $N_{jk}(t, A) = \emptyset$, then $g_{jk}(t, A) < f_k(t, A)$.*

Property 6. *If $g_{jk}(t, A) > g_{jk}(t', A)$, then there exists an index $p \in t' \backslash t$ such that $g_{jk}(t', A) = g_{jk}(t' \backslash t, A) = a_{pjk}$.*

For any index $k \in N_s$ we define several binary relations on the set of trajectories T

$$t \underset{A,k}{\vdash} t' \;\Leftrightarrow\; t \underset{A,k}{\models\!\!\sim} t' \underset{A,k}{\models\!\!\approx} t,$$

$$t \underset{A,k}{\models\!\!\sim} t' \;\Leftrightarrow\; \forall j \in J_k(t, A) \;\; (N_{jk}(t, A) \supseteq N_{jk}(t', A)),$$

$$t' \underset{A,k}{\models\!\!\approx} t \;\Leftrightarrow\; J_k(t', A) \supseteq J_k(t, A).$$

Furthermore, we will use binary relations

$$t \underset{A}{\vdash} t' \;\Leftrightarrow\; \forall k \in N_s \;\; (t \underset{A,k}{\vdash} t'),$$

$$t \underset{A}{\sim} t' \;\Leftrightarrow\; f(t, A) = f(t', A).$$

By virtue of continuity of the function $g_{jk}(t, A)$ in parameters space $\mathbb{R}^m$ from the relations $N_{jk}(t, A) \supseteq N_{jk}(t', A) \neq \emptyset$ the formula follows

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (g_{jk}(t, A + A') \leq g_{jk}(t', A + A')). \tag{3}$$

Therefore the following property holds

Property 7. *If for any index $k \in N_s$ the relation $t \underset{A,k}{\vdash\!\sim} t'$ holds, then*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; \forall k \in N_s \;\; \forall j \in J_k(t, A + A') \;\; (g_{jk}(t, A + A') \leq g_{jk}(t', A + A')).$$

Property 8. *If $t \underset{A}{\vdash} t'$, then*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; \forall k \in N_s \;\; (f_k(t, A + A') \leq f_k(t', A + A')).$$

Property 9. *If $t \underset{A}{\vdash} t'$, then there exists a number $\varepsilon > 0$ such that for any perturbing matrix $A' \in \Omega(\varepsilon)$ the following relation holds*

$$t \underset{A+A',P}{\overline{\succ}} t'.$$

Property 10. *If any of the following conclusions holds for trajectories t and t'*

$$(i) \;\; f_1(t', A) > f_1(t, A),$$

$$(ii) \; \exists r \in N_{s-1} \;\; (f_{r+1}(t', A) > f_{r+1}(t, A) \;\&\; \forall k \in N_r \;\; (t \underset{A,k}{\vdash} t')),$$

then the formula

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \; (t \underset{A+A',L}{\overline{\succ}} t') \tag{4}$$

is true

Proof. If $f_1(t', A) > f_1(t, A)$, then in view of continuity of the function $f_k(t, A)$ in parameters space $\mathbb{R}^{m \times n}$ we have

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (f_1(t', A + A') > f_1(t, A + A')).$$

Hence (4) holds. $\square$

Now let condition (ii) hold. Then, using $t \underset{A,k}{\mid\!\sim} t'$, $k \in N_r$, in view of (3) we get

$$\exists \varepsilon' > 0 \ \ \forall A' \in \Omega(\varepsilon') \ \ \forall k \in N_r \ \ \forall j \in J_k(t, A+A') \ \ (g_{jk}(t, A+A') \leq g_{jk}(t', A+A')).$$

Therefore

$$\exists \varepsilon' > 0 \ \ \forall A' \in \Omega(\varepsilon') \ \ \forall k \in N_r \ \ (f_k(t, A+A') \leq f_k(t', A+A')). \tag{5}$$

In addition, since $f_{r+1}(t', A) > f_{r+1}(t, A)$ it follows that

$$\exists \varepsilon'' > 0 \ \ \forall A' \in \Omega(\varepsilon'') \ \ (f_{r+1}(t', A+A') > f_{r+1}(t, A+A')). \tag{6}$$

Assuming $\varepsilon = \min\{\varepsilon', \varepsilon''\}$, we derive (4) from (5) and (6).

4 Lemmas

Set

$$\overline{P^s}(A) = T \setminus P^s(A).$$

Lemma 1. *If $t^0 \in P^s(A)$, $t^0 \underset{A}{\sim} t$ and there exists an index $r \in N_s$ such that $t \underset{A,r}{\overline{\mid\!\approx}} t^0$, then the trajectory t^0 is not stable.*

Proof. From $t \underset{A,r}{\overline{\mid\!\approx}} t^0$ it follows that there exists an index $q \in J_r(t^0, A) \backslash J_r(t, A)$. Therefore according to property 4 $N_{qr}(t, A) = \emptyset$. Hence using property 5 we have $g_{qr}(t, A) < f_r(t, A)$ and applying property 2 we derive $f_r(t^0, A) = g_{qr}(t^0, A)$. Thus, taking into account $t^0 \underset{A}{\sim} t$ we obtain $g_{qr}(t^0, A) > g_{qr}(t, A)$. Hence in view of property 6 there exists an index $p \in t \backslash t^0$, such that

$$g_{qr}(t, A) = g_{qr}(t \backslash t^0, A) = a_{pqr}. \tag{7}$$

For any number $\varepsilon > 0$ we build elements of the perturbing matrix $A^0 = (a^0_{ijk}) \in \Omega(\varepsilon)$ of size $m \times n \times s$ by the rule

$$a^0_{ijk} = \begin{cases} \alpha, & \text{if } i \in t^0, \ j = q, \ k = r, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$. We show that $t^0 \in \overline{P^s}(A + A^0)$. According to the matrix construction the following equalities hold

$$g_{qr}(t^0, A + A^0) = g_{qr}(t^0, A) + \alpha,$$

$$g_{jr}(t^0, A + A^0) = g_{jr}(t^0, A) \text{ for } j \neq q,$$

$$g_{jr}(t, A + A^0) = g_{jr}(t, A) \text{ for } j \neq q,$$

and by (7) it follows that

$$g_{qr}(t, A + A^0) = g_{qr}(t, A).$$

Hence we derive

$$f_r(t^0, A+A^0) = \max_{j\in N_n} g_{jr}(t^0, A+A^0) = \max\{g_{qr}(t^0, A+A^0),\ \max_{j\neq q} g_{jr}(t^0, A+A^0)\} =$$

$$= \max\{g_{qr}(t^0, A) + \alpha,\ \max_{j\neq q} \min_{i\in t^0} a_{ijq}\} = f_r(t^0, A) + \alpha,$$

$$f_r(t, A+A^0) = \max_{j\in N_n} g_{jr}(t, A+A^0) = \max\{g_{qr}(t, A),\ \max_{j\neq q} \min_{i\in t} a_{ijr}\} = f_r(t, A).$$

It follows from these equalities that

$$f_r(t^0, A+A^0) > f_r(t, A+A^0). \tag{8}$$

Furthermore, taking into account the construction of the perturbing matrix A^0 and the relation $t^0 \underset{A}{\sim} t$ the following equalities are evident

$$f_k(t^0, A+A^0) = f_k(t, A+A^0) \ \text{ for } \ k \neq r. \tag{9}$$

Therefore

$$t^0 \underset{A+A^0,P}{\succ} t.$$

Thus we have

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ \ (t^0 \in \overline{P^s}(A+A^0)), \tag{10}$$

i. e. trajectory t^0 is not stable. $\square$

Lemma 2. *If $t^0 \in P^s(A)$, $t^0 \underset{A}{\sim} t$ and there exists an index $r \in N_s$ such that $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$, then trajectory t^0 is not stable.*

Proof. We assume that $t \underset{A,r}{\mid\!\approx} t^0$. Otherwise t^0 is not stable by virtue of Lemma 1. Since $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$, then in view of $t \underset{A,r}{\mid\!\approx} t^0$ there exists an index $q \in J_r(t, A) \supseteq J_r(t^0, A)$ such that $p \in N_{qr}(t, A)\backslash N_{qr}(t^0, A)$.

For any number $\varepsilon > 0$ we build elements of the perturbing matrix $A^0 = (a^0_{ijk}) \in \Omega(\varepsilon)$ of size $m \times n \times s$ by the rule

$$a^0_{ijk} = \begin{cases} -\alpha, & \text{if } i = p,\ j = q,\ k = r, \\ -\alpha, & \text{if } i \in t,\ j \in N_n\backslash\{q\},\ k = r, \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < \alpha < \varepsilon$.

Let us prove that $t^0 \in \overline{P^s}(A+A^0)$. It suffices to prove that relations (8) and (9) are valid. Taking into account the construction of matrix A^0 and the relation $t^0 \underset{A}{\sim} t$ equalities (9) are evident.

Further let us prove inequalities (8). Since $p \in N_{qr}(t, A)$, then using properties 2 and 3 we obtain $f_r(t, A) = g_{qr}(t, A) = a_{pqr}$. Hence according to the construction of matrix A^0 it follows that

$$g_{qr}(t, A + A^0) = g_{qr}(t, A) - \alpha = f_r(t, A) - \alpha,$$

$$g_{jr}(t, A + A^0) = g_{jr}(t, A) - \alpha \quad \text{for} \quad j \neq q.$$

Therefore we derive

$$f_r(t, A + A^0) = \max_{j \neq N_n} g_{jr}(t, A + A^0) = \max\{g_{qr}(t, A + A^0),\ \max_{j \neq q} g_{jr}(t, A + A^0)\} =$$

$$= \max\{f_r(t, A) - \alpha,\ \max_{j \neq q}(g_{jr}(t, A) - \alpha)\} = f_r(t, A) - \alpha = f_r(t^0, A) - \alpha. \tag{11}$$

Further let us prove that $f_r(t^0, A + A^0) = f_r(t^0, A)$.

Taking into account the construction of matrix A^0 the following inequalities are evident

$$g_{jr}(t^0, A + A^0) \leq g_{jr}(t^0, A), \quad j \in N_n.$$

Furthermore, using $p \notin N_{qr}(t^0, A)$ and $q \in J_r(t^0, A)$, we have

$$g_{qr}(t^0, A + A^0) = g_{qr}(t^0, A) = f_r(t^0, A).$$

Thus in view of $f_r(t^0, A) \geq g_{jr}(t^0, A) \geq g_{jr}(t^0, A + A^0)$ for $j \in N_n$ we derive

$$\begin{gathered} f_r(t^0, A + A^0) = \max_{j \in N_n} g_{jr}(t^0, A + A^0) = \\ = \max\{g_{qr}(t^0, A),\ \max_{j \neq q} g_{jr}(t^0, A + A^0)\} = f_r(t^0, A). \end{gathered} \tag{12}$$

Combining (11) and (12), we obtain inequality (8). Thus we derive formula (10). Consequently the trajectory t^0 is not stable. □

Set

$$\overline{L^s}(A) = T \setminus L^s(A).$$

Lemma 3. *If $t^0 \in L^s(A)$ and there exist $r \in N_s$ and $t \in L^s_r(A)$ such that $t \underset{A,r}{\overline{\approx}} t^0$, then the trajectory t^0 is not stable.*

Proof. This lemma con be proved in analogous way as Lemma 1. It can be done by constructing a perturbing matrix A^0 the same way as in proof of lemma 1 and repeating all arguments. Thus the inequality (8) is true.

Moreover, since $t^0, t \in L^s_r(A)$, then the following inequalities hold for $r > 1$

$$f_k(t^0, A) = f_k(t, A),\ k \in N_{r-1}.$$

Therefore, taking into account the construction of matrix A^0, we obtain

$$f_k(t^0, A + A^0) = f_k(t, A + A^0),\ \in N_{r-1}. \tag{13}$$

Hence

$$t^0 \underset{A+A^0,L}{\succ} t, \tag{14}$$

Summarizing we derive the formula

$$\forall \varepsilon > 0 \;\; \exists A^0 \in \Omega(\varepsilon) \;\; (t^0 \in \overline{L^s}(A + A^0)), \tag{15}$$

i. e. the trajectory $t^0 \in L^s(A)$ is not stable. □

Lemma 4. *If $t^0 \in L^s(A)$ and there exist $r \in N_s$ and $t \in L^s_r(A)$ such that $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$, then the trajectory t^0 is not stable.*

Proof. If we construct a matrix A^0 by the same rules as in lemma 2 and carry out the same reasoning, then we conclude that the inequalities (8) are true. Moreover, taking into account $t^0, t \in L^s_r(A)$ we obtain equalities (13). Hence we have (14).

Thus, formula (15) is valid, i. e. the trajectory $t^0 \in L^s(A)$ is not stable. □

5 Theorems

For any trajectory t^0 set

$$Q^s(t^0, A) = \{t \in T \;:\; t^0 \underset{A}{\sim} t\}.$$

Theorem 1. *A trajectory $t^0 \in P^s(A)$ is stable if and only if the formula*

$$\forall t \in Q^s(t^0, A) \;\; (t^0 \underset{A}{\vdash} t) \tag{16}$$

is valid.

Proof. Necessity. Let a trajectory $t^0 \in P^s(A)$ be stable. Assume that formula (16) is not true. Then there exist $r \in N_s$ and $t \underset{A}{\sim} t^0$ such that $t^0 \underset{A}{\overline{\vdash}} t$, i. e. one of the following relations holds: $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$ or $t \underset{A,r}{\overline{|\!\approx}} t^0$. Therefore according to lemmas 1 and 2 the trajectory t^0 is not stable. Contradiction.

Sufficiency. Let formula (16) hold. Let us show that trajectory $t^0 \in P^s(A)$ is stable. We consider two possible cases for an arbitrary trajectory $t \in T$.

Case 1. $t \in Q^s(t^0, A)$. Then according to the theorem condition $t^0 \underset{A}{\vdash} t$. Hence from property 9 it follows that the formula

$$\exists \varepsilon(t) > 0 \;\; \forall A' \in \Omega(\varepsilon(t)) \; (t^0 \underset{A+A',P}{\overline{\succ}} t) \tag{17}$$

is true.

Case 2. $t \in T \setminus Q^s(t^0, A)$. Therefore the relation $t^0 \underset{A}{\sim} t$ does not hold. Then there exists an index $r \in N_s$ such that $f_r(t^0, A) < f_r(t, A)$. Hence by virtue of continuity of the function $f_r(t, A)$ in $\mathbb{R}^{m \times n}$ there exists a number $\varepsilon(t)$ such that formula (17) is valid.

Summarizing both cases, we obtain

$$\exists \varepsilon^* > 0 \ \ \forall t \in T \ \ \forall A' \in \Omega(\varepsilon^*) \ \ (t^0 \underset{A+A',L}{\overline{\succsim}} t),$$

where $\varepsilon^* = \min\{\varepsilon(t) : t \in T\}$, i. e. trajectory $t^0 \in L^s(A)$ is stable. $\square$

Theorem 2. *A trajectory $t^0 \in L^s(A)$ is stable if and only if the formula*

$$\forall k \in N_s \ \ \forall t \in L_k^s(A) \ \ (t^0 \underset{A,k}{\vdash} t) \tag{18}$$

is valid.

Proof. Necessity. Let a trajectory $t^0 \in L^s(A)$ be stable. Assume that formula (18) does not hold. Then there exist $r \in N_s$ and $t \in L_r^s(A)$ such that $t^0 \underset{A,r}{\overline{\vdash}} t$. Therefore one of the following relations holds: $t^0 \overline{\underset{A,r}{\vdash\!\sim}} t$ or $t \overline{\underset{A,r}{\vdash\!\approx}} t^0$. Further using Lemmas 3 and 4 we conclude that trajectory $t^0 \in L^s(A)$ is not stable. Contradiction.

Sufficiency. Let formula (18) hold. We show that a trajectory $t^0 \in L^s(A)$ is stable. We consider two possible cases for an arbitrary trajectory $t \in T$.

Case 1. $t \in L_1^s(A)$. First, let $t \in L^s(A)$. Then according to the theorem condition for any index $k \in N_s$ the relation $t^0 \underset{A,r}{\vdash} t$ is valid. Therefore from properties 1 and 9 it follows that the following formula holds

$$\exists \varepsilon(t) > 0 \ \ \forall A' \in \Omega(\varepsilon(t)) \ (t^0 \underset{A+A',L}{\overline{\succsim}} t). \tag{19}$$

Now, let $t \in L_1^s(A) \setminus L^s(A)$. Then there exists an index $r = r(t) \in N_s \setminus \{1\}$ such that $t \notin L_r^s(A)$ and $t \in L_r^s(A)$ for $k \in N_{r-1}$. Hence we have

$$f_{r+1}(t, A) > f_{r+1}(t^0, A) \ \& \ \forall k \in N_{r-1} \ \ (t^0 \underset{A,k}{\vdash} t).$$

Taking into account these facts and property $10(ii)$, we conclude that the following formula holds

$$\exists \varepsilon(t) > 0 \ \ \forall A' \in \Omega(\varepsilon(t)) \ \ (t \underset{A+A',L}{\succ} t^0).$$

Thus we obtain (19).

Case 2. $t \in T \setminus L_1^s(A)$. Therefore the relation

$$f_1(t, A) > f_1(t^0, A)$$

is valid. Hence formula (19) follows from property $10(i)$.

Summarizing both cases, we obtain

$$\exists \varepsilon^* > 0 \ \ \forall t \in T \ \ \forall A' \in \Omega(\varepsilon^*) \ \ (t^0 \underset{A+A',L}{\not\succ} t),$$

where $\varepsilon^* = \min\{\varepsilon(t) : t \in T\}$, i. e. trajectory $t^0 \in L^s(A)$ is stable. □

6 Corollaries

Next corollaries follow from Theorems 1 and 2.

Corollary 1. *The equality $Q^s(t^0, A) = \{t^0\}$ is the sufficient condition for a trajectory $t^0 \in P^s(A)$ to be stable.*

Corollary 2. *The formula*

$$\forall t \in Q^s(t^0, A) \ \ \forall k \in N_s \ \ (t \underset{A,k}{\models\approx} t^0)$$

is the necessary condition for trajectory $t^0 \in P^s(A)$ to be stable.

Corollary 3. *A sufficient condition for a trajectory $t^0 \in P^s(A)$ to be stable is that for any trajectory $t \in Q^s(t^0, A)$ and any index $k \in N_s$ the following equalities hold*

$$J_k(t^0, A) = J_k(t, A),$$

$$N_{jk}(t^0, A) = N_{jk}(t, A), \quad j \in J_k(t^0, A).$$

It is evident that the problem under consideration turns to the vector combinatorial problem with partial criteria of the form MINMIN for $n = 1$ ($A \in \mathbb{R}^{m\times s}$). Hence the following well-known result follows from Theorem 1.

Corollary 4. *[29] A trajectory $t^0 \in P^s(A)$ of the problem with partial criteria of the form MINMIN ($n = 1$) is stable if and only if the following formula holds*

$$\forall t \in Q^s(t^0, A) \ \ \forall k \in N_s \ \ (N_k(t^0, A) \supseteq N_k(t, A)),$$

where $N_k(t, A) = \mathrm{Arg}\min\{a_{ik} \ : \ i \in t\}, \ A = (a_{ik}) \in \mathbb{R}^{m\times s}$.

Corollary 5. *If $|t| = 1$ for any trajectory $t \in T$ ($p = 1$), then the equality $Q^s(t^0, A) = \{t^0\}$ is the necessary and sufficient condition for trajectory of a vector 1-center problem $t^0 \in P^s(A)$ to be stable.*

Corollary 6. *The equality $L^s_1(A) = \{t^0\}$ is the sufficient condition for a trajectory t^0 to be stable.*

Corollary 7. *If $p = 1$ (a vector 1-center problem), then the equality $L^s_1(A) = \{t^0\}$ is the necessary and sufficient condition for trajectory $t^0 \in L^s(A)$ to be stable.*

Corollary 8. *The formula*

$$\forall k \in N_s \quad \forall t \in L_k^s(A) \quad (t \underset{A,k}{\not\approx} t^0)$$

is the necessary condition for trajectory $t^0 \in L^s(A)$ to be stable.

Corollary 9. *For a trajectory $t^0 \in L^s(A)$ to be stable it is sufficient for any index $k \in N_s$ and any trajectory $t \in L_k^s(A)$ to have*

$$J_k(t^0, A) = J_k(t, A),$$

$$N_{jk}(t^0, A) = N_{jk}(t, A), \quad j \in J_k(t^0, A).$$

Corollary 10. *[30] A trajectory $t^0 \in L^s(A)$ of the problem with partial criteria of the form MINMIN ($n = 1$) is stable if and only if the following formula holds*

$$\forall k \in N_s \quad \forall t \in L_k^s(A) \quad (N_k(t^0, A) \supseteq N_k(t, A)),$$

$N_k(t, A) = Arg\min\{a_{ik} \ : \ i \in t\}, \ A = (a_{ik}) \in \mathbb{R}^{m \times s}$.

Corollary 11. *A trajectory $t^0 \in L^s(A)$ is not stable if*

$$\exists k \in N_s \quad \exists t \in L_k^s(A) \quad (J_k(t^0, A) \cap J_k(t, A) = \emptyset).$$

Corollary 12. *A trajectory $t^0 \in L^s(A)$ is not stable if*

$$\exists k \in N_s \quad \exists t \in L_k^s(A) \quad \exists j \in J_k(t^0, A) \quad (N_{jk}(t^0, A) \not\supseteq N_{jk}(t, A)).$$

7 Examples

Let us give several examples which illustrate results stated above. First, consider the example of the problem, in which each Pareto optimal trajectory is stable.

Example 1. Let $m = 2, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{1, 2\}$, $t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1), \ f(t^2, A) = (0, 1), \ f(t^3, A) = (2, 2)$. Hence $P^2(A) = \{t^1, t^2\}$, $t^1 \underset{A}{\sim} t^2$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = J_2(t^2, A) = \{2\},$$

$$N_{21}(t^1, A) = N_{21}(t^2, A) = \{2\},$$

$$N_{22}(t^1, A) = N_{22}(t^2, A) = \{2\}.$$

Therefore we have

$$\forall k \in N_2 \quad (t^2 \underset{A,k}{\vdash\!\sim} t^1 \underset{A,k}{\vdash\!\approx} t^2),$$

$$\forall k \in N_2 \quad (t^1 \underset{A,k}{\vdash\!\sim} t^2 \underset{A,k}{\vdash\!\approx} t^1),$$

i. e. $t^2 \underset{A}{\vdash} t^1 \underset{A}{\vdash} t^2$. Hence formula (16) is valid for trajectories t^1 and t^2. Thus, by virtue of Theorem 1 trajectories t^1 and t^2 are stable.

The following example illustrates the situation when both stable and nonstable trajectories exist among Pareto optimal trajectories.

Example 2. Let $m = 3, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3, t^4\}, \quad t^1 = \{1, 2\}, \quad t^2 = \{1, 3\}, \quad t^3 = \{2, 3\}, \ t^4 = \{1\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1), \ f(t^2, A) = (0, 1), \ f(t^3, A) = (1, 0), \ f(t^4, A) = (0, 2)$. Therefore $P^2(A) = \{t^1, t^2, t^3\}, \ t^1 \underset{A}{\sim} t^2, \ Q^2(t^3) = \{t^3\}$. Taking into account the last equality and Corollary 1 we derive that trajectory t^3 is stable. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = \{1\}, \ J_2(t^2, A) = \{2\}.$$

Hence we conclude that there exists index $k = 2$ such that $J_2(t^1, A) \not\subseteq J_2(t^2, A)$ and $J_2(t^2, A) \not\subseteq J_2(t^1, A)$. Hence $t^1 \underset{A,2}{\overline{\vdash\!\approx}} t^2 \underset{A,2}{\overline{\vdash\!\approx}} t^1$, i. e. $t^2 \underset{A}{\overline{\vdash}} t^1 \underset{A}{\overline{\vdash}} t^2$. Thus, by virtue of Theorem 1 trajectories $t^1, \ t^2$ are not stable.

Further we consider the example of the problem in which each Pareto optimal trajectory is nonstable.

Example 3. Let $m = 3, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{2, 3\}, \ t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1), \ f(t^2, A) = (0, 1), \ f(t^3, A) = (2, 1)$. Therefore $P^2(A) = \{t^1, t^2\}, \ t^1 \underset{A}{\sim} t^2$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = J_2(t^2, A) = \{2\},$$

$$N_{21}(t^1, A) = \{1\},\ N_{21}(t^2, A) = \{2\}.$$

Hence $N_{21}(t^1, A) \not\subseteq N_{21}(t^2, A)$, $N_{21}(t^2, A) \not\subseteq N_{21}(t^1, A)$, i. e. there exist $k = 1$ and $j = 2$ such that $t^1 \underset{A,1}{\overline{\mid\!\sim}} t^2 \underset{A,1}{\overline{\mid\!\sim}} t^1$. Therefore $t^1 \underset{A}{\overline{\vdash}} t^2 \underset{A}{\overline{\vdash}} t^1$. Hence formula (16) is not valid for trajectories t^1 and t^2. Thus, by virtue of Theorem 1 trajectories t^1 and t^2 are not stable.

Now consider the example of the problem in which each lexicographically optimal trajectory is stable.

Example 4. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2\}$, $t^2 = \{2, 3\}$, $t^3 = \{1, 2, 3\}$ and

$$A_1 = \begin{pmatrix} -2 & -1 & 0 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (-1, -1)$, $f(t^2, A) = (-1, 1)$, $f(t^3, A) = (-1, -1)$. Therefore $L_1^2(A) = \{t^1, t^2, t^3\} = T$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = J_1(t^3, A) = \{2, 3\},$$

$$J_2(t^1, A) = J_2(t^3, A) = \{1\},\ J_2(t^2, A) = \{3\},$$

$$\{1, 2\} = N_{21}(t^1, A) = N_{21}(t^3, A) \subset\ N_{21}(t^2, A) = \{2\},$$

$$N_{31}(t^1, A) = N_{31}(t^2, A) = N_{31}(t^3, A) = \{2\},$$

$$N_{12}(t^1, A) = N_{12}(t^3, A) = \{1\}.$$

Hence the following relations hold

$$t^1 \underset{A,1}{\mid\!\sim} t^2 \underset{A,1}{\mid\!\approx} t^1,\ t^1 \underset{A,1}{\mid\!\sim} t^3 \underset{A,1}{\mid\!\approx} t^1,\ t^3 \underset{A,1}{\mid\!\sim} t^1 \underset{A,1}{\mid\!\approx} t^3,\ t^3 \underset{A,1}{\mid\!\sim} t^2 \underset{A,1}{\mid\!\approx} t^3,$$

$$t^1 \underset{A,2}{\mid\!\sim} t^3 \underset{A,2}{\mid\!\approx} t^1,\ t^3 \underset{A,2}{\mid\!\sim} t^1 \underset{A,2}{\mid\!\approx} t^3.$$

Thus,

$$\forall k \in N_2 \ \ \forall t \in L_k^2(A) \ \ (t^1 \underset{A}{\vdash} t),$$

$$\forall k \in N_2 \ \ \forall t \in L_k^2(A) \ \ (t^3 \underset{A}{\vdash} t),$$

i. e. formula (18) is true. Therefore, by virtue of Theorem 2 trajectories t^1, t^3 are stable.

Further, we consider the problem in which each lexicographically optimal trajectory is not stable.

Example 5. Let $m = 2,\ n = 3,\ s = 2,\ T = \{t^1, t^2\},\ t^1 = \{1\},\ t^2 = \{2\}$ and

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & -2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -2 & -1 \end{pmatrix}.$$

Then $f(t^1, A) = (0, -1),\ f(t^2, A) = (0, -1)$. Therefore $L_1^2(A) = \{t^1, t^2\} = T$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{3\},$$
$$\{1, 3\} = J_2(t^1, A) \not\subseteq\ J_2(t^2, A) = \{3\},$$
$$N_{31}(t^1, A) = \{1\},\ N_{31}(t^2, A) = \{2\}.$$

Hence we have

$$N_{31}(t^1, A) \not\subseteq N_{31}(t^2, A),\ N_{31}(t^2, A) \not\subseteq N_{31}(t^1, A),$$

i. e. $t^1 \underset{A,1}{\overline{\vdash\!\!\sim}} t^2 \underset{A,1}{\overline{\vdash\!\!\sim}} t^1,\ t^2 \underset{A,1}{\overline{\vdash\!\!\approx}} t^1$. Therefore $t^1 \underset{A}{\overline{\vdash}} t^2 \underset{A}{\overline{\vdash}} t^1$. Hence formula (18) is not valid for both lexicographically optimal trajectories t^1 and t^2. Thus, in view of Theorem 2 they are not stable.

The following example illustrates situation when both stable and nonstable trajectories exist among lexicographically optimal trajectories.

Example 6. Let $m = 2, \quad n = 3, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{1, 2\}, \quad t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & -1 & -2 \\ 0 & -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (-1, 1),\ f(t^2, A) = (-1, 1),\ f(t^3) = (0, 2)$. Therefore $L_1^2(A) = \{t^1, t^2\}$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{1, 2\},$$
$$J_2(t^1, A) = J_2(t^2, A) = \{1, 3\},$$
$$N_{11}(t^1, A) = N_{11}(t^2, A) = \{1\},$$
$$\{1\} = N_{21}(t^1, A) \subset\ N_{21}(t^2, A) = \{1, 2\},$$
$$N_{12}(t^1, A) = N_{12}(t^2, A) = \{1\},$$
$$\{1\} = N_{32}(t^1, A) \subset\ N_{32}(t^2, A) = \{1, 2\}.$$

Hence we derive

$$\forall k \in N_2 \quad (t^2 \underset{A}{\vdash} t^1).$$

Therefore formula (18) is valid and by virtue of Theorem 2 trajectory t^2 is stable. But

$$N_{21}(t^1, A) \not\supseteq N_{21}(t^2, A),$$

i. e. there exist index $k = 1$ and trajectory $t^2 \in L_1^2(A)$ such that $t^1 \underset{A,1}{\overline{\vdash\!\!\sim}} t^2$. Hence $t^1 \underset{A}{\overline{\vdash}} t^2$. Thus, formula (18) does not hold and by virtue of Theorem 2 trajectory t^1 is nonstable.

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Vladimir Emelichev, Olga Karelkina
Belarusian State University
ave. Independence, 4, Minsk 220030
Belarus
E-mail: *emelichev@bsu.by, okarel@mail.ru*

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Eberhard Girlich
Otto-von-Guericke-Universitat
Univestitatsplatz, 2, Magdeburg 39106
Germany
E-mail: *eberhard.girlich@mathematik.uni-magdeburg.de*