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# On $\pi$-quasigroups isotopic to abelian groups

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**Abstract.** A $\pi$-quasigroup is a quasigroup satisfying one of the seven minimal identities from the V.Belousov's classification given in [1]. Some general results about $\pi$-quasigroups isotopic to groups are obtained by V. Belousov and A. Gwaramija in [1] and [2]. $\pi$-Quasigroups isotopic to abelian groups are investigated in this paper.

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Let $Q$ be a nonempty set and let $\Sigma(Q)$ be the set of all binary quasigroup operations defined on $Q$. V. Belousov (see [1]) found all nontrivial identities $w_1 = w_2$ in $Q(\Sigma)$ having the length $|w_1| + |w_2| = 5$ ($|w|$ is the number of free elements in the word $w$), called minimal identities. He proved that, using transformation to inverse operations, every minimal identity can be transformed into the form:

$$A(x, B(x, C(x, y))) = y.$$

Using multiplication of operations, the last identity can be rewritten in abbreviated form as $ABC = E$, where $E(x, y) = y, \forall x, y \in Q$, is the right selector.

Minimal nontrivial identities imply the orthogonality of participating operations. It is known that two quasigroup operations $A$ and $B$, defined on a set $Q$, are orthogonal if and only if there exists a quasigroup operation $C$ on $Q$, such that $CBA^{-1} = E$ [1]. Hence, if $A, B, C \in Q(\Sigma)$ and $ABC = E$,we have $A \perp B^{-1}$, $B \perp C^{-1}$ and $C \perp A^{-1}$.

A quasigroup $Q(A)$ is called a $\pi$-quasigroup of type $[\alpha, \beta, \gamma]$, where $\alpha, \beta, \gamma \in S_3$, if it satisfies the identity ${}^{\alpha}A^{\beta}A^{\gamma}A = E$.

V. Belousov considered the following transformations of types on $S_3^3$: $f[\alpha, \beta, \gamma] = [\beta, \gamma, \alpha]$ and $h[\alpha, \beta, \gamma] = [r\gamma, r\beta, r\alpha]$, where $r = (23)$. The transformations $f$ and $h$ generate the group $S^0 = \{\varepsilon, f, f^2, h, fh, f^2h\} \cong S_3$. Two types $T = [\alpha, \beta, \gamma]$ and $T' = [\alpha', \beta', \gamma']$ are called parastrophically equivalent if there exist $g \in S^0$ and $\theta \in S_3$ such that $T' = gT\theta$. If the types $T$ and $T'$ are parastrophically equivalent then we'll denote $T \sim T'$. The binary relation " $\sim$ " is an equivalence on $S_3^3$ and $S_3^3/\sim$ consists of 7 classes [1]. A system of representatives of the seven equivalence classes is:

$T_1 = [\varepsilon, \varepsilon, \varepsilon]$, $T_2 = [\varepsilon, \varepsilon, l]$, $T_4 = [\varepsilon, \varepsilon, lr]$, $T_6 = [\varepsilon, l, lr]$, $T_{10} = [\varepsilon, lr, l]$,
$T_8 = [\varepsilon, rl, lr]$, $T_{11} = [\varepsilon, lr, rl]$, where $l = (13)$, $r = (23)$, $s = (12)$.

Two minimal identities

$${}^{\alpha}A(x, {}^{\beta}A(x, {}^{\gamma}A(x, y))) = y,$$

$$^{\alpha'}A(x,^{\beta'}A(x,^{\gamma'}A(x,y))) = y,$$

where $A \in \Sigma(Q)$, are called parastrophically equivalent if the types $T = [\alpha, \beta, \gamma]$ and $T' = [\alpha', \beta', \gamma']$ are parastrophically equivalent.

Denoting $A = " \cdot "$ the following identities, which correspond to the seven types, respectively, were obtained by V. Belousov in [1]:

| No. | Type | Identity in $Q(\cdot)$ | |
|---|---|---|---|
| 1 | $T_1 = [\varepsilon, \varepsilon, \varepsilon]$ | $x(x \cdot xy) = y$ | |
| 2 | $T_2 = [\varepsilon, \varepsilon, l]$ | $x(y \cdot yx) = y$ | |
| 3 | $T_4 = [\varepsilon, \varepsilon, lr]$ | $x \cdot xy = yx$ | Stein's 1st law |
| 4 | $T_6 = [\varepsilon, l, lr]$ | $xy \cdot x = y \cdot xy$ | Stein's 2nd law |
| 5 | $T_{10} = [\varepsilon, lr, l]$ | $xy \cdot yx = y$ | Stein's 3d law |
| 6 | $T_8 = [\varepsilon, rl, lr]$ | $xy \cdot y = x \cdot xy$ | Schröder's 1st law |
| 7 | $T_{11} = [\varepsilon, lr, rl]$ | $yx \cdot xy = y$ | Schröder's 2nd law |

The same classification was obtained independently by F. E. Bennett in 1989 (see, for example, [3] and [4]).

$\pi$-Quasigroups isotopic to groups have been investigated by V. Belousov in [1] and by V. Belousov and A. Gwaramija in [2]. In particular, they proved that the groups which are isotopic to $\pi$-quasigroups of type $T_4 = [\varepsilon, \varepsilon, lr]$ (i.e. to Stein quasigroups) or to $\pi$-quasigroups of type $T_6 = [\varepsilon, l, lr]$, are metabelian. Also it is proved in [1] that if a group $Q(+)$ is isotopic to a $\pi$-quasigroup of type $T_8 = [\varepsilon, rl, lr]$ then $Q(+)$ is abelian of exponent 2. More, every finite group of exponent 2 is isotopic to a $\pi$-quasigroup of type $T_8$. $\pi$-Quasigroups of other types, isotopic to groups, are considered in [1] as well. We'll consider below $\pi$-quasigroups isotopic to abelian groups.

Let $Q(\cdot)$ be a $\pi$-quasigroups of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$, i.e. a quasigroup satisfying the identity

$$x(x \cdot xy) = y. \tag{1}$$

Such quasigroups are also called $C_3$-quasigroups. Suppose that $Q(\cdot)$ is isotopic to an abelian group, and for $a, b \in Q$ consider the $LP$-isotopes $(\cdot)^{(R_a^{-1}, L_b^{-1}, \varepsilon)}$ and $(+) = (\cdot)^{(R_0^{-1}, L_{f_0}^{-1}, \varepsilon)}$ where $0 = b \cdot a$ and $f_0 \cdot 0 = 0$. According to Albert's theorem, these two $LP$-isotopes are abelian groups (as loops which are isotopic to groups), so $Q(+)$ where $x + y = R_0^{-1}(x) \cdot L_{f_0}^{-1}(y)$, for every $x, y \in Q$, is an abelian group with the neutral element $0 = f_0 \cdot 0$. Let denote now $L_{f_0}^{-1}$ by $\lambda$. Then $x + y = R_0^{-1}(x) \cdot \lambda(y)$ and $x \cdot y = R_0(x) + \lambda^{-1}(y)$, for every $x, y \in Q$, so the identity (1) takes the form $R_0(x) + \lambda^{-1}(R_0(x) + \lambda^{-1}(R_0(x) + \lambda^{-1}(y))) = y$ or, after replacing $R_0(x)$ by $x$:

$$x + \lambda^{-1}(x + \lambda^{-1}(x + \lambda^{-1}(y))) = y. \tag{2}$$

Taking $x = 0$, from (2) it follows $\lambda^3 = \varepsilon$. Also the equality (2) implies $x + \lambda^{-1}(x + \lambda^{-1}(y)) = \lambda(I(x) + y)$ or, replacing $\lambda^{-1}(y)$ by $y$:

$$x + \lambda^{-1}(x + y) = \lambda(I(x) + \lambda(y)), \tag{3}$$

where $I : Q \to Q$, $I(x) = -x$ (in the abelian group $Q(+)$). Taking $y = 0$, (3) implies

$$x + \lambda^{-1}(x) = \lambda I(x), \tag{4}$$

for every $x \in Q$, as $\lambda(0) = L_{f_0}^{-1}(0) = 0$.

Let consider now a new operation on $Q$ denoted by " $\circ$ " and defined as follows:

$$x \circ y = \lambda(x) + x + I(y), \tag{5}$$

$\forall x, y \in Q$.

**Proposition 1.** *The grupoid $Q(\circ)$ is a quasigroup isotopic to $Q(+)$.*

*Proof.* From (4) it follows $\lambda^{-1}(x) = \lambda I(x) + I(x)$, $\forall x \in Q$ so $\lambda^{-1}I(x) = \lambda(x) + x, \forall x \in Q$, and then $x \circ y = \lambda(x) + x + I(y) = \lambda^{-1}I(x) + I(y)$, $\forall x, y \in Q$, i.e. $(\circ) = (+)^{(\lambda^{-1}I, I, \varepsilon)}$. □

**Proposition 2.** *Let $Q(\cdot)$ be a $\pi$-quasigroup of type $T_1$, isotopic to an abelian group and let $Q(+)$ and $Q(\circ)$ be its isotopes defined above. The following conditions are equivalent:*

1. *$\lambda I = I\lambda$;*
2. *$\lambda \in AutQ(+)$;*
3. *$\lambda \in AutQ(\circ)$;*
4. *$I \in AutQ(\circ)$;*
5. *$Q(+)$ satisfies the equality $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$;*
6. *$Q(\circ)$ is a medial quasigroup.*

*Proof.* 1.⇒ 2.: If $\lambda I = I\lambda$ then from (3) and (4) it follows $x+\lambda^{-1}(x+y) = \lambda(I(x)+\lambda(y)) = \lambda I(x+I\lambda(y)) = \lambda I(x+\lambda I(y)) = \lambda I(x+y+\lambda^{-1}(y)) = x+y+\lambda^{-1}y+\lambda^{-1}(x+y+\lambda^{-1}(y)) = x+\lambda I(y)+\lambda^{-1}(x+\lambda I(y))$, so $\lambda^{-1}(x+y) = \lambda I(y)+\lambda^{-1}(x+\lambda I(y))$, which implies

$$\lambda^{-1}(x + y) + \lambda(y) = \lambda^{-1}(x + \lambda I(y)).$$

Denoting $x + y$ by $z$, from the last equality it follows $\lambda^{-1}(z) + \lambda(y) = \lambda^{-1}(z + I(y) + \lambda I(y)) = \lambda^{-1}(z + \lambda^{-1}(y))$, so replacing $y$ by $\lambda(y)$ and using the equality $\lambda^3 = \varepsilon$, we get: $\lambda^{-1}(z) + \lambda^{-1}(y) = \lambda^{-1}(z + y)$, $\forall z, y \in Q$, i.e. $\lambda \in \mathrm{Aut}Q(+)$.

2.⇒ 1.: If $\lambda \in \mathrm{Aut}Q(+)$ then $\lambda(-x) = -x$, $\forall x \in Q$, i.e. $\lambda I = I\lambda$.

1.⇒ 3.: Using Proposition 1, we get: $\lambda I = I\lambda \Rightarrow \lambda \in \mathrm{Aut}Q(+)$, so $\lambda(x \circ y) = \lambda(\lambda^{-1}I(x) + I(y)) = I(x) + \lambda I(y) = \lambda^{-1}I\lambda(x) + I\lambda(y) = \lambda(x) \circ \lambda(y)$, $\forall x, y \in Q$, so $\lambda \in \mathrm{Aut}Q(\circ)$.

3.⇒ 1.: $\lambda \in \mathrm{Aut}Q(\circ) \Leftrightarrow \lambda(x \circ y) = \lambda(x) \circ \lambda(y)$, $\forall x, y \in Q \Leftrightarrow \lambda(\lambda^{-1}I(x) + I(y)) = \lambda^{-1}I\lambda(x) + I\lambda(y)$, $\forall x, y \in Q$. Taking $x = 0$, the last equality implies $\lambda I(y) = I\lambda(y)$, $\forall y \in Q$, i.e. $\lambda I = I\lambda$.

1.⇔ 5.: Using (4), we have: $\lambda I = I\lambda \Leftrightarrow x + \lambda^{-1}(x) = I\lambda(x)$, $\forall x \in Q$, $\Leftrightarrow$ $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.

1.⇒ 4.: According to Proposition 1, $x \circ y = \lambda^{-1}I(x) + I(y)$, $\forall x, y \in Q$. If $\lambda I = I\lambda$, then $I(x\circ y) = I(\lambda^{-1}I(x)+I(y)) = \lambda^{-1}(x)+y$, $\forall x, y \in Q$, and $I(x)\circ I(y) = \lambda^{-1}I(I(x)) + I(I(y)) = \lambda^{-1}(x) + y$, $\forall x, y \in Q$, so $I(x \circ y) = I(x) \circ I(y)$, $\forall x, y \in Q$, i.e. $I \in \mathrm{Aut}Q(\circ)$.

4.⇒ 1.: If $I \in \mathrm{Aut}Q(\circ)$, then $I(x \circ y) = I(x) \circ I(y)$, $\forall x, y \in Q$, $\Rightarrow I(\lambda^{-1}I(x) + I(y)) = \lambda^{-1}\, II(x) + I(I(y)) = \lambda^{-1}(x) + y, \Rightarrow I\lambda^{-1}I(x) + y = \lambda^{-1}(x) + y$, $\forall x, y \in Q$, $\Rightarrow I\lambda^{-1}I = \lambda^{-1}, \Rightarrow I\lambda = \lambda I$.

6.⇒ 1.: Remark that from (5) it follows $x \circ x = \lambda(x), \forall x \in Q$. If $Q(\circ)$ is a medial quasigroup, i.e. if $Q(\circ)$ satisfies the identity $(x \circ y) \circ (u \circ v) = (x \circ u) \circ (y \circ v)$, then $\lambda(x \circ y) = (x \circ y) \circ (x \circ y) = \lambda(x) \circ \lambda(y)$, $\Rightarrow \lambda \in \mathrm{Aut}Q(\circ) \Rightarrow \lambda I = I\lambda$.

1.⇒ 6.: If $\lambda I = I\lambda$, then $\lambda \in \mathrm{Aut}Q(+)$, so $\lambda I, I \in \mathrm{Aut}Q(+)$ where $Q(+)$ is an abelian group and $(\lambda^{-1}I)I = \lambda^{-1} = I(\lambda^{-1}I)$, i.e. $Q(\circ)$, where $x\circ y = \lambda^{-1}I(x)+I(y)$, $\forall x, y \in Q$, is a medial quasigroup. □

**Proposition 3.** *Let $Q(\cdot)$ be an isotope of an abelian group, $0 \in Q$, $f_0 \cdot 0 = 0$, $\lambda = L_{f_0}^{-1}$, $(+) = (\cdot)^{(R_0^{-1},\lambda,\varepsilon)}$, and let $\lambda I = I\lambda$, where $I : Q \to Q, I(x) + x = 0$. Then $Q(\cdot)$ is a $\pi$-quasigroup of type $T_1$ if and only if $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.*

*Proof.* If $Q(\cdot)$ is a $\pi$-quasigroup of type $T_1$, isotopic to an abelian group and $\lambda I = I\lambda$ then, according to Proposition 2, $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.

Conversely, if $\lambda^2(x)+\lambda(x)+x = 0$, $\forall x \in Q$, then $\lambda \in \mathrm{Aut}Q(+)$ and $\lambda^2+\lambda+\varepsilon = \omega$, where $\omega : Q \to Q, \omega(x) = 0$, $\forall x \in Q$, $\Rightarrow \lambda^3 - \varepsilon = (\lambda - \varepsilon)(\lambda^2 + \lambda + \varepsilon) = (\lambda_\varepsilon)\omega = \omega$ (in the ring of endomorphisms of $Q(+)$), as $\lambda^3 = \varepsilon$. Moreover, $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q \Rightarrow \lambda^2(x) + \lambda(x) + x + y = y$, $\forall x, y \in Q$, $\Rightarrow x + \lambda^{-1}(x + \lambda^{-1}(x + \lambda^{-1}(y))) = y$, $\forall x, y \in Q$, $\Rightarrow x(x \cdot xy) = y$, $\forall x, y \in Q$ (see (2)), so $Q(\cdot)$ is a $\pi$-quasigroup of type $T_1$. □ □

**Corollary.** *Let $Q(\cdot)$ be an isotope of an abelian group $Q(+) \cong Z_2^k$, for some positive integer $k$, with the isotopy $(R_0^{-1}, L_{f_0}^{-1}, \varepsilon)$, where $0$ is the neutral element of $Q(+)$ and $f_0 \cdot 0 = 0$. Then $Q(\cdot)$ is a $\pi$-quasigroup of type $T_1$ if and only if $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$, where $\lambda = L_{f_0}^{-1}$.*

*Proof.* Indeed, in this case $I = \varepsilon$, so $\lambda I = I\lambda$. □

**Proposition 4.** *Let $Q(\cdot)$ be a $\pi$-quasigroup of type $T_1$, isotopic to an abelian group $Q(+)$, $(+) = (\cdot)^{(R_0^{-1},\lambda,\varepsilon)}$ where $0 \in Q$, $f_0 \cdot 0 = 0$, $\lambda = L_{f_0}^{-1}$. The following conditions are equivalent:*

1. *$Q(\cdot)$ has a left unit;*
2. *$Q(\circ)$, where "$\circ$" is defined in (5), is idempotent;*
3. *$Q(+)$ satisfies the equality $x + x + x = 0$, $\forall x \in Q$.*

*Proof.* 1.$\Leftrightarrow$ 2.: According to the definition (5), $x \circ x = \lambda(x) + x + I(x)$, $\forall x \in Q$. So $x \circ x = x, \forall x \in Q \Leftrightarrow \lambda = \varepsilon \Leftrightarrow \lambda^{-1} = \varepsilon \Leftrightarrow L_{f_0}(x) = x, \forall x \in Q \Leftrightarrow f_0 \cdot x = x, \forall x \in Q$, i.e. $Q(\cdot)$ has the left unit $f_0$.

2.$\Leftrightarrow$ 3.: $x \circ x = x, \forall x \in Q \Leftrightarrow \lambda = \varepsilon \Leftrightarrow x + x = x + \lambda^{-1}(x) = \lambda I(x) = I(x)$, $\forall x \in Q$ (see (4)), i.e. $x + x + x = 0, \forall x \in Q$. □

Denote Id$Q(\circ) = \{x \in Q \mid x \circ x = x\}$, i.e. the set of all idempotents of $Q(\circ)$.

**Proposition 5.** *If $\lambda I = I\lambda$, then Id$Q(\circ)$ is a subquasigroup of $Q(\circ)$.*

*Proof.* If $\lambda I = I\lambda$, then $\lambda \in$ Aut$Q(+)$, so for every $x, y \in$ Id$Q(\circ)$ we have:

$$(x \circ y) \circ (x \circ y) = \lambda(x \circ y) = \lambda(x) \circ \lambda(y) = (x \circ x) \circ (y \circ y) = x \circ y,$$

i.e. $x \circ y \in$ Id$Q(\circ)$. Moreover, if $a, b \in$ Id$Q(\circ)$ and $a \circ x = b$, then (as $Q(\circ)$ is a medial quasigroup) we have:

$$a \circ (x \circ x) = (a \circ a) \circ (x \circ x) = (a \circ x) \circ (a \circ x) = b \circ b = b,$$

hence $x \circ x = x$, i.e. the solution $x$ of the equation $a \circ x = b$ is in Id$Q(\circ)$, for every $a, b \in$ Id$Q(\circ)$. Analogously we get that the solution of the equation $x \circ a = b$ belongs to Id$Q(\circ)$, for every $a, b \in$ Id$Q(\circ)$. □

*Remark.* If $\lambda I = I\lambda$, then Id$Q(\circ) \subseteq \{x \in Q \mid x + x + x = 0\}$.

*Proof.* Indeed, if $x \in$ Id$Q(\circ)$, then $x = x \circ x = \lambda(x) + x + I(x) = \lambda(x)$, $\forall x \in Q$. On the other hand, from (4) it follows $x + x = x + \lambda^{-1}(x) = \lambda I(x) = I(x)$, $\forall x \in Q$ $\Rightarrow x + x + x = 0$, $\forall x \in Q$. □

**Proposition 6.** *If $|Q| < \infty$, then Id$Q(\circ) = \{0\}$ if and only if $\lambda I = I\lambda$.*

*Proof.* If $\lambda I = I\lambda$ and $x \in$ Id$Q(\circ)\backslash\{0\}$, then $x$ is an element of order 3 in $Q(+)$ (see the remark above). But it is known that there exist the following possibilities for the order $|Q|$ of a finite $\pi$-quasigroup of type $T_1$: $|Q| = 4$, $|Q| \equiv 1$ or $4(\text{mod}12)$, or $|Q| \equiv 1(\text{mod}3)$, i.e. $|Q|$ is not divisible by 3. Consequently, if $\lambda I = I\lambda$, then Id$Q(\circ) = \{0\}$.

Conversely, let Id$Q(\circ) = \{0\}$ and $|Q| < \infty$. As Ker$(\lambda - \varepsilon) = \{x \in Q | \lambda(x) = x\}$ $= \{x \in Q | x \circ x = x\}$, we have: $(\lambda - \varepsilon)(x) = (\lambda - \varepsilon)(y) \Rightarrow \lambda(x - y) = x - y \Rightarrow x - y \in$ Ker$(\lambda - \varepsilon) \Rightarrow x - y = 0 \Rightarrow x = y$, hence $\lambda - \varepsilon$ is injective and, as $Q$ is finite, it follows that $\lambda - \varepsilon$ is a bijection. On the other hand, $\lambda^3 = \varepsilon \Rightarrow \omega = \lambda^3 - \varepsilon = (\lambda - \varepsilon)(\lambda^2 + \lambda + \varepsilon)$ $\Rightarrow \lambda^2 + \lambda + \varepsilon = (\lambda - \varepsilon)^{-1}\omega = \omega$, where $\omega : Q \to Q, \omega(x) = 0, \forall x \in Q$, hence according to Proposition 2, $\lambda I = I\lambda$. □

**Example.** The quasigroup $Q(\cdot)$, where $Q = \{0, 1, 2, 3\}$ and

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 3 | 1 | 0 | 2 |
| 1 | 0 | 2 | 3 | 1 |
| 2 | 1 | 3 | 2 | 0 |
| 3 | 2 | 0 | 1 | 3 |

is a $\pi$-quasigroup of type $T_1$ and is isotopic to the Klein group $K_4 = Q(+)$ (0 is the neutral element of $Q(+)$): $x \cdot y = R_0(x) + \lambda^{-1}(y)$, where $R_0 = (0321), \lambda = (132)$. Remark that the quasigroup $Q(\circ)$, where $x \circ y = \lambda(x) + x + I(y)$, is defined by the following table:

| $\circ$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 0 | 1 |
| 2 | 3 | 2 | 1 | 0 |
| 3 | 1 | 0 | 3 | 2 |

As $I = \varepsilon$ we have $\lambda I = I\lambda$, hence $\lambda \in \mathrm{Aut}Q(+)$ and $\lambda \in \mathrm{Aut}Q(\circ)$. The conditions $|Q| < \infty$ and $\lambda I = I\lambda$ give $\mathrm{Id}Q(\circ) = \{0\}$.

**Proposition 7.** *If a $\pi$-quasigroup $Q(\cdot)$ of type $T_2 = [\varepsilon, \varepsilon, l]$ is isotopic to an abelian group $Q(\oplus)$, then for every $b \in Q$ there exists an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = IL_b^3(x) + L_b(y) + b$, $\forall x, y \in Q$, where $I : Q \to Q$, $I(x) = -x$, $\forall x \in Q$.*

*Proof.* Let $Q(\cdot)$ be a $\pi$-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$, isotopic to an abelian group. Then, for every $a, b \in Q$, the LP-isotope $Q(+)$, where $x + y = R_a^{-1}(x) + L_b^{-1}(y)$, $\forall x, y \in Q$, is an abelian group as well. Denote its neutral element $b \cdot a$ by 0. The quasigroup $Q(\cdot)$ satisfies the identity

$$x(y \cdot yx) = y. \tag{6}$$

Using the definition of " + ", the identity (6) takes the form $R_a(x) + L_b(R_a(y) + L_b(R_a(y) + L_b(x))) = y$ or, after replacing $y \to R_a^{-1}(y)$ and $x \to L_b^{-1}(x)$: $R_aL_b^{-1}(x) + L_b(y + L_b(y + x)) = R_a^{-1}(y)$, which implies:

$$L_b(y + L_b(y + x)) = R_a^{-1}(y) + IR_aL_b^{-1}(x). \tag{7}$$

Taking $y = 0$ in (7) we get $L_b^2(x) = b + IR_aL_b^{-1}(x) \Rightarrow L_b^3(x) = b + IR_a(x), \forall x \in Q, \Rightarrow R_a(x) = b + IL_b^3(x), \forall x \in Q$, $x \cdot y = IL_b^3(x) + L_b(y) + b$, $\forall x, y \in Q$. □

**Proposition 8.** *A quasigroup $Q(\cdot)$, isotopic to a group $Q(\oplus)$ and having an idempotent 0, is a $\pi$-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$ if and only if there exists an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = IL_0^3(x) + L_0(y)$ and $L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y)$, for every $x, y \in Q$.*

*Proof.* If $Q(\cdot)$ is a $\pi$-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$ and 0 is an idempotent of $Q(\cdot)$, then the LP-isotope $(+) = (\cdot)^{(R_0^{-1}, L_0^{-1}, \varepsilon)}$ is a group with unit 0. Using the definition of " + " the identity $x(y \cdot yx) = y$ takes the form:

$$R_0(x) + L_0(R_0(y) + L_0(R_0(y) + L_0(x))) = y,$$

or, after replacing $R_0(y)$ by $y$ and $L_0(x)$ by $x$:

$$L_0(y + L_0(y + x)) = IR_0L_0^{-1}(x) + R_0^{-1}(y). \tag{8}$$

For $y = 0$ the last equality implies $L_0^2(x) = IR_0L_0^{-1}(x)$ for every $x \in Q$, so $R_0 = IL_0^3$ and then (8) implies

$$L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y).$$

At the same time we get that $x \cdot y = IL_0^3(x) + L_0(y)$.

Conversely, let $Q(\cdot)$ be the quasigroup defined by the last equality, where $Q(+)$ is a group, 0 is an idempotent of $Q(\cdot)$ and let the equality $L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y)$ holds. Then $x(y \cdot yx) = IL_0^3(x) + L_0(IL_0^3(y) + L_0(IL_0^3(y) + L_0(x))) = IL_0^3(x) + L_0^3(x) + L_0^{-3}I^2L_0^3(y) = y$, i.e. $Q(\cdot)$ is a $\pi$-quasigroup of type $T_2$. □

It was proved in [2] by V.Belousov and A.Gwaramiya that every group $G$ which is isotopic to a $\pi$-quasigroup of type $T_4 = [\varepsilon, \varepsilon, lr]$ (i.e. to a Stein quasigroup) is metabelian (i.e. $[x, y] \in Z$ for every $x, y \in G$). It was also proved by V.Belousov in [1] that if a group $Q(\cdot)$ is isotopic to a $\pi$-quasigroup of type $T_6 = [\varepsilon, l, lr]$, then $Q(\cdot)$ is metabelian.

**Proposition 9.** *If a $\pi$-quasigroup $Q(\cdot)$ of type $T_6 = [\varepsilon, l, lr]$ is isotopic to an abelian group $Q(\oplus)$, then there exists an element $0 \in Q$ and an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = R_0(x) + \varphi R_0(y), \forall x, y \in Q$, where $\varphi \in AutQ(+)$.*

*Proof.* Let $Q(\cdot)$ be a $\pi$-quasigroup of type $T_6 = [\varepsilon, l, lr]$, i.e. let $Q(\cdot)$ be a quasigroup with the identity

$$x \cdot yx = yx \cdot y. \tag{9}$$

Then for $y = f_x$, where $f_x x = x, \forall x \in Q$, we have $x^2 = x \cdot f_x \Rightarrow f_x = x \Rightarrow x = f_x \cdot x = x \cdot x$, i.e. $Q(\cdot)$ is idempotent. Let $0 \in Q$ and consider the LP-isotope $(+) = (\cdot)^{(R_0^{-1}, L_0^{-1}, \varepsilon)}$. It is clear that $Q(+)$ is an abelian group with the neutral element $0 = 0 \cdot 0$. Now, using the equality $x \cdot y = R_0(x) + L_0(y)$, the identity (9) takes the form

$$R_0(x) + L_0(R_0(y) + L_0(x)) = R_0(R_0(y) + L_0(x)) + L_0(y),$$

$\forall x, y \in Q$, hence replacing $R_0(y) \to y$ and $L_0(x) \to x$, we get

$$R_0L_0^{-1}(x) + L_0(y + x) = R_0(y + x) + L_0R_0^{-1}(y),$$

which implies

$$L_0(y+x)+IR_0(y+x)=IR_0L_0^{-1}(x)+L_0R_0^{-1}(y). \tag{10}$$

Taking $x=0$ in (10) we get:

$$L_0(y)+IR_0(y)=L_0R_0^{-1}(y) \tag{11}$$

for all $y \in Q$. For $y=I(x)$, the equality (10) implies $0=IR_0L_0^{-1}(x)+L_0R_0^{-1}I(x), \forall x \in Q$, i.e. $R_0L_0^{-1}=L_0R_0^{-1}I$. Denoting $L_0R_0^{-1}$ by $\varphi$, we get $\varphi I=\varphi^{-1}$ and $L_0=\varphi R_0$, so $x \cdot y = R_0(x)+\varphi R_0(y), \forall x,y \in Q$. On the other hand, using (11), the equality (10) takes the form $L_0R_0^{-1}(y+x)=IR_0L_0^{-1}(x)+L_0R_0^{-1}(y)$, i.e. $\varphi(x+y)=I\varphi^{-1}(x)+\varphi(y)=I\varphi I(x)+\varphi(y)$, $\forall x,y \in Q$.

As $\varphi(0)=L_0R_0^{-1}(0)$, taking $y=0$ in the last equality, we get $\varphi=I\varphi^{-1}$, so $\varphi(x+y)=\varphi(x)+\varphi(y), \forall x,y \in Q$, i.e. $\varphi \in \mathrm{Aut}Q(+)$. □

**Proposition 10.** *Let $Q(+)$ be an abelian group with the neutral element $0$, $\varphi \in AutQ(+)$ and $\varphi^2=I$, where $I(x)=-x, \forall x \in Q$. If the isotope $Q(\cdot)$, where $(+)=(\cdot)^{(R_0^{-1},R_0^{-1}\varphi^{-1},\varepsilon)}$, is idempotent then $Q(\cdot)$ is a $\pi$-quasigroup of type $T_6$.*

*Proof.* Using the definition of " $\cdot$ ", we have $x \cdot y = R_0(x)+\varphi R_0(y)$, $\forall x,y \in Q$. If $Q(\cdot)$ is idempotent then $z \cdot z = z, \forall z \in Q$, so $R_0(z)+\varphi R_0(z)=z, \Rightarrow z+IR_0(z)=\varphi R_0(z), \forall z \in Q$. Taking $z=y+\varphi(x)$ in the last equality, we get $y+\varphi(x)+IR_0(y+\varphi(x))=\varphi R_0(y+\varphi(x)) \Rightarrow \varphi(x)+IR_0(y+\varphi(x))=\varphi R_0(y+\varphi(x))+I(y) \Rightarrow \varphi(x)+\varphi^2R_0(y+\varphi(x))=\varphi R_0(y+\varphi(x))+\varphi^2(y) \Rightarrow x+\varphi R_0(y+\varphi(x))=R_0(y+\varphi(x))+\varphi(y)$, $\forall x,y \in Q$. Now, replacing $x \to R_0(x)$ and $y \to R_0(y)$, we get:

$$R_0(x)+\varphi R_0(R_0(y)+\varphi R_0(x))=R_0(R_0(y)+\varphi R_0(x))+\varphi R_0(y),$$

$\forall x,y \in Q$, i.e. $x \cdot yx = yx \cdot y, \forall x,y \in Q$. So $Q(\cdot)$ is a $\pi$-quasigroup of type $T_6$. □

**Proposition 11.** *If $Q(\cdot)$ is a $\pi$-quasigroup of type $T_{10}=[\varepsilon,lr,l]$, isotopic to an abelian group, $a \in Q$ and $(+)=(\cdot)^{(R_a^{-1},L_a^{-1},\varepsilon)}$, then there exists a complete substitution $\theta$ of $Q(+)$ such that $x \cdot y = R_ax+R_a^{-1}I\theta y$, for every $x,y \in Q$, where $Ix=-x$, $\forall x \in Q$.*

*Proof.* The quasigroup $Q(\cdot)$ satisfies the identity $xy \cdot yx = y$ so, using the equality $x \cdot y = R_ax+L_ay$, we get $R_a(R_ax+L_ay)+L_a(R_ay+L_ax)=y$ or, after replacing $R_ax$ by $x$ and $L_ay$ by $y$:

$$R_a(x+y)+L_a(R_aL_a^{-1}(y)+L_aR_a^{-1}(x))=L_a^{-1}(y). \tag{12}$$

Taking $x=a^2$ (the unit of the group $Q(+)$), from (12) it follows:

$$R_a(y)+L_aR_aL_a^{-1}(y)=L_a^{-1}(y), \tag{13}$$

or, replacing $y$ by $L_a(y)$:

$$R_aL_a(y) + L_aR_a(y) = y. \qquad (14)$$

Now, taking $y = a^2$ in (12), we have: $R_ax + L_a^2R_a^{-1}x = a$ and, replacing $x$ by $R_ax$ in the last equality, we get $R_a^2x + L_a^2x = a$. From (14) it follows $y + IR_aL_a(y) = L_aR_a(y)$, $\forall y \in Q$, where $I(x) = -x$, $\forall x \in Q$, so $IR_aL_a$ is a complete substitution of $Q(+)$. Finally, denoting $IR_aL_a$ by $\theta$, we get $L_a = R_a^{-1}I\theta$ and $x \cdot y = R_ax + R_a^{-1}I\theta y$, $\forall x, y \in Q$. □

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