On a Generalization of Hardy-Hilbert’s Integral Inequality

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Abstract. A generalization of Hardy-Hilbert’s integral inequality was given by B. Yang in [18]. The main purpose of the present article is to generalize the inequality. As applications, the reverse, the equivalent form of the inequality, some particular results and the generalization of Hardy-Littlewood inequalities are derived.

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1 Introduction

Let \( \frac{1}{p} + \frac{1}{q} = 1 \) (\( p > 1 \)), \( f, g \geq 0 \). Suppose \( 0 < \int_0^\infty f^p(x)dx < \infty \) and \( 0 < \int_0^\infty g^q(x)dx < \infty \). The well known Hardy-Hilbert’s integral inequality (see [1]) is given by

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1}
\]

and an equivalent form is given by

\[
\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y}dx \right)^pdy \leq \left[ \frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x)dx, \tag{2}
\]

where the constant factor \( \pi/\sin(\pi/p) \) and \( \left[ \pi/\sin(\pi/p) \right]^p \) are the best possible. Recently many generalizations and refinements of these inequalities were also obtained. Some of them are given in [4]–[27]. One of the generalizations given by Yang [18] is the following:

Theorem 1. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \phi_r > 0 \) (\( r = p, q \)), \( \phi_p + \phi_q = \lambda \), \( u(x) \) is a differentiable strictly increasing function in \( (a, b) \) (\( -\infty \leq a < b \leq \infty \)) such that \( u(a+) = 0 \) and \( u(b-) = \infty \), \( f, g \geq 0 \) satisfy \( 0 < \int_a^b \frac{(u(x))^{p(1-\phi_p)-1}}{(u'(x))^{p-1}} f^p(x)dx < \infty \) and \( 0 < \int_a^b \frac{(u(x))^{q(1-\phi_q)-1}}{(u'(x))^{q-1}} g^q(x)dx < \infty \) then

\[
\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda}dxdy \leq B(\phi_p, \phi_q) \left( \int_a^b \frac{(u(x))^{p(1-\phi_p)-1}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \times \\times \left( \int_a^b \frac{(u(x))^{q(1-\phi_q)-1}}{(u'(x))^{q-1}} g^q(x)dx \right)^{\frac{1}{q}}, \tag{3}
\]
where the constant factor $B(\phi_p, \phi_q)$ is the best possible. If $p < 1(p \neq 0), \{\lambda : \phi_\nu > 0, (r = p, q), \phi_p + \phi_q = \lambda \} \neq \Phi$, with the above assumption, the reverse of (3) holds and the constant factor is still the best possible.

In this paper, we have generalized the inequality (3), where we have weakened the normalized condition $\phi_p + \phi_q = \lambda$ and considered two different functions $u(x)$ and $v(x)$, which is more generalized inequality and from which most of the recent results are obtained by specialising the parameters and the functions $u(x)$ and $v(x)$. We have also given the generalization of Hardy–Littlewood inequality.

2 Some Lemmas

We first set the following notations: Suppose $p \not\in \{0,1\}, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi_p < \lambda$ ($r = p, q$), $u(x)$ and $v(x)$ are differentiable strictly increasing function in $(a, b)$ ($-\infty \leq a < b \leq \infty$) and $(c, d)$ ($-\infty \leq c < d \leq \infty$) respectively such that $u(a+) = v(c+) = 0$ and $u(b-) = v(d-) = \infty$.

We need the formula of the $\beta$–function as (cf. Wang et al. [3]):

$$B(p, q) = \int_0^\infty \frac{1}{(1 + t)^{p+q}} dt = B(q, p)$$

(4)

**Lemma 1.** (cf. Kuang [2]). If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \omega(t) > 0, f, g \geq 0, f \in L_p^p(E)$ and $g \in L_q^q(E)$, then one has the H"older’s inequality with weight as:

$$\int_E \omega(t)f(t)g(t)dt \leq \left( \int_E \omega(t)f^p(t)dt \right)^{\frac{1}{p}} \left( \int_E \omega(t)g^q(t)dt \right)^{\frac{1}{q}}.$$  

(5)

If $p < 1(p \neq 0)$, with the above assumption, the reverse of (5) holds, where the equality in the above two cases holds if and only if there exists non-negative real numbers $c_1$ and $c_2$ such that they are not all zero and

$$c_1f^p(t) = c_2g^q(t), a.e. \text{ in } E.$$

**Lemma 2.** Define $\omega_\lambda(u, v, p, x)$ and $\omega_\lambda(v, u, q, y)$ as

$$\omega_\lambda(u, v, p, x) = \int_c^d \frac{(v(y))^{\phi_v-1}v'(y)}{(u(x) + v(y))^{\lambda}} dy, x \in (a, b),$$  

(6)

$$\omega_\lambda(v, u, q, y) = \int_a^b \frac{(u(x))^{\phi_u-1}u'(x)}{(u(x) + v(y))^{\lambda}} dx, y \in (c, d).$$  

(7)

Then

$$\omega_\lambda(u, v, p, x) = B(\phi_p, \lambda - \phi_p)(u(x))^{\phi_v - \lambda}, x \in (a, b),$$  

(8)

$$\omega_\lambda(v, u, q, y) = B(\phi_q, \lambda - \phi_q)(v(y))^{\phi_u - \lambda}, y \in (c, d).$$  

(9)
Proof. Setting \( t = \frac{v(y)}{u(x)} \) in (6), we have

\[
\omega_\lambda(u, v, p, x) = \int_0^\infty \frac{(tu(x))^{\phi_p-1}u(x)}{(u(x) + tu(x))^{\lambda}} dt = (u(x))^{\phi_p-\lambda} \int_0^\infty \frac{1}{(1+t)^\lambda t^{\phi_p-1}} dt.
\]

By (4), we get (8). Similarly, (9) can be proved. The lemma is proved. \( \square \)

**Lemma 3.** Suppose \( \phi_p + \phi_q = \lambda \). Take \( a_1 = u^{-1}(1), c_1 = v^{-1}(1) \).

(i) If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( 0 < \varepsilon < q\phi_p \), then

\[
I : = \int_{a_1}^b \int_{c_1}^d \frac{(u(x))^{\phi_q-\frac{\varepsilon}{q} - 1}u'(x)(v(y))^{\phi_p-\frac{\varepsilon}{q} - 1}v'(y)}{(u(x) + v(y))^\lambda} dx dy > \frac{1}{\varepsilon} B \left( \phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q} \right) - \bigcirc(1).
\]

(ii) If \( 0 < p < 1 \) (or \( p < 0 \)) and \( 0 < \varepsilon < -q\phi_q \) (or \( 0 < \varepsilon < q\phi_p \)), then

\[
I < \frac{1}{\varepsilon} B \left( \phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q} \right).
\]

**Proof.** For fixed \( x \in (a_1, b) \), setting \( t = \frac{v(y)}{u(x)} \) in (10), we have

\[
I : = \int_{a_1}^b (u(x))^{-1-\varepsilon} u'(x) \left( \int_0^\infty \frac{1}{t^{(1+t)^\lambda} t^{\phi_p-\frac{\varepsilon}{q} - 1}} dt \right) dx =
\]

\[
= \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} \frac{1}{(u(x))^{1+\varepsilon}} \frac{1}{(1+t)^\lambda t^{\phi_p-\frac{\varepsilon}{q} - 1}} dt -
\]

\[
- \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} \left( \int_0^{u(x)} \frac{1}{(1+t)^\lambda t^{\phi_p-\frac{\varepsilon}{q} - 1}} dt \right) dx >
\]

\[
> \frac{1}{\varepsilon} \int_0^\infty \frac{1}{(1+t)^\lambda t^{\phi_p - \frac{\varepsilon}{q} - 1}} dt - \int_{a_1}^b \frac{u'(x)}{u(x)} \left( \int_0^{u(x)} \frac{1}{t^{\phi_p - \frac{\varepsilon}{q} - 1}} dt \right) dx =
\]

\[
= \frac{1}{\varepsilon} \int_0^\infty \frac{1}{(1+t)^\lambda t^{\phi_p - \frac{\varepsilon}{q} - 1}} dt - \left( \phi_p - \frac{\varepsilon}{q} \right)^{-2}.
\]

By (4), inequality (10) is valid. If \( 0 < p < 1 \) (or \( p < 0 \)), by (12) we get

\[
I < \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_0^\infty \frac{1}{(1+t)^\lambda t^{\phi_p - \frac{\varepsilon}{q} - 1}} dt
\]

and then by (4), inequality (11) is valid. The lemma is proved. \( \square \)
3 Main Results

Theorem 2. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi_r < \lambda \) \((r = p, q)\) and \( f, g \geq 0\) satisfy

\[
0 < \int_a^b \frac{(u(x))^{\phi_r - \lambda + (p - 1)(1 - \phi_q)}}{(u(x))^{\phi_q - 1}} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_c^d \frac{(v(x))^{\phi_r - \lambda + (q - 1)(1 - \phi_p)}}{(v(x))^{\phi_p - 1}} g^q(x) dx \leq \infty
\]

then

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} dxdy < H_X(\phi_p, \phi_q) \left( \int_a^b \frac{(u(x))^{\phi_r - \lambda + (p - 1)(1 - \phi_q)}}{(u(x))^{\phi_q - 1}} f^p(x) dx \right)^{\frac{1}{p}} \times \left( \int_c^d \frac{(v(x))^{\phi_r - \lambda + (q - 1)(1 - \phi_p)}}{(v(x))^{\phi_p - 1}} g^q(x) dx \right)^{\frac{1}{q}}
\]

where \( H_X(\phi_p, \phi_q) = B^\frac{1}{2}(\phi_p, \lambda - \phi_p)B^\frac{1}{2}(\phi_q, \lambda - \phi_q) \).

If \( p < 1(p \neq 0), \{ \lambda: 0 < \phi_r < \lambda, r = p, q \} \neq \Phi \), with the above assumption, the reverse of (13) holds.

Proof. By \((5)\), we have

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} dxdy = \int_a^b \int_c^d \frac{1}{(u(x) + v(y))^\lambda} \left[ \frac{(v(y))^{(\phi_r - 1)/p}(v'(y))^{1/p}}{(u(x))^{(\phi_q - 1)/q}(v'(x))^{1/q}f(x)} \right] \times \left[ \frac{(u(x))^{(\phi_q - 1)/q}(u'(x))^{1/q}}{(v(y))^{(\phi_r - 1)/p}(v'(y))^{1/p}g(y)} \right] dxdy \leq \left\{ \int_a^b \left[ \int_c^d \frac{(v(y))^{\phi_r - 1}v'(y)}{(u(x) + v(y))^\lambda} dy \right] \frac{(u(x))^{(p - 1)(1 - \phi_q)}}{(u'(x))^{p - 1}} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_c^d \left[ \int_a^b \frac{(u(x))^{\phi_q - 1}u'(x)}{(u(x) + v(y))^\lambda} dx \right] \frac{(v(y))^{(q - 1)(1 - \phi_p)}}{(v'(y))^{q - 1}} g^q(y) dy \right\}^{\frac{1}{q}}.
\]

If (14) takes the form of equality, then by \((5)\) there exist non negative numbers \(c_1\) and \(c_2\) such that they are not all zero and

\[
c_1 \frac{(v(y))^{\phi_r - 1}v'(y)(u(x))^{(p - 1)(1 - \phi_q)}}{(u'(x))^{p - 1}} f^p(x) = c_2 \frac{(u(x))^{\phi_q - 1}u'(x)(v(y))^{(q - 1)(1 - \phi_p)}}{(v'(y))^{q - 1}} g^q(y), \quad \text{a.e. \ in } (a, b) \times (c, d).
\]

It follows that

\[
c_1 \frac{(u(x))^{(1 - \phi_q)}}{(u'(x))^{p - 1}} f^p(x) = c_2 \frac{(v(y))^{(1 - \phi_p)}}{(v'(y))^{q - 1}} g^q(y) = c_3, \quad \text{a.e. \ in } (a, b) \times (c, d)
\]
where \( c_3 \) is a constant. Without loss of generality, suppose that \( c_1 \neq 0 \). Then we have
\[
\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(\phi_q-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx = \frac{c_3}{c_1} \int_a^b (u(x))^{\phi_p+\phi_q-\lambda-1} u'(x)dx = \frac{c_3}{c_1} \left\{ \int_0^1 t^{\phi_p+\phi_q-\lambda-1} dt + \int_1^\infty t^{\phi_p+\phi_q-\lambda-1} dt \right\} = \infty
\]
which contradicts to
\[
0 < \int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(\phi_q-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx < \infty.
\]
Then by (6) and (7), we have
\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda} dxdy < \left\{ \int_a^b \omega_\lambda(u,v,p,x) \frac{(u(x))^{(p-1)(\phi_q-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx \right\} \frac{1}{\tilde{p}} \times \left\{ \int_c^d \omega_\lambda(v,u,q,y) \frac{(v(y))^{(q-1)(\phi_p-\lambda)}}{(v'(y))^{q-1}} g^q(y)dy \right\} \]
\[
\text{(15)}
\]
and in view of (8) and (9), it follows that (13) is valid.

For \( 0 < p < 1 \) (or \( p > 0 \)), by the reverse of (5) and using the same procedure, we can obtain the reverse of (13). The theorem is proved. \( \square \)

**Theorem 3.** Let the assumptions of Theorem 2 hold.

(i) If \( p > 1, 1/p + 1/q = 1 \), we obtain the equivalent inequality of (13) as follows:
\[
\int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[ \int_a^b \frac{f(x)}{(u(x)+v(y))^{\lambda}} dx \right]^p dy < \left[ H_\lambda(\phi_p, \phi_q) \right]^p \int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(\phi_q-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx;
\]
\[
\text{(16)}
\]
(ii) If \( 0 < p < 1 \), we obtain the reverse of (16) equivalent to the reverse of (13);
(iii) If \( p < 0 \), we obtain inequality (16) equivalent to the reverse of (13).

**Proof.** Set \( g(y) = \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[ \int_a^b \frac{f(x)}{(u(x)+v(y))^{\lambda}} dx \right]^{p-1} \). By (13), we have
\[
0 < \int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}} g^q(y)dy = \int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[ \int_a^b \frac{f(x)}{(u(x)+v(y))^{\lambda}} dx \right]^p dy = \int_a^b \int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}} dxdy \leq
\]
\[
\leq H_\lambda(\phi_p, \phi_q) \left( \int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_c^d \frac{(v(x))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(x))^{q-1}} g^q(x)dx \right)^{\frac{1}{q}},
\]

(17)

then

\[
0 < \left\{ \int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}} g^q(y)dy \right\}^{\frac{1}{p}} = \\
= \left\{ \int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)} } \left[ \int_a^b \frac{f(x)}{(u(x) + v(y))^{\lambda}} dx \right] dy \right\}^{\frac{1}{p}} \leq \\
\leq H_\lambda(\phi_p, \phi_q) \left\{ \int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx \right\}^{\frac{1}{p}} < \infty.
\]

(18)

It follows that (17) takes the form of strict inequality by using (13); so, does (18). Hence we can get (16).

On the other hand, if (16) holds, then by (5), we have

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^{\lambda}} dxdy = \\
= \int_c^d \left[ \frac{(v'(y))^{1/p}}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)/p} } \int_a^b \frac{f(x)}{(u(x) + v(y))^{\lambda}} dx \right] \times \\
\times \left[ \frac{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)/p}}{(v'(y))^{1/p}} g(y) \right] dy \leq \\
\leq \left\{ \int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)} } \left[ \int_a^b \frac{f(x)}{(u(x) + v(y))^{\lambda}} dx \right]^{p} dy \right\}^{\frac{1}{p}} \times \\
\times \left\{ \int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}} g^q(y)dy \right\}^{\frac{1}{q}}.
\]

Hence by (16), (13) yields. Thus it follows that (13) and (16) are equivalent. The theorem is proved. \(\square\)
Theorem 4. If \( p > 1, \ 1/p + 1/q = 1, \ \phi_r > 0(r = p, q), \ \phi_p + \phi_q = \lambda \) and \( f, g \geq 0 \) satisfy
\[
0 < \int_a^b \frac{(u(x))^{\phi_p - 1}}{(u'(x))^{p-1}} f^p(x) \, dx < \infty \quad \text{and} \quad 0 < \int_c^d \frac{(v(x))^{\phi_q - 1}}{(v'(x))^{q-1}} g^q(x) \, dx < \infty
\]
then
\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} \, dxdy < B(\phi_p, \phi_q) \left( \int_a^b \frac{(u(x))^{\phi_p - 1}}{(u'(x))^{p-1}} f^p(x) \, dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_c^d \frac{(v(x))^{\phi_q - 1}}{(v'(x))^{q-1}} g^q(x) \, dx \right)^{\frac{1}{q}}
\]
where the constant factor \( B(\phi_p, \phi_q) \) is the best possible.

If \( p < 1(p \neq 0), \{\lambda: \phi_r > 0, (r = p, q), \phi_p + \phi_q = \lambda\} \neq \Phi, \) with the above assumption, the reverse of (19) holds and the constant is still the best possible.

Proof. Since \( \phi_p + \phi_q = \lambda, \) then by Theorem 2, (19) and its inverse are valid. For \( 0 < \varepsilon < q\phi_p, \) setting
\[
f_{\varepsilon}(x) = \begin{cases} 
0 & \text{if } x \in (a, a_1) (a_1 = u^{-1}(1)), \\
(u(x))^{\phi_q - \frac{\varepsilon}{q} - 1}u'(x) & \text{if } x \in [a_1, b),
\end{cases}
\]
\[
g_{\varepsilon}(x) = \begin{cases} 
0 & \text{if } x \in (c, c_1) (c_1 = v^{-1}(1)), \\
(v(x))^{\phi_p - \frac{\varepsilon}{q} - 1}v'(x) & \text{if } x \in [c_1, d),
\end{cases}
\]
we have
\[
\left( \int_a^b \frac{(u(x))^{\phi_p - 1}}{(u'(x))^{p-1}} f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_c^d \frac{(v(x))^{\phi_q - 1}}{(v'(x))^{q-1}} g^q(x) \, dx \right)^{\frac{1}{q}} = \frac{1}{\varepsilon}
\]
(20)

If the constant factor \( B(\phi_p, \phi_q) \) in (19) is not the best possible, then there exists a positive constant \( K < B(\phi_p, \phi_q) \) such that (19) is still valid if we replace \( B(\phi_p, \phi_q) \) by \( K. \) In particular, by (10) and (20), we have
\[
B \left( \phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q} \right) - \varepsilon \bigcirc (1) < \\
< \varepsilon \int_a^b \int_c^d \frac{f_{\varepsilon}(x)g_{\varepsilon}(y)}{(u(x) + v(y))^\lambda} \, dxdy < \\
< \varepsilon K \left( \int_a^b \frac{(u(x))^{\phi_p - 1}}{(u'(x))^{p-1}} f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_c^d \frac{(v(x))^{\phi_q - 1}}{(v'(x))^{q-1}} g^q(x) \, dx \right)^{\frac{1}{q}} = K,
\]
and then \( B(\phi_p, \phi_q) \leq K (\varepsilon \to 0^+). \) This contradiction leads to the conclusion that the constant factor in (19) is the best possible.

For the best constant factor in the reverse of (19), for \( 0 < p < 1 (p < 0), \) we set \( f_{\varepsilon}(x) \) and \( g_{\varepsilon}(x) \), for \( 0 < \varepsilon < -q\phi_q \) (or \( 0 < \varepsilon < q\phi_p \), as the above; we still have (20).
If the constant factor $B(\phi_p, \phi_q)$ in the reverse of (19) is not the best possible, then there exists a positive constant $K > B(\phi_p, \phi_q)$ such that the reverse of (19) is still valid if we replace $B(\phi_p, \phi_q)$ by $K$. In particular, by (11) and (20), we have

$$B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) >$$

$$> \varepsilon \int_{a}^{b} \int_{c}^{d} \frac{f_{\varepsilon}(x)g_{\varepsilon}(y)}{(u(x) + v(y))^\lambda} \, dx \, dy >$$

$$> \varepsilon K \left( \int_{a}^{b} \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}f^p(x)dx} \right)^{\frac{1}{p}} \left( \int_{c}^{d} \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}g^q(x)dx} \right)^{\frac{1}{q}} = K,$$

and then $B(\phi_p, \phi_q) \geq K (\varepsilon \to 0^+)$. This contradiction leads to the conclusion that the constant factor in the reverse of (19) is the best possible. The theorem is proved. \[\Box\]

**Corollary 1.** For $f = g$, $u = v$, $\lambda = 1$, $\phi_r = \frac{1}{r} (r = p, q)$, if $0 < \int_{a}^{b} (u'(x))^{1-p} f^r(x) dx < \infty (r = p, q)$ then

$$\int_{a}^{b} \int_{a}^{b} \frac{f(x)f(y)}{u(x) + u(y)} \, dx \, dy <$$

$$< \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_{a}^{b} (u'(x))^{1-p} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} (u'(x))^{1-q} f^q(x) dx \right)^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

**Corollary 2.** For $f = g$, $u = v$, $\lambda = 1$, $\phi_r = \frac{1}{r} (r = p, q)$, if $0 < \int_{a}^{b} (u(x))^{\frac{q}{p}-1} f^r(x) dx < \infty (r = p, q)$ then

$$\int_{a}^{b} \int_{a}^{b} \frac{f(x)f(y)}{u(x) + u(y)} \, dx \, dy <$$

$$< \pi \left( \int_{a}^{b} \frac{(u(x))^{\frac{q}{p}-1}}{(u'(x))^{p-1}f^p(x)dx} \right)^{\frac{1}{p}} \left( \int_{a}^{b} \frac{(u(x))^{\frac{q}{p}-1}}{(u'(x))^{q-1}f^q(x)dx} \right)^{\frac{1}{q}},$$

where the constant factor $\pi$ is the best possible.

**Theorem 5.** Let the assumptions of Theorem 4 hold.

(i) If $p > 1, 1/p + 1/q = 1$, we obtain the equivalent inequality of (19) as follows:

$$\int_{c}^{d} \frac{v'(y)}{(v(y))^{1-pq}} \left[ \int_{a}^{b} \frac{f(x)}{(u(x) + v(y))^\lambda} \, dx \right]^{p} \, dy <$$

$$< [B(\phi_p, \phi_q)]^p \int_{a}^{b} \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}f^p(x)dx};$$
(ii) If \(0 < p < 1\), we obtain the reverse of (23) equivalent to the reverse of (19);
(iii) If \(p < 0\), we obtain inequality (23) equivalent to the reverse of (19), where
the constants in the above inequalities are all the best possible.

Proof. Since \(\phi_p + \phi_q = \lambda\), then by Theorem 3, we get inequality (23) and its inverse
which are equivalent to (19) and its inverse accordingly. By Theorem-4, the constants
in (19) and its inverse are best possible, hence the constants in (23) and its inverse
are best possible. The theorem is proved. \(\qed\)

4 Some Particular Inequalities

Theorem 6. If \(p > 1, \frac{1}{p} + \frac{1}{p} = 1, \lambda > \max\{\frac{1}{p}, \frac{1}{q}\}\), \(0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u(x))^{p-1}} f^p(x)dx < \infty\) and
\(0 < \int_c^d \frac{(v(x))^{1-\lambda}}{(v(x))^{p-1}} g^q(x)dx < \infty\), then we have the following two equivalent inequalities:

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^{\lambda}} dxdy <
\]

\[
< \bar{H}_\lambda \left(\frac{1}{p}, \frac{1}{q}\right) \left(\int_a^b \frac{(u(x))^{1-\lambda}}{(u(x))^{p-1}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_c^d \frac{(v(x))^{1-\lambda}}{(v(x))^{p-1}} g^q(x)dx\right)^{\frac{1}{q}},
\]

\[
\int_c^d \frac{v'(y)}{(v(y))^{p-1}} \left[ \int_a^b \frac{f(x)}{(u(x) + v(y))^{\lambda}} dx \right]^p dy <
\]

\[
< \left[ \bar{H}_\lambda \left(\frac{1}{p}, \frac{1}{q}\right) \right]^p \int_a^b \frac{(u(x))^{1-\lambda}}{(u(x))^{p-1}} f^p(x)dx,
\]

where \(\bar{H}_\lambda \left(\frac{1}{p}, \frac{1}{q}\right) = B^{\frac{1}{p}} \left(\frac{1}{p}, \lambda - \frac{1}{p}\right) B^{\frac{1}{q}} \left(\frac{1}{q}, \lambda - \frac{1}{q}\right)\).

Proof. Setting \(\phi_r = \frac{1}{r} (r = p, q)\), in Theorem 2 and Theorem 3, we get the inequalities (24) and (25) respectively. \(\qed\)

We discuss a number of special cases of inequality (24). Similar examples apply
also to inequality (25).

Example 1. Set \(u(x) = Ax + C (A > 0), x \in (-C/A, \infty)\) and \(v(x) = Bx + C
(B > 0), x \in (-C/B, \infty)\) in Theorem 6. Then (24) becomes

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(Ax + By + 2C)^\lambda} dxdy <
\]

\[
< \frac{1}{A^{1/q}B^{1/p}} \bar{H}_\lambda \left(\frac{1}{p}, \frac{1}{q}\right) \left(\int_{-\infty}^{\infty} (Ax + C)^{1-\lambda} f^p(x)dx\right)^{\frac{1}{p}} \times
\]

\[
\times \left(\int_{-\infty}^{\infty} (Bx + C)^{1-\lambda} g^q(x)dx\right)^{\frac{1}{q}}.
\]
For \( A = B = 1, C = -\alpha \), we recover the result of Yang [7].

**Example 2.** Set \( u(x) = x^\alpha (\alpha > 0), x \in (0, \infty) \) and \( v(x) = x^\beta (\beta > 0), x \in (0, \infty) \) in Theorem 6. Then (24) becomes
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\beta)^\lambda} \, dx \, dy < \\
\times \left( \int_0^\infty x^\beta(2-\lambda-q)+q-1 g^q(x) \, dx \right)^{\frac{1}{q}}.
\]

Taking \( \lambda = 1 \) and \( \alpha = \beta \), we get the result of Yang [12].

**Example 3.** Set \( u(x) = v(x) = \ln x, x \in (1, \infty) \) in Theorem 6. Then (24) becomes
\[
\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{\ln x + \ln y} \, dx \, dy < \tilde{H}_\lambda \left( \frac{1}{p}, \frac{1}{q} \right) \left( \int_1^\infty (\ln x)^{1-\lambda} x^{p-1} f^p(x) \, dx \right)^{\frac{1}{p}} \times \\
-x \left( \int_1^\infty (\ln x)^{1-\lambda} x^{q-1} g^q(x) \, dx \right)^{\frac{1}{q}}.
\]

**Theorem 7.** Suppose \( f, g \geq 0 \) satisfy \( 0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x) \, dx < \infty \) and \( 0 < \int_c^d \frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}} g^q(x) \, dx < \infty \).

(i) If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\} \), then we have the following two equivalent inequalities:
\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} \, dx \, dy < \\
k_\lambda(p) \left( \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_c^d \frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}} g^q(x) \, dx \right)^{\frac{1}{q}}
\]

(ii) If \( 0 < p < 1 \) and \( 2 - p < \lambda < 2 - q \), we have two equivalent reverses of (29) and (30).

(iii) If \( p < 0 \) and \( 2 - q < \lambda < 2 - p \), we have reverse of (29) and the inequality (30), which are equivalent; where the constants in the above inequalities are all the best possible.
Proof. Setting $\phi_r = 1 + (1 - \frac{1}{r})(\lambda - 2)$ ($r = p, q$), in Theorem 4 and Theorem 5, we get the inequalities (29) and (30) respectively.

Example 4. Set $u(x) = Ax + C$ ($A > 0$), $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ ($B > 0$), $x \in (-C/B, \infty)$ in Theorem 7. Then (29) becomes

$$\int_{-C/A}^{\infty} \int_{-C/B}^{\infty} \frac{f(x)g(y)}{(Ax + By + 2C)^{\lambda}} dxdy < \frac{k\lambda(p)}{A^{1/q}B^{1/p}} \left( \int_{-C/A}^{\infty} (Ax + C)^{1-\lambda} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_{-C/B}^{\infty} (Bx + C)^{1-\lambda} g^q(x)dx \right)^{\frac{1}{q}}. \tag{31}$$

For $C = 0$ and $p = q = 2$ this is the result of Yang [11] and for $C = 0$ we get the result of Yang and Debnath [15]. Setting $A = B = 1, C = -\alpha$, we recover the result of Yang [9]. Taking $A = B = 1, C = 0$, we get the result of Yang [10].

Example 5. Set $u(x) = x^\alpha$ ($\alpha > 0$), $x \in (0, \infty)$ and $v(x) = x^\beta$ ($\beta > 0$), $x \in (0, \infty)$, in Theorem 7. Then (29) becomes

$$\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{(x^\alpha + y^\beta)^{\lambda}} dxdy < k\lambda(p) \left( \int_{0}^{\infty} x^{\alpha(2-\lambda-p)+p-1} f^p(x)dx \right)^{\frac{1}{p}} \times \left( \int_{0}^{\infty} x^{\beta(2-\lambda-q)+q-1} g^q(x)dx \right)^{\frac{1}{q}}. \tag{32}$$

For $\alpha = \beta = 1$, this is the result of Yang [10]. Taking $\lambda = 1$ and $\alpha = \beta$, we get the result of Yang [12, Theorem 3].

Example 6. Set $u(x) = v(x) = \ln x$, $x \in (1, \infty)$ in Theorem 7. Then (29) becomes

$$\int_{1}^{\infty} \int_{1}^{\infty} \frac{f(x)g(y)}{(\ln x + \ln y)^{\lambda}} dxdy < k\lambda(p) \left( \int_{1}^{\infty} (\ln x)^{1-\lambda} f^p(x)dx \right)^{\frac{1}{p}} \times \left( \int_{1}^{\infty} (\ln x)^{1-\lambda} g^q(x)dx \right)^{\frac{1}{q}}. \tag{33}$$

For $\lambda = 1$ this is the result of Yang [16, Theorem 3.1].

Theorem 8. Suppose $f, g \geq 0$ satisfy $0 < \int_{a}^{b} \frac{f(u(x))g^{(1-\lambda)/2}}{(u(x))^{\lambda-1}} f^p(x)dx < \infty$ and $0 < \int_{c}^{d} \frac{f(u(x))g^{(1-\lambda)/2}}{(v(x))^{\lambda-1}} g^q(x)dx < \infty$. 


(i) If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 1 - 2 \min\{\frac{1}{p}, \frac{1}{q}\} \), then we have the following two equivalent inequalities:

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} dxdy < \tilde{k}_\lambda(p) \left( \int_a^b \frac{(u(x))^{p(1-\lambda)/2}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_c^d \frac{v'(y)}{(v(y))^{\lambda/2}} g^q(x)dx \right)^{\frac{1}{q}},
\]

where \( \tilde{k}_\lambda(p) = B \left( \frac{\lambda-1}{2} + \frac{1}{p}, \frac{\lambda-1}{2} + \frac{1}{q} \right) \).

(ii) If \( 0 < p < 1 \) and \( 1 - \frac{2}{p} < \lambda < 1 - \frac{2}{q} \), we have two equivalent reverses of (34) and (35).

(iii) If \( p < 0 \) and \( 1 - \frac{2}{q} < \lambda < 1 - \frac{2}{p} \), we have reverse of (34) and the inequality (35), which are equivalent; where the constants in the above inequalities are all the best possible.

Proof. Setting \( \phi_r = \frac{\lambda-1}{2} + \frac{1}{r} \) (\( r = p, q \)), in Theorem 4 and Theorem 5, we get the inequalities (34) and (35) respectively. \qed

**Example 7.** Set \( u(x) = Ax + C \) (\( A > 0 \)), \( x \in (-C/A, \infty) \) and \( v(x) = Bx + C \) (\( B > 0 \)), \( x \in (-C/B, \infty) \) in Theorem 8. Then (34) becomes

\[
\int_{-C/B}^C \int_{-C/A}^C \frac{f(x)g(y)}{(Ax + By + 2C)^\lambda} dxdy < \\
< \frac{\tilde{k}_\lambda(p)}{A^{1/q}B^{1/p}} \left( \int_{-C/A}^C \frac{(Ax + C)^{p(1-\lambda)/2}}{f^p(x)dx} \right)^{\frac{1}{p}} \times \\
\times \left( \int_{-C/B}^C \frac{g^q(x)dx}{(By+C)^{\lambda/2}} \right)^{\frac{1}{q}}.
\]

For \( C = 0 \) and \( p = q = 2 \) this is the result of Yang [11].

**Example 8.** Set \( u(x) = x^\alpha \) (\( \alpha > 0 \)), \( x \in (0, \infty) \) and \( v(x) = x^\beta \) (\( \beta > 0 \)), \( x \in (0, \infty) \) in Theorem 8. Then (34) becomes

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\beta)^\lambda} dxdy < \tilde{k}_\lambda(p) \left( \int_0^\infty \frac{x^{p-1+\alpha(2-\lambda-p)/2}}{f^p(x)dx} \right)^{\frac{1}{p}} \times \\
\times \left( \int_0^\infty \frac{y^{q-1+\beta(2-\lambda-q)/2}}{g^q(x)dx} \right)^{\frac{1}{q}}.
\]
Taking $\lambda = 1$ and $\alpha = \beta$, we get the result of Yang [12, Theorem 3].

**Example 9.** Set $u(x) = v(x) = \ln x$, $x \in (1, \infty)$ in Theorem 8. Then (34) becomes

\[
\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x + \ln y)^\lambda} \, dx \, dy < k\lambda(p) \left( \int_1^\infty (\ln x)^{p(1-\lambda)/2} x^{p-1} f(x) \, dx \right)^{\lambda \frac{p}{q}} \times
\]

\[
\left( \int_1^\infty (\ln y)^{(1-\lambda)/2} y^{q-1} g^q(y) \, dy \right)^{\frac{1}{q}}. \tag{38}
\]

For $\lambda = 1$ this is the result of Yang [16, Theorem 3.1].

**Theorem 9.** If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $f, g \geq 0$ satisfy $0 < \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) \, dx < \infty$ and $0 < \int_c^d \frac{(v(x))^{(q-1)(1-\lambda)}}{(v'(x))^{q-1}} g^q(x) \, dx < \infty$, then we have the following two equivalent inequalities:

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} \, dx \, dy < B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) \, dx \right)^{\frac{1}{p}} \times
\]

\[
\left( \int_c^d \frac{(v(x))^{(q-1)(1-\lambda)}}{(v'(x))^{q-1}} g^q(x) \, dx \right)^{\frac{1}{q}}, \tag{39}
\]

\[
\int_c^d \frac{v'(y)}{(v(y))^{1-\lambda}} \left[ \int_a^b \frac{f(x)}{(u(x) + v(y))^{\lambda}} \, dx \right]^p \, dy < \left[ B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \right]^{p} \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) \, dx, \tag{40}
\]

where the constants in the above inequalities are all the best possible.

**Proof.** Setting $\phi_r = \frac{\lambda}{r}$ $(r = p, q)$, in Theorem 4 and Theorem 5, we get the inequalities (39) and (40) respectively.

**Example 10.** Set $u(x) = Ax + C \ (A > 0)$, $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ $(B > 0)$, $x \in (-C/B, \infty)$ in Theorem 9. Then (39) becomes

\[
\int_{-\infty}^\infty \int_{-C}^C \frac{f(x)g(y)}{(Ax + By + 2C)^\lambda} \, dx \, dy <
\]

\[
< \frac{1}{A^{1/q} B^{1/p}} \frac{B}{A} \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \int_{-\infty}^\infty (Ax + C)^{(p-1)(1-\lambda)} f^p(x) \, dx \right)^{\frac{1}{p}} \times
\]

\[
\left( \int_{-C}^C (Bx + C)^{(q-1)(1-\lambda)} g^q(x) \, dx \right)^{\frac{1}{q}}. \tag{41}
\]

For $C = 0$ and $p = q = 2$ this is the result of Yang [11].
Example 11. Set \( u(x) = x^\alpha \) \((\alpha > 0), x \in (0, \infty)\) and \( v(x) = x^\beta \) \((\beta > 0), x \in (0, \infty)\) in Theorem 9. Then (39) becomes

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\beta)\lambda} dy dx < \frac{1}{\alpha^\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \int_0^\infty x^{(p-1)(1-\alpha)} f^p(x) dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_0^\infty x^{(q-1)(1-\beta)} g^q(x) dx \right)^{\frac{1}{q}}.
\] 

(42)

For \( \alpha = \beta = 1 \), this is the result of Yang [17]; for \( \alpha = \beta, \lambda = 1 \) this gives the result of Yang [14]; for \( \alpha = \beta = 2, \lambda = \frac{1}{2} \) this gives the result of Hong [5].

Example 12. Set \( u(x) = ax^{1+x}, v(x) = bx^{1+x}, x \in (0, \infty) \) and \( \lambda = 1 \) in Theorem 9. Then (39) becomes

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} dy dx < \\
< \frac{1}{\alpha^p \beta^q} B \left( \frac{1}{p}, \frac{1}{q} \right) \left( \int_0^\infty (x^p(1 + x + xlnx))^{1-p} f^p(x) dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_0^\infty (x^q(1 + x + xlnx))^{1-q} g^q(x) dx \right)^{\frac{1}{q}}.
\] 

(43)

This is the result of Jia and Gao [19].

Example 13. Set \( u(x) = v(x) = lnx, x \in (1, \infty) \) in Theorem 9. Then (39) becomes

\[
\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(lnx + lny)\lambda} dy dx < B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \int_1^\infty (ln x)^{(p-1)(1-\lambda)} x^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_1^\infty (ln x)^{(q-1)(1-\lambda)} x^{q-1} g^q(x) dx \right)^{\frac{1}{q}}.
\] 

(44)

For \( \lambda = 1 \) this is the result of Yang [16, Theorem 3.1].

Theorem 10. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, f, g \geq 0 \) satisfy \( 0 < \int_a^b \frac{(u(x))^{p-1}}{f^p(x) dx < \infty \text{ and } 0 < \int_c^d \frac{(v(x))^{q-1}}{g^q(x) dx < \infty \text{, then we have the following two equivalent inequalities:}}

\[
\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))\lambda} dy dx < B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \int_a^b \frac{(u(x))^{p-1}}{f^p(x) dx \right)^{\frac{1}{p}} \times \\
\times \left( \int_c^d \frac{(v(x))^{q-1}}{g^q(x) dx \right)^{\frac{1}{q}},
\] 

(45)
\[
\int_{c}^{d} \frac{v'(y)}{(v(y))^{1-\lambda(p-1)}} \left[ \int_{a}^{b} \frac{f(x)}{(u(x) + v(y))^\lambda} dx \right]^p dy < \left[ B \left( \frac{\lambda}{p} \frac{\lambda}{q} \right)^p \int_{a}^{b} \frac{(u(x))^{p-\lambda-1}}{(u'(x))^{p-1}} f^p(x) dx, \right.
\]

where the constants in the above inequalities are all the best possible.

Proof. Setting \( \phi_r = \lambda (1 - \frac{1}{r}) \) \( (r = p, q) \), in Theorem 4 and Theorem 5, we get the inequalities (45) and (46) respectively.

\( \Box \)

Example 14. Set \( u(x) = Ax + C \) \( (A > 0) \), \( x \in (-C/A, \infty) \) and \( v(x) = Bx + C \) \( (B > 0), x \in (-C/B, \infty) \) in Theorem 10. Then (45) becomes

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(Ax + By + 2C)^\lambda} dxdy < \frac{1}{A^{1/q}B^{1/p}} B \left( \frac{\lambda}{p} \frac{\lambda}{q} \right) \left( \int_{-\infty}^{\infty} (Ax + C)^{p-\lambda-1} f^p(x) dx \right)^\frac{1}{p} \times \left( \int_{-\infty}^{\infty} (Bx + C)^{q-\lambda-1} g^q(x) dx \right)^\frac{1}{q}.
\]

For \( C = 0 \) and \( p = q = 2 \) this is the result of Yang [11].

Example 15. Set \( u(x) = x^\alpha \) \( (\alpha > 0), x \in (0, \infty) \) and \( v(x) = x^\beta \) \( (\beta > 0), x \in (0, \infty) \) in Theorem 10. Then (45) becomes

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{(x^\alpha + y^\beta)^\lambda} dxdy < \frac{1}{\alpha^{1/\beta} \beta^{1/\alpha}} B \left( \frac{\lambda}{p} \frac{\lambda}{q} \right) \times \left( \int_{0}^{\infty} x^{p-\alpha\lambda-1} f^p(x) dx \right)^\frac{1}{p} \times \left( \int_{0}^{\infty} x^{q-\beta\lambda-1} g^q(x) dx \right)^\frac{1}{q}.
\]

This is the result of Azar [23, Theorem 1], with the constant factor \( \frac{1}{\alpha^{1/\beta} \beta^{1/\alpha}} B \left( \frac{\lambda}{p} \frac{\lambda}{q} \right) \) is the best possible for \( \alpha = \beta \), but we proved that the constant factor is the best possible for all \( \alpha \) and \( \beta \). For \( \alpha = \beta = 1 \), we get the result of Yang [17].

Example 16. Set \( u(x) = v(x) = \ln x, \ x \in (1, \infty) \) in Theorem 10. Then (45) becomes

\[
\int_{1}^{\infty} \int_{1}^{\infty} \frac{f(x)g(y)}{(\ln x + \ln y)^\lambda} dxdy < B \left( \frac{\lambda}{p} \frac{\lambda}{q} \right) \left( \int_{1}^{\infty} (\ln x)^{p-\lambda-1} x^{p-1} f^p(x) dx \right)^\frac{1}{p} \times \left( \int_{1}^{\infty} (\ln x)^{q-\lambda-1} x^{q-1} g^q(x) dx \right)^\frac{1}{q}.
\]
Remark 1. For $\lambda = 1, u(x) = v(x) = x^\alpha, \phi_r = \frac{1}{\alpha r} \ (r = p, q)$, Theorem 2 gives

$$
\int_0^\infty \int_0^\infty f(x)g(y) \frac{dxdy}{x^\alpha + y^\alpha} < \frac{\pi}{\alpha \sin \frac{\pi}{\alpha p} \sin \frac{\pi}{\alpha q}} \times \left( \int_0^\infty x^{1-\alpha} f^p(x)dx \right) \frac{1}{p} \left( \int_0^\infty x^{1-\alpha} g^q(x)dx \right) \frac{1}{q}, \tag{50}
$$

which is the result of Kuang[4].

Remark 2. For $\lambda = b + c + 1, u(x) = v(x) = x, \phi_p = c + 1 - \frac{1}{p}, \phi_q = b + 1 - \frac{1}{q}$, Theorem 2 gives

$$
\int_0^\infty \int_0^\infty f(x)g(y) \frac{dxdy}{(x+y)^{p+c+1}} < B \left( b + \frac{1}{p}, c + \frac{1}{q} \right) \times \left( \int_0^\infty x^{p(1-b)-2} f^p(x)dx \right) \frac{1}{p} \left( \int_0^\infty x^{q(1-c)-2} g^q(x)dx \right) \frac{1}{q}, \tag{51}
$$

which is given by Peachey [24].

Remark 3. For $u(x) = v(x) = x^\alpha, \phi_p = \frac{1-mp}{\alpha}, \phi_q = \frac{1-nq}{\alpha}$, Theorem 2 gives the following results:

If $p > 1, 1/p + 1/q = 1, \alpha > 0, \lambda > 0, m, n \in \mathbb{R}$ such that $0 < 1 - mp < \alpha \lambda, 0 < 1 - nq < \alpha \lambda$ and $p \geq 0, q \geq 0$ satisfy $0 < \int_0^\infty x^{(1-\alpha \lambda)+p(n-m)} f^p(x)dx < \infty$ and $0 < \int_0^\infty y^{(1-\alpha \lambda)+q(m-n)} g^q(x)dx < \infty$ then

$$
\int_0^\infty \int_0^\infty f(x)g(y) \frac{dxdy}{(x^\alpha + y^\alpha)^\lambda} < H_{\lambda,\alpha}(m, n, p, q) \left( \int_0^\infty x^{(1-\alpha \lambda)+p(n-m)} f^p(x)dx \right) \frac{1}{p} \times \left( \int_0^\infty y^{(1-\alpha \lambda)+q(m-n)} g^q(y)dy \right) \frac{1}{q}, \tag{52}
$$

where $H_{\lambda,\alpha}(m, n, p, q) = \frac{1}{\alpha} B^\frac{\alpha}{\beta} \left( \frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) B^\frac{\alpha}{\gamma} \left( \frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha} \right)$.

Further if $mp + nq = 2 - \alpha \lambda$, then Theorem 4 gives

$$
\int_0^\infty \int_0^\infty f(x)g(y) \frac{dxdy}{(x^\alpha + y^\alpha)^\lambda} < \frac{1}{\alpha} B \left( \frac{1-mp}{\alpha}, \frac{1-nq}{\alpha} \right) \left( \int_0^\infty x^{n(p+q)-1} f^p(x)dx \right) \frac{1}{p} \times \left( \int_0^\infty y^{m(p+q)-1} g^q(y)dy \right) \frac{1}{q}, \tag{53}
$$

where the constant factor $\frac{1}{\alpha} B \left( \frac{1-mp}{\alpha}, \frac{1-nq}{\alpha} \right)$ is the best possible. These two inequalities are given by Hong [6].
Remark 4. Replacing \( u(x) \) by \( xu(x) \) and \( v(x) \) by \( xv(x) \) and taking \( \phi_p = 1 - A_2p \), \( \phi_q = 1 - A_1q \) in Theorem 2, we get the following result given by Mario Krnic et. al [27]:

If \( p > 1, 1/p + 1/q = 1, \lambda > 0, A_1 \in \left( \frac{1-\lambda}{q}, \frac{1}{q} \right) \) and \( A_2 \in \left( \frac{1-\lambda}{p}, \frac{1}{p} \right) \) then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(xu(x) + yv(y))^\lambda} dxdy < L \left( \int_0^\infty (xu(x))^{1-\lambda+p(A_1-A_2)}(u(x) + xu'(x))^{1-p}f^p(x)dx \right)^{1/p} \times (54)
\]

\[
\times \left( \int_0^\infty (xv(x))^{1-\lambda+q(A_2-A_1)}(v(x) + xv'(x))^{1-q}g^q(x)dx \right)^{1/q},
\]

where \( L = (B (1 - A_2p, \lambda - 1 + A_2p))^\frac{1}{p} (B (1 - A_1q, \lambda - 1 + A_1q))^\frac{1}{q} \).

Remark 5. For \( u(x) = Ax^r, v(x) = Bb^x, \phi_r = 1 + (1-\frac{1}{r})(\lambda-2) (r = p, q) \), Theorem 4 gives the following inequality:

If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, A > 0, B > 0, a > 1, b > 1 \) and \( f, g \geq 0 \) satisfy \( 0 < \int_\infty^\infty a^{(2-\lambda)p}x f^p(x)dx < \infty \) and \( 0 < \int_\infty^\infty b^{(2-\lambda)q}x g^q(x)dx < \infty \), then

\[
\int_\infty^\infty \int_\infty^\infty \frac{f(x)g(y)}{(Ax^r + Bb^y)^\lambda} dxdy < C \left( \int_\infty^\infty a^{(2-\lambda)p}x f^p(x)dx \right)^\frac{1}{p} \left( \int_\infty^\infty b^{(2-\lambda)q}x g^q(x)dx \right)^\frac{1}{q}, (55)
\]

where the constant factor \( C = \left( A^{1-\lambda} \right)^\frac{1}{p} \left( B^{1-\lambda} \right)^\frac{1}{q} B \left( \frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \) is the best possible. This inequality is an extension of the result of Zhou et.al[22], where they consider the parameter \( p \geq q > 1, 1 - \frac{2}{\lambda} < \lambda \leq 2 \).

Remark 6. For \( u(x) = v(x) \), Theorem 4 gives (52).

For other appropriate values of \( \lambda, \phi_p, \phi_q, u(x) \) and \( v(x) \) taken in Theorem 2–5, many new inequalities can be obtained.

5 Applications

In this section, we will give the generalizations of Hardy-Littlewood’s inequality.

Let \( f \in L^2(0, 1) \) and \( f(x) \neq 0 \). If

\[
a_n = \int_0^1 x^n f(x)dx, \quad n = 0, 1, 2, 3, \ldots
\]

then we have the Hardy-Littlewood’s inequality (see [1]) of the form

\[
\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)dx \quad (56)
\]
where the constant factor $\pi$ is the best possible.

In [20, 21], Gao gave the integral version of Hardy-Littlewood’s inequality as follows:

Let $h \in L^2(0, 1)$ and $h \neq 0$. If

$$f(x) = \int_0^1 t^x |h(t)| dt, \quad x \in [0, \infty),$$

then

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 h^2(t) dt \quad (57)$$

and

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 th^2(t) dt. \quad (58)$$

**Theorem 11.** Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h \in L^2(0, 1)$ and $h(t) \neq 0$. Define a function $f(x)$ by

$$f(x) = (u'(x))^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| dt, \quad x \in (a, b).$$

If $0 < \int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx < \infty$, then

$$\left( \int_a^b f^p(x) dx \right)^{1+\frac{1}{p}} < \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx \right)^{\frac{1}{p}} \int_0^1 th^2(t) dt. \quad (59)$$

**Proof.** We can write

$$f^p(x) = f^{p-1}(x) u'(x)^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| dt.$$

Now applying, Schwartz inequality and Corollary-1, we have

$$\left( \int_a^b f^p(x) dx \right)^{2} = \left( \int_0^1 \left( \int_a^b f^{p-1}(x) u'(x)^{\frac{1}{p}} t^{u(x)-\frac{1}{p}} dx \right) t^{\frac{1}{p}} |h(t)| dt \right)^{2} \leq \int_0^1 \left( \int_a^b f^{p-1}(x) u'(x)^{\frac{1}{p}} t^{u(x)-\frac{1}{p}} dx \right)^{2} dt \times \int_0^1 th^2(t) dt = \left( \int_a^b f^{p-1}(x) u'(x)^{\frac{1}{p}} u(x) + u(y)^{\frac{1}{p}} dxdy \right) \times \int_0^1 th^2(t) dt \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_a^b (u'(x))^{1-p} f^{p(p-1)}(x) dx \right)^{\frac{1}{p}} \times \left( \int_a^b (u'(x))^{1-q} f^{q(p-1)}(x) dx \right)^{\frac{1}{q}} \times \int_0^1 th^2(t) dt = \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx \right)^{\frac{1}{p}} \left( \int_a^b f^p(x) dx \right)^{\frac{1}{p}} \times \int_0^1 th^2(t) dt.$$
Since \( h(t) \neq 0 \), so, \( f(x) \neq 0 \). Hence it is impossible for equality in (60) and then we get the inequality (59). This completes the theorem. \( \square \)

**Theorem 12.** Let \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( h \in L^2(0,1) \) and \( h(t) \neq 0 \). Define a function \( f(x) \) by

\[
f(x) = (u(x))^{\frac{1}{2} - \frac{1}{p}} (u'(x))^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| \, dt, \quad x \in (a,b).
\]

If \( 0 < \int_a^b \left( \frac{u'(x)}{u(x)} \right)^{2-p} f^{p(p-1)}(x) \, dx < \infty \), then

\[
\left( \int_a^b f^p(x) \, dx \right)^{\frac{1}{p} + \frac{1}{p}} < \pi \left( \int_a^b \left( \frac{u'(x)}{u(x)} \right)^{2-p} f^{p(p-1)}(x) \, dx \right)^{\frac{1}{p}} \int_0^1 t h^2(t) \, dt.
\] (61)

**Proof.** Proceeding as in Theorem refhlthm1 and using Corollary 2, we complete the theorem. \( \square \)

**Remark 7.** Taking \( p = 2 \) in Theorem 11 and Theorem 12, we get

\[
\int_0^\infty f^2(x) \, dx < \pi \int_0^1 t h^2(t) \, dt
\] (62)

which is a generalization of Hardy-Littlewood inequality (58).

**References**


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