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On a Generalization of Hardy-Hilbert's Integral Inequality

Namita Das, Srinibas Sahoo

Abstract. A generalization of Hardy-Hilbert's integral inequality was given by B.Yang in [18]. The main purpose of the present article is to generalize the inequality. As applications, the reverse, the equivalent form of the inequality, some particular results and the generalization of Hardy-Littlewood inequalities are derived.

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1 Introduction

Let $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $f,g\geq 0$. Suppose $0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(x)dx<\infty$. The well known Hardy-Hilbert's integral inequality (see [1]) is given by

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \quad (1)$$

and an equivalent form is given by

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy<\left[\frac{\pi}{sin(\pi/p)}\right]^p\int_0^\infty f^p(x)dx, \quad (2)$$

where the constant factor $\pi/sin(\pi/p)$ and $[\pi/sin(\pi/p)]^p$ are the best possible. Recently many generalizations and refinements of these inequalities were also obtained. Some of them are given in [4]–[27]. One of the generalizations given by Yang [18] is the following:

Theorem 1. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\phi_r>0$ $(r=p,q)$, $\phi_p+\phi_q=\lambda$, $u(x)$ is a differentiable strictly increasing function in (a,b) $(-\infty\leq a<b\leq\infty)$ such that $u(a+)=0$ and $u(b-)=\infty, f,g\geq 0$ satisfy $0<\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}}f^p(x)dx<\infty$ and $0<\int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}}g^q(x)dx<\infty$ then*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda}dxdy<B(\phi_p,\phi_q)\left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\times$$
$$\times\left(\int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}}g^q(x)dx\right)^{\frac{1}{q}}; \quad (3)$$

where the constant factor $B(\phi_p,\phi_q)$ *is the best possible. If* $p<1(p\neq 0), \{\lambda : \phi_r > 0, (r=p,q), \phi_p+\phi_q=\lambda\}\neq\Phi$*, with the above assumption, the reverse of* (3) *holds and the constant factor is still the best possible.*

In this paper, we have generalized the inequality (3), where we have weakened the normalized condition $\phi_p+\phi_q=\lambda$ and considered two different functions $u(x)$ and $v(x)$, which is more generalized inequality and from which most of the recent results are obtained by specialising the parameters and the functions $u(x)$ and $v(x)$. We have also given the generalization of Hardy–Littlewood inequality.

2 Some Lemmas

We first set the following notations: Suppose $p\notin\{0,1\},\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\phi_r<\lambda\ (r=p,q),\ u(x)$ and $v(x)$ are differentiable strictly increasing function in $(a,b)\ (-\infty\leq a<b\leq\infty)$ and $(c,d)\ (-\infty\leq c<d\leq\infty)$ respectively such that $u(a+)=v(c+)=0$ and $u(b-)=v(d-)=\infty$.

We need the formula of the $\beta-$function as (cf. Wang et al. [3]):

$$B(p,q)=\int_0^\infty\frac{1}{(1+t)^{p+q}}t^{p-1}dt=B(q,p) \tag{4}$$

Lemma 1. (cf. Kuang [2]). *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \omega(t)>0,\ f,g\geq 0, f\in L^p_\omega(E)$ *and* $g\in L^q_\omega(E)$*, then one has the Hölder's inequality with weight as:*

$$\int_E\omega(t)f(t)g(t)dt\leq\left\{\int_E\omega(t)f^p(t)dt\right\}^{\frac{1}{p}}\left\{\int_E\omega(t)g^q(t)dt\right\}^{\frac{1}{q}}. \tag{5}$$

If $p<1(p\neq 0)$*, with the above assumption, the reverse of* (5) *holds, where the equality in the above two cases holds if and only if there exists non-negative real numbers* c_1 *and* c_2 *such that they are not all zero and*

$$c_1f^p(t)=c_2g^q(t),\quad a.e.\ in\ \ E.$$

Lemma 2. *Define* $\omega_\lambda(u,v,p,x)$ *and* $\omega_\lambda(v,u,q,y)$ *as*

$$\omega_\lambda(u,v,p,x)=\int_c^d\frac{(v(y))^{\phi_p-1}v'(y)}{(u(x)+v(y))^\lambda}dy,\ x\in(a,b), \tag{6}$$

$$\omega_\lambda(v,u,q,y)=\int_a^b\frac{(u(x))^{\phi_q-1}u'(x)}{(u(x)+v(y))^\lambda}dx,\ y\in(c,d). \tag{7}$$

Then

$$\omega_\lambda(u,v,p,x)=B(\phi_p,\lambda-\phi_p)(u(x))^{\phi_p-\lambda},\ x\in(a,b), \tag{8}$$

$$\omega_\lambda(v,u,q,y)=B(\phi_q,\lambda-\phi_q)(v(y))^{\phi_q-\lambda},\ y\in(c,d). \tag{9}$$

Proof. Setting $t = \frac{v(y)}{u(x)}$ in (6), we have

$$\omega_\lambda(u,v,p,x) = \int_0^\infty \frac{(tu(x))^{\phi_p - 1} u(x)}{(u(x)+tu(x))^\lambda} dt = \\ = (u(x))^{\phi_p - \lambda} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - 1} dt.$$

By (4), we get (8). Similarly, (9) can be proved. The lemma is proved. ☐

Lemma 3. *Suppose $\phi_p + \phi_q = \lambda$. Take $a_1 = u^{-1}(1),\ c_1 = v^{-1}(1)$.*
(i) *If $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and $0 < \varepsilon < q\phi_p$, then*

$$I := \int_{a_1}^b \int_{c_1}^d \frac{(u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x) (v(y))^{\phi_p - \frac{\varepsilon}{q} - 1} v'(y)}{(u(x)+v(y))^\lambda} dxdy > \\ > \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \bigcirc(1). \tag{10}$$

(ii) *If $0 < p < 1$ (or $p < 0$) and $0 < \varepsilon < -q\phi_q$ (or $0 < \varepsilon < q\phi_p$), then*

$$I < \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right). \tag{11}$$

Proof. For fixed $x \in (a_1, b)$, setting $t = \frac{v(y)}{u(x)}$ in (10), we have

$$I := \int_{a_1}^b (u(x))^{-1-\varepsilon} u'(x) \left(\int_{\frac{1}{u(x)}}^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt \right) dx = \\ = \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt - \\ - \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} \left(\int_0^{\frac{1}{u(x)}} \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt \right) dx > \\ > \frac{1}{\varepsilon} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt - \int_{a_1}^b \frac{u'(x)}{u(x)} \left(\int_0^{\frac{1}{u(x)}} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt \right) dx = \\ = \frac{1}{\varepsilon} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt - \left(\phi_p - \frac{\varepsilon}{q}\right)^{-2}. \tag{12}$$

By (4), inequality (10) is valid. If $0 < p < 1$ (or $p < 0$), by (12) we get

$$I < \int_{a_1}^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\phi_p - \frac{\varepsilon}{q} - 1} dt$$

and then by (4), inequality (11) is valid. The lemma is proved. ☐

3 Main Results

Theorem 2. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\phi_r<\lambda\ (r=p,q)$ *and* $f,g\geq 0$ *satisfy* $0<\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)dx<\infty$ *and* $0<\int_c^d \frac{(v(x))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(x))^{q-1}}g^q(x)dx<\infty$ *then*

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy<H_\lambda(\phi_p,\phi_q)\left(\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\times$$
$$\times\left(\int_c^d \frac{(v(x))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(x))^{q-1}}g^q(x)dx\right)^{\frac{1}{q}} \tag{13}$$

where $H_\lambda(\phi_p,\phi_q)=B^{\frac{1}{p}}(\phi_p,\lambda-\phi_p)B^{\frac{1}{q}}(\phi_q,\lambda-\phi_q)$.

If $p<1(p\neq 0),\{\lambda:0<\phi_r<\lambda,r=p,q\}\neq\Phi$, *with the above assumption, the reverse of* (13) *holds.*

Proof. By (5), we have

$$\begin{aligned}
&\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy=\\
&=\int_a^b\int_c^d \frac{1}{(u(x)+v(y))^\lambda}\left[\frac{(v(y))^{(\phi_p-1)/p}(v'(y))^{1/p}}{(u(x))^{(\phi_q-1)/q}(u'(x))^{1/q}}f(x)\right]\times\\
&\times\left[\frac{(u(x))^{(\phi_q-1)/q}(u'(x))^{1/q}}{(v(y))^{(\phi_p-1)/p}(v'(y))^{1/p}}g(y)\right]dxdy\leq \qquad (14)\\
&\leq\left\{\int_a^b\left[\int_c^d \frac{(v(y))^{\phi_p-1}v'(y)}{(u(x)+v(y))^\lambda}dy\right]\frac{(u(x))^{(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}}\times\\
&\times\left\{\int_c^d\left[\int_a^b \frac{(u(x))^{\phi_q-1}u'(x)}{(u(x)+v(y))^\lambda}dx\right]\frac{(v(y))^{(q-1)(1-\phi_p)}}{(v'(y))^{q-1}}g^q(y)dy\right\}^{\frac{1}{q}}.
\end{aligned}$$

If (14) takes the form of equality, then by (5) there exist non negative numbers c_1 and c_2 such that they are not all zero and

$$\begin{aligned}
&c_1\frac{(v(y))^{\phi_p-1}v'(y)(u(x))^{(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)=\\
&=c_2\frac{(u(x))^{\phi_q-1}u'(x)(v(y))^{(q-1)(1-\phi_p)}}{(v'(y))^{q-1}}g^q(y),\quad \text{a. e. in } (a,b)\times(c,d).
\end{aligned}$$

It follows that

$$c_1\frac{(u(x))^{p(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)=c_2\frac{(v(y))^{q(1-\phi_p)}}{(v'(y))^{q-1}}g^q(y)=c_3,\quad \text{a. e. in } (a,b)\times(c,d)$$

where c_3 is a constant. Without loss of generality, suppose that $c_1 \neq 0$. Then we have

$$\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx = \frac{c_3}{c_1}\int_a^b (u(x))^{\phi_p+\phi_q-\lambda-1}u'(x)dx =$$
$$= \frac{c_3}{c_1}\left\{\int_0^1 t^{\phi_p+\phi_q-\lambda-1}dt + \int_1^\infty t^{\phi_p+\phi_q-\lambda-1}dt\right\} = \infty$$

which contradicts to

$$0 < \int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx < \infty.$$

Then by (6) and (7), we have

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy < \left\{\int_a^b \omega_\lambda(u,v,p,x)\frac{(u(x))^{(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}} \times$$
$$\times \left\{\int_c^d \omega_\lambda(v,u,q,y)\frac{(v(y))^{(q-1)(1-\phi_p)}}{(v'(y))^{q-1}}g^q(y)dy\right\}^{\frac{1}{q}} \tag{15}$$

and in view of (8) and (9), it follows that (13) is valid.

For $0 < p < 1$ (or $p < 0$), by the reverse of (5) and using the same procedure, we can obtain the reverse of (13). The theorem is proved. □

Theorem 3. *Let the assumptions of Theorem 2 hold.*

(i) If $p > 1, 1/p+1/q = 1$, we obtain the equivalent inequality of (13) as follows:

$$\begin{aligned}\int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^p dy <\\ < [H_\lambda(\phi_p,\phi_q)]^p\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}}f^p(x)dx;\end{aligned} \tag{16}$$

(ii) If $0 < p < 1$, we obtain the reverse of (16) equivalent to the reverse of (13);

(iii) If $p < 0$, we obtain inequality (16) equivalent to the reverse of (13).

Proof. Set $g(y) = \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^{p-1}$. By (13), we have

$$0 < \int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}}g^q(y)dy =$$
$$= \int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^p dy =$$
$$= \int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy \leq$$

$$\leq H_\lambda(\phi_p,\phi_q)\left(\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx\right)^{\frac{1}{p}} \times \\ \times \left(\int_c^d \frac{(v(x))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(x))^{q-1}} g^q(x)dx\right)^{\frac{1}{q}}, \tag{17}$$

then

$$0 < \left\{\int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{p}} = \\ = \left\{\int_c^d \frac{v'(y)}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^p dy\right\}^{\frac{1}{p}} \leq \tag{18} \\ \leq H_\lambda(\phi_p,\phi_q)\left\{\int_a^b \frac{(u(x))^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x)dx\right\}^{\frac{1}{p}} < \infty.$$

It follows that (17) takes the form of strict inequality by using (13); so, does (18). Hence we can get (16).

On the other hand, if (16) holds, then by (5), we have

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy = \\ = \int_c^d \left[\frac{(v'(y))^{1/p}}{(v(y))^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}}\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right] \times \\ \times \left[\frac{(v(y))^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}}{(v'(y))^{1/p}} g(y)\right] dy \leq \\ \leq \left\{\int_c^d \frac{v'(y}{(v(y))^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^p dy\right\}^{\frac{1}{p}} \times \\ \times \left\{\int_c^d \frac{(v(y))^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{q}}.$$

Hence by (16), (13) yields. Thus it follows that (13) and (16) are equivalent. The theorem is proved. □

Theorem 4. *If* $p > 1,\ 1/p + 1/q = 1,\ \phi_r > 0 (r = p, q),\ \phi_p + \phi_q = \lambda$ *and* $f, g \geq 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ *and* $0 < \int_c^d \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}} g^q(x)dx < \infty$ *then*

$$\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda} dxdy < B(\phi_p, \phi_q) \left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \times \times \left(\int_c^d \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}} g^q(x)dx \right)^{\frac{1}{q}} \tag{19}$$

where the constant factor $B(\phi_p, \phi_q)$ *is the best possible.*

If $p < 1 (p \neq 0), \{\lambda : \phi_r > 0, (r = p, q), \phi_p + \phi_q = \lambda\} \neq \Phi$*, with the above assumption, the reverse of* (19) *holds and the constant is still the best possible.*

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 2, (19) and its inverse are valid. For $0 < \varepsilon < q\phi_p$, setting

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (a, a_1)\ (a_1 = u^{-1}(1)), \\ (u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x) & \text{if } x \in [a_1, b), \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (c, c_1)\ (c_1 = v^{-1}(1)), \\ (v(x))^{\phi_p - \frac{\varepsilon}{q} - 1} v'(x) & \text{if } x \in [c_1, d), \end{cases}$$

we have

$$\left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x)dx \right)^{\frac{1}{p}} \left(\int_c^d \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}} g_\varepsilon^q(x)dx \right)^{\frac{1}{q}} = \frac{1}{\varepsilon} \tag{20}$$

If the constant factor $B(\phi_p, \phi_q)$ in (19) is not the best possible, then there exists a positive constant $K < B(\phi_p, \phi_q)$ such that (19) is still valid if we replace $B(\phi_p, \phi_q)$ by K. In particular, by (10) and (20), we have

$$\begin{aligned} & B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \varepsilon \bigcirc (1) < \\ & < \varepsilon \int_a^b \int_c^d \frac{f_\varepsilon(x) g_\varepsilon(y)}{(u(x)+v(y))^\lambda} dxdy < \\ & < \varepsilon K \left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x)dx \right)^{\frac{1}{p}} \left(\int_c^d \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}} g_\varepsilon^q(x)dx \right)^{\frac{1}{q}} = K, \end{aligned}$$

and then $B(\phi_p, \phi_q) \leq K\ (\varepsilon \to 0^+)$. This contradiction leads to the conclusion that the constant factor in (19) is the best possible.

For the best constant factor in the reverse of (19), for $0 < p < 1\ (p < 0)$, we set $f_\varepsilon(x)$ and $g_\varepsilon(x)$, for $0 < \varepsilon < -q\phi_q$ (or $0 < \varepsilon < q\phi_p$), as the above; we still have (20).

If the constant factor $B(\phi_p,\phi_q)$ in the reverse of (19) is not the best possible, then there exists a positive constant $K > B(\phi_p,\phi_q)$ such that the reverse of (19) is still valid if we replace $B(\phi_p,\phi_q)$ by K. In particular, by (11) and (20), we have

$$\begin{aligned} &B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) > \\ &> \varepsilon \int_a^b \int_c^d \frac{f_\varepsilon(x) g_\varepsilon(y)}{(u(x)+v(y))^\lambda} dxdy > \\ &> \varepsilon K \left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x)dx\right)^{\frac{1}{p}} \left(\int_c^d \frac{(v(x))^{q(1-\phi_p)-1}}{(v'(x))^{q-1}} g_\varepsilon^q(x)dx\right)^{\frac{1}{q}} = K, \end{aligned}$$

and then $B(\phi_p,\phi_q) \geq K$ ($\varepsilon \to 0^+$). This contradiction leads to the conclusion that the constant factor in the reverse of (19) is the best possible. The theorem is proved. $\square$

Corollary 1. *For* $f = g$, $u = v$, $\lambda = 1$, $\phi_r = \frac{1}{r}$ $(r = p,q)$, *if* $0 < \int_a^b (u'(x))^{1-r} f^r(x)dx < \infty$ $(r = p,q)$ *then*

$$\begin{aligned} &\int_a^b \int_a^b \frac{f(x)f(y)}{u(x)+u(y)} dxdy < \\ &< \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_a^b (u'(x))^{1-p} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b (u'(x))^{1-q} f^q(x)dx\right)^{\frac{1}{q}}, \end{aligned} \tag{21}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ *is the best possible.*

Corollary 2. *For* $f = g$, $u = v$, $\lambda = 1$, $\phi_r = \frac{1}{2}$ $(r = p,q)$, *if* $0 < \int_a^b \frac{(u(x))^{\frac{r}{2}-1}}{(u'(x))^{r-1}} f^r(x)dx < \infty$ $(r = p,q)$ *then*

$$\begin{aligned} &\int_a^b \int_a^b \frac{f(x)f(y)}{u(x)+u(y)} dxdy < \\ &< \pi \left(\int_a^b \frac{(u(x))^{\frac{p}{2}-1}}{(u'(x))^{p-1}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b \frac{(u(x))^{\frac{q}{2}-1}}{(u'(x))^{q-1}} f^q(x)dx\right)^{\frac{1}{q}}, \end{aligned} \tag{22}$$

where the constant factor π *is the best possible.*

Theorem 5. *Let the assumptions of Theorem* 4 *hold.*

(i) If $p > 1, 1/p + 1/q = 1$, *we obtain the equivalent inequality of* (19) *as follows:*

$$\begin{aligned} &\int_c^d \frac{v'(y)}{(v(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda} dx\right]^p dy < \\ &< [B(\phi_p,\phi_q)]^p \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x)dx; \end{aligned} \tag{23}$$

(ii) If $0 < p < 1$, we obtain the reverse of (23) equivalent to the reverse of (19);

(iii) If $p < 0$, we obtain inequality (23) equivalent to the reverse of (19), where the constants in the above inequalities are all the best possible.

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 3, we get inequality (23) and its inverse which are equivalent to (19) and its inverse accordingly. By Theorem-4, the constants in (19) and its inverse are best possible, hence the constants in (23) and it's inverse are best possible. The theorem is proved. $\square$

4 Some Particular Inequalities

Theorem 6. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > \max\{\frac{1}{p}, \frac{1}{q}\}, 0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ and $0 < \int_c^d \frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}} g^q(x)dx < \infty$, then we have the following two equivalent inequalities:*

$$\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda} dxdy < \\ < \tilde{H}_\lambda\left(\frac{1}{p}, \frac{1}{q}\right) \left(\int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_c^d \frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}} g^q(x)dx\right)^{\frac{1}{q}}, \tag{24}$$

$$\int_c^d \frac{v'(y)}{(v(y))^{(p-1)(1-\lambda)}} \left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda} dx\right]^p dy < \\ < \left[\tilde{H}_\lambda\left(\frac{1}{p}, \frac{1}{q}\right)\right]^p \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x)dx, \tag{25}$$

where $\tilde{H}_\lambda\left(\frac{1}{p}, \frac{1}{q}\right) = B^{\frac{1}{p}}\left(\frac{1}{p}, \lambda - \frac{1}{p}\right) B^{\frac{1}{q}}\left(\frac{1}{q}, \lambda - \frac{1}{q}\right)$.

Proof. Setting $\phi_r = \frac{1}{r}$ $(r = p, q)$, in Theorem 2 and Theorem 3, we get the inequalities (24) and (25) respectively. $\square$

We discuss a number of special cases of inequality (24). Similar examples apply also to inequality (25).

Example 1. Set $u(x) = Ax + C$ $(A > 0)$, $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ $(B > 0)$, $x \in (-C/B, \infty)$ in Theorem 6. Then (24) becomes

$$\int_{-\frac{C}{A}}^\infty \int_{-\frac{C}{B}}^\infty \frac{f(x)g(y)}{(Ax+By+2C)^\lambda} dxdy < \\ < \frac{1}{A^{1/q}B^{1/p}} \tilde{H}_\lambda\left(\frac{1}{p}, \frac{1}{q}\right) \left(\int_{-\frac{C}{A}}^\infty (Ax+C)^{1-\lambda} f^p(x)dx\right)^{\frac{1}{p}} \times \\ \times \left(\int_{-\frac{C}{B}}^\infty (Bx+C)^{1-\lambda} g^q(x)dx\right)^{\frac{1}{q}}. \tag{26}$$

For $A = B = 1, C = -\alpha$, we recover the result of Yang [7].

Example 2. Set $u(x) = x^{\alpha}$ $(\alpha > 0)$, $x \in (0,\infty)$ and $v(x) = x^{\beta}$ $(\beta > 0)$, $x \in (0,\infty)$ in Theorem 6. Then (24) becomes

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x^{\alpha}+y^{\beta})^{\lambda}}dxdy <\\ &< \frac{1}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}\tilde{H}_{\lambda}\left(\frac{1}{p},\frac{1}{q}\right)\left(\int_0^{\infty}x^{\alpha(2-\lambda-p)+p-1}f^p(x)dx\right)^{\frac{1}{p}}\times\\ &\times\left(\int_0^{\infty}x^{\beta(2-\lambda-q)+q-1}g^q(x)dx\right)^{\frac{1}{q}}.\end{aligned} \tag{27}$$

Taking $\lambda = 1$ and $\alpha = \beta$, we get the result of Yang [12].

Example 3. Set $u(x) = v(x) = lnx$, $x \in (1,\infty)$ in Theorem 6. Then (24) becomes

$$\begin{aligned}&\int_1^{\infty}\int_1^{\infty}\frac{f(x)g(y)}{(\ln x+\ln y)^{\lambda}}dxdy < \tilde{H}_{\lambda}\left(\frac{1}{p},\frac{1}{q}\right)\left(\int_1^{\infty}(\ln x)^{1-\lambda}x^{p-1}f^p(x)dx\right)^{\frac{1}{p}}\times\\ &\times\left(\int_1^{\infty}(\ln x)^{1-\lambda}x^{q-1}g^q(x)dx\right)^{\frac{1}{q}}.\end{aligned} \tag{28}$$

Theorem 7. *Suppose* $f,\ g \geq 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx < \infty$ *and* $0 < \int_c^d \frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}}g^q(x)dx < \infty$.

(i) *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda > 2-\min\{p,q\}$, *then we have the following two equivalent inequalities:*

$$\begin{aligned}&\int_a^b\int_c^d\frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}}dxdy <\\ &< k_{\lambda}(p)\left(\int_a^b\frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_c^d\frac{(v(x))^{1-\lambda}}{(v'(x))^{q-1}}g^q(x)dx\right)^{\frac{1}{q}},\end{aligned} \tag{29}$$

$$\begin{aligned}&\int_c^d\frac{v'(y)}{(v(y))^{(p-1)(1-\lambda)}}\left[\int_a^b\frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right]^p dy <\\ &< [k_{\lambda}(p)]^p\int_a^b\frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx\ ,\end{aligned} \tag{30}$$

where $k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$.

(ii) *If* $0 < p < 1$ *and* $2-p < \lambda < 2-q$, *we have two equivalent reverses of* (29) *and* (30).

(iii) *If* $p < 0$ *and* $2-q < \lambda < 2-p$, *we have reverse of* (29) *and the inequality* (30), *which are equivalent; where the constants in the above inequalities are all the best possible.*

Proof. Setting $\phi_r = 1 + (1 - \frac{1}{r})(\lambda - 2)$ $(r = p, q)$, in Theorem 4 and Theorem 5, we get the inequalities (29) and (30) respectively. $\square$

Example 4. Set $u(x) = Ax + C$ $(A > 0)$, $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ $(B > 0)$, $x \in (-C/B, \infty)$ in Theorem 7. Then (29) becomes

$$\begin{aligned} &\int_{-\frac{C}{A}}^{\infty} \int_{-\frac{C}{B}}^{\infty} \frac{f(x)g(y)}{(Ax + By + 2C)^{\lambda}} dxdy < \\ &< \frac{k_{\lambda}(p)}{A^{1/q} B^{1/p}} \left(\int_{-\frac{C}{A}}^{\infty} (Ax + C)^{1-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\frac{C}{B}}^{\infty} (Bx + C)^{1-\lambda} g^q(x) dx \right)^{\frac{1}{q}}. \end{aligned} \tag{31}$$

For $C = 0$ and $p = q = 2$ this is the result of Yang [11] and for $C = 0$ we get the result of Yang and Debnath [15]. Setting $A = B = 1, C = -\alpha$, we recover the result of Yang [9].Taking $A = B = 1, C = 0$, we get the result of Yang [10].

Example 5. Set $u(x) = x^{\alpha}$ $(\alpha > 0)$, $x \in (0, \infty)$ and $v(x) = x^{\beta}$ $(\beta > 0)$, $x \in (0, \infty)$, in Theorem 7. Then (29) becomes

$$\begin{aligned} &\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{(x^{\alpha} + y^{\beta})^{\lambda}} dxdy < \frac{k_{\lambda}(p)}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}} \left(\int_{0}^{\infty} x^{\alpha(2-\lambda-p)+p-1} f^p(x) dx \right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_{0}^{\infty} x^{\beta(2-\lambda-q)+q-1} g^q(x) dx \right)^{\frac{1}{q}}. \end{aligned} \tag{32}$$

For $\alpha = \beta = 1$, this is the result of Yang [10]. Taking $\lambda = 1$ and $\alpha = \beta$, we get the result of Yang [12, Theorem 3].

Example 6. Set $u(x) = v(x) = lnx$, $x \in (1, \infty)$ in Theorem 7. Then (29) becomes

$$\begin{aligned} &\int_{1}^{\infty} \int_{1}^{\infty} \frac{f(x)g(y)}{(\ln x + \ln y)^{\lambda}} dxdy < k_{\lambda}(p) \left(\int_{1}^{\infty} (\ln x)^{1-\lambda} x^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_{1}^{\infty} (\ln x)^{1-\lambda} x^{q-1} g^q(x) dx \right)^{\frac{1}{q}}. \end{aligned} \tag{33}$$

For $\lambda = 1$ this is the result of Yang [16, Theorem 3.1].

Theorem 8. *Suppose* f, $g \geq 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{p(1-\lambda)/2}}{(u'(x))^{p-1}} f^p(x) dx < \infty$ *and* $0 < \int_c^d \frac{(v(x))^{q(1-\lambda)/2}}{(v'(x))^{q-1}} g^q(x) dx < \infty$.

(i) *If* $p>1, \frac{1}{p}+\frac{1}{q}=1, \lambda>1-2\min\{\frac{1}{p},\frac{1}{q}\}$, *then we have the following two equivalent inequalities:*

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy < \tilde{k}_\lambda(p)\left(\int_a^b \frac{(u(x))^{p(1-\lambda)/2}}{(u'(x))^{p-1}}f^p(x)dx\right)^{\frac{1}{p}} \times \times\left(\int_c^d \frac{(v(x))^{q(1-\lambda)/2}}{(v'(x))^{q-1}}g^q(x)dx\right)^{\frac{1}{q}}, \tag{34}$$

$$\int_c^d \frac{v'(y)}{(v(y))^{p(1-\lambda)/2}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right]^p dy < < \left[\tilde{k}_\lambda(p)\right]^p \int_a^b \frac{(u(x))^{p(1-\lambda)/2}}{(u'(x))^{p-1}}f^p(x)dx\ , \tag{35}$$

where $\tilde{k}_\lambda(p)=B\left(\frac{\lambda-1}{2}+\frac{1}{p}, \frac{\lambda-1}{2}+\frac{1}{q}\right)$.

(ii) *If* $0<p<1$ *and* $1-\frac{2}{p}<\lambda<1-\frac{2}{q}$, *we have two equivalent reverses of* (34) *and* (35).

(iii) *If* $p<0$ *and* $1-\frac{2}{q}<\lambda<1-\frac{2}{p}$, *we have reverse of* (34) *and the inequality* (35), *which are equivalent; where the constants in the above inequalities are all the best possible.*

Proof. Setting $\phi_r=\frac{\lambda-1}{2}+\frac{1}{r}$ $(r=p,q)$, in Theorem 4 and Theorem 5, we get the inequalities (34) and (35) respectively. $\square$

Example 7. Set $u(x)=Ax+C$ $(A>0)$, $x\in(-C/A,\infty)$ and $v(x)=Bx+C$ $(B>0)$, $x\in(-C/B,\infty)$ in Theorem 8. Then (34) becomes

$$\int_{-\frac{C}{A}}^\infty\int_{-\frac{C}{B}}^\infty \frac{f(x)g(y)}{(Ax+By+2C)^\lambda}dxdy < < \frac{\tilde{k}_\lambda(p)}{A^{1/q}B^{1/p}}\left(\int_{-\frac{C}{A}}^\infty (Ax+C)^{p(1-\lambda)/2}f^p(x)dx\right)^{\frac{1}{p}} \times \times\left(\int_{-\frac{C}{B}}^\infty (Bx+C)^{q(1-\lambda)/2}g^q(x)dx\right)^{\frac{1}{q}}. \tag{36}$$

For $C=0$ and $p=q=2$ this is the result of Yang [11].

Example 8. Set $u(x)=x^\alpha$ $(\alpha>0)$, $x\in(0,\infty)$ and $v(x)=x^\beta$ $(\beta>0)$, $x\in(0,\infty)$ in Theorem 8. Then (34) becomes

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\beta)^\lambda}dxdy < \frac{\tilde{k}_\lambda(p)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}\left(\int_0^\infty x^{p-1+\alpha(2-p\lambda-p)/2}f^p(x)dx\right)^{\frac{1}{p}} \times \times\left(\int_0^\infty x^{q-1+\beta(2-q\lambda-q)/2}g^q(x)dx\right)^{\frac{1}{q}}. \tag{37}$$

Taking $\lambda = 1$ and $\alpha = \beta$, we get the result of Yang [12, Theorem 3].

Example 9. Set $u(x) = v(x) = lnx,\ x \in (1, \infty)$ in Theorem 8. Then (34) becomes

$$\begin{aligned} &\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x + \ln y)^\lambda} dxdy < \tilde{k}_\lambda(p) \left(\int_1^\infty (\ln x)^{p(1-\lambda)/2} x^{p-1} f^p(x)dx \right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_1^\infty (\ln x)^{q(1-\lambda)/2} x^{q-1} g^q(x)dx \right)^{\frac{1}{q}}. \end{aligned} \tag{38}$$

For $\lambda = 1$ this is the result of Yang [16, Theorem 3.1].

Theorem 9. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, f, g \geq 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ *and* $0 < \int_c^d \frac{(v(x))^{(q-1)(1-\lambda)}}{(v'(x))^{q-1}} g^q(x)dx < \infty$, *then we have the following two equivalent inequalities :*

$$\begin{aligned} &\int_a^b \int_c^d \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_c^d \frac{(v(x))^{(q-1)(1-\lambda)}}{(v'(x))^{q-1}} g^q(x)dx \right)^{\frac{1}{q}}, \end{aligned} \tag{39}$$

$$\begin{aligned} &\int_c^d \frac{v'(y)}{(v(y))^{1-\lambda}} \left[\int_a^b \frac{f(x)}{(u(x) + v(y))^\lambda} dx \right]^p dy < \\ &\quad < \left[B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \right]^p \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx, \end{aligned} \tag{40}$$

where the constants in the above inequalities are all the best possible.

Proof. Setting $\phi_r = \frac{\lambda}{r}$ $(r = p, q)$, in Theorem 4 and Theorem 5, we get the inequalities (39) and (40) respectively. $\square$

Example 10. Set $u(x) = Ax + C$ $(A > 0)$, $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ $(B > 0)$, $x \in (-C/B, \infty)$ in Theorem 9. Then (39) becomes

$$\begin{aligned} &\int_{-\frac{C}{A}}^\infty \int_{-\frac{C}{B}}^\infty \frac{f(x)g(y)}{(Ax + By + 2C)^\lambda} dxdy < \\ &< \frac{1}{A^{1/q} B^{1/p}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_{-\frac{C}{A}}^\infty (Ax + C)^{(p-1)(1-\lambda)} f^p(x)dx \right)^{\frac{1}{p}} \times \\ &\times \left(\int_{-\frac{C}{B}}^\infty (Bx + C)^{(q-1)(1-\lambda)} g^q(x)dx \right)^{\frac{1}{q}}. \end{aligned} \tag{41}$$

For $C = 0$ and $p = q = 2$ this is the result of Yang [11].

Example 11. Set $u(x) = x^{\alpha}\,(\alpha > 0),\ x \in (0, \infty)$ and $v(x) = x^{\beta}\,(\beta > 0),\ x \in (0, \infty)$ in Theorem 9. Then (39) becomes

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x^{\alpha}+y^{\beta})^{\lambda}}dxdy < \frac{1}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_0^{\infty} x^{(p-1)(1-\alpha\lambda)} f^p(x)dx\right)^{\frac{1}{p}} \times \times \left(\int_0^{\infty} x^{(q-1)(1-\beta\lambda)} g^q(x)dx\right)^{\frac{1}{q}}. \tag{42}$$

For $\alpha = \beta = 1$, this is the result of Yang [17]; for $\alpha = \beta, \lambda = 1$ this gives the result of Yang [14]; for $\alpha = \beta = 2, \lambda = \frac{1}{2}$ this gives the result of Hong [5].

Example 12. Set $u(x) = ax^{1+x}, v(x) = bx^{1+x},\ x \in (0, \infty)$ and $\lambda = 1$ in Theorem 9. Then (39) becomes

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy < < \frac{1}{a^{\frac{1}{q}}b^{\frac{1}{p}}} B\left(\frac{1}{p}, \frac{1}{q}\right) \left(\int_0^{\infty} (x^x(1+x+xlnx))^{1-p} f^p(x)dx\right)^{\frac{1}{p}} \times \times \left(\int_0^{\infty} (x^x(1+x+xlnx))^{1-q} g^q(x)dx\right)^{\frac{1}{q}}. \tag{43}$$

This is the result of Jia and Gao [19].

Example 13. Set $u(x) = v(x) = lnx,\ x \in (1, \infty)$ in Theorem 9. Then (39) becomes

$$\int_1^{\infty}\int_1^{\infty} \frac{f(x)g(y)}{(\ln x+\ln y)^{\lambda}}dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_1^{\infty} (\ln x)^{(p-1)(1-\lambda)} x^{p-1} f^p(x)dx\right)^{\frac{1}{p}} \times \times \left(\int_1^{\infty} (\ln x)^{(q-1)(1-\lambda)} x^{q-1} g^q(x)dx\right)^{\frac{1}{q}}. \tag{44}$$

For $\lambda = 1$ this is the result of Yang [16, Theorem 3.1].

Theorem 10. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 0,\ f, g \geq 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{p-\lambda-1}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ *and* $0 < \int_c^d \frac{(v(x))^{q-\lambda-1}}{(v'(x))^{q-1}} g^q(x)dx < \infty$*, then we have the following two equivalent inequalities:*

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}}dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_a^b \frac{(u(x))^{p-\lambda-1}}{(u'(x))^{p-1}} f^p(x)dx\right)^{\frac{1}{p}} \times \times \left(\int_c^d \frac{(v(x))^{q-\lambda-1}}{(v'(x))^{q-1}} g^q(x)dx\right)^{\frac{1}{q}}, \tag{45}$$

$$\begin{aligned}&\int_c^d \frac{v'(y)}{(v(y))^{1-\lambda(p-1)}}\left[\int_a^b \frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right]^p dy < \\ &\quad < \left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p \int_a^b \frac{(u(x))^{p-\lambda-1}}{(u'(x))^{p-1}} f^p(x)dx,\end{aligned} \tag{46}$$

where the constants in the above inequalities are all the best possible.

Proof. Setting $\phi_r = \lambda\left(1-\frac{1}{r}\right)$ $(r = p, q)$, in Theorem 4 and Theorem 5, we get the inequalities (45) and (46) respectively. □

Example 14. Set $u(x) = Ax + C$ $(A > 0)$, $x \in (-C/A, \infty)$ and $v(x) = Bx + C$ $(B > 0)$, $x \in (-C/B, \infty)$ in Theorem-10. Then (45) becomes

$$\begin{aligned}&\int_{-\frac{C}{A}}^{\infty}\int_{-\frac{C}{B}}^{\infty} \frac{f(x)g(y)}{(Ax+By+2C)^{\lambda}}dxdy < \\ &< \frac{1}{A^{1/q}B^{1/p}} B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_{-\frac{C}{A}}^{\infty}(Ax+C)^{p-\lambda-1}f^p(x)dx\right)^{\frac{1}{p}} \times \\ &\times \left(\int_{-\frac{C}{B}}^{\infty}(Bx+C)^{q-\lambda-1}g^q(x)dx\right)^{\frac{1}{q}}.\end{aligned} \tag{47}$$

For $C = 0$ and $p = q = 2$ this is the result of Yang [11].

Example 15. Set $u(x) = x^{\alpha}\,(\alpha > 0)$, $x \in (0, \infty)$ and $v(x) = x^{\beta}\,(\beta > 0)$, $x \in (0, \infty)$ in Theorem 10. Then (45) becomes

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x^{\alpha}+y^{\beta})^{\lambda}}dxdy < \frac{1}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}} B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) \times \\ &\quad \times \left(\int_0^{\infty} x^{p-\alpha\lambda-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} x^{q-\beta\lambda-1}g^q(x)dx\right)^{\frac{1}{q}}.\end{aligned} \tag{48}$$

This is the result of Azar [23, Theorem 1], with the constant factor $\frac{1}{\alpha^{1/q}\beta^{1/p}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ is the best possible for $\alpha = \beta$, but we proved that the constant factor is the best possible for all α and β . For $\alpha = \beta = 1$, we get the result of Yang [17].

Example 16. Set $u(x) = v(x) = lnx$, $x \in (1, \infty)$ in Theorem 10. Then (45) becomes

$$\begin{aligned}&\int_1^{\infty}\int_1^{\infty} \frac{f(x)g(y)}{(\ln x+\ln y)^{\lambda}}dxdy < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_1^{\infty}(\ln x)^{p-\lambda-1}x^{p-1}f^p(x)dx\right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_1^{\infty}(\ln x)^{q-\lambda-1}x^{q-1}g^q(x)dx\right)^{\frac{1}{q}}.\end{aligned} \tag{49}$$

Remark 1. For $\lambda = 1, u(x) = v(x) = x^\alpha, \phi_r = \frac{1}{\alpha r}$ $(r = p, q)$, Theorem 2 gives

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} dxdy &< \frac{\pi}{\alpha \sin^{\frac{1}{p}}(\pi/\alpha p) \sin^{\frac{1}{q}}(\pi/\alpha q)} \times \\ &\times \left(\int_0^\infty x^{1-\alpha} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\alpha} g^q(x) dx \right)^{\frac{1}{q}}, \end{aligned} \tag{50}$$

which is the result of Kuang[4].

Remark 2. For $\lambda = b + c + 1, u(x) = v(x) = x, \phi_p = c + 1 - \frac{1}{p}, \phi_q = b + 1 - \frac{1}{q}$, Theorem 2 gives

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+1}} dxdy &< B\left(b + \frac{1}{p}, c + \frac{1}{q}\right) \times \\ &\times \left(\int_0^\infty x^{p(1-b)-2} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q(1-c)-2} g^q(x) dx \right)^{\frac{1}{q}}, \end{aligned} \tag{51}$$

which is given by Peachey [24].

Remark 3. For $u(x) = v(x) = x^\alpha$, $\phi_p = \frac{1-mp}{\alpha}$, $\phi_q = \frac{1-nq}{\alpha}$, Theorem 2 gives the following results:

If $p > 1, 1/p + 1/q = 1, \alpha > 0, \lambda > 0, m, n \in \mathbb{R}$ such that $0 < 1 - mp < \alpha\lambda$, $0 < 1 - nq < \alpha\lambda$ and $f \geq 0, g \geq 0$ satisfy $0 < \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx < \infty$ and $0 < \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(x) dx < \infty$ then

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dxdy &< H_{\lambda,\alpha}(m,n,p,q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx \right)^{\frac{1}{p}} \times \\ &\times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \tag{52}$$

where $H_{\lambda,\alpha}(m,n,p,q) = \frac{1}{\alpha} B^{\frac{1}{p}}(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}) B^{\frac{1}{q}}(\frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha})$.

Further if $mp + nq = 2 - \alpha\lambda$, then Theorem 4 gives

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dxdy &< \frac{1}{\alpha} B\left(\frac{1-mp}{\alpha}, \frac{1-nq}{\alpha}\right) \left(\int_0^\infty x^{n(p+q)-1} f^p(x) dx \right)^{\frac{1}{p}} \times \\ &\times \left(\int_0^\infty y^{m(p+q)-1} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \tag{53}$$

where the constant factor $\frac{1}{\alpha} B(\frac{1-mp}{\alpha}, \frac{1-nq}{\alpha})$ is the best possible. These two inequalities are given by Hong [6].

Remark 4. Replacing $u(x)$ by $xu(x)$ and $v(x)$ by $xv(x)$ and taking $\phi_p = 1 - A_2 p$, $\phi_q = 1 - A_1 q$ in Theorem 2, we get the following result given by Mario Krnic et. al [27]:

If $p > 1, 1/p + 1/q = 1, \lambda > 0, A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$ *and* $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$ then

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(xu(x)+yv(y))^\lambda} dxdy < \\ &< L\left(\int_0^\infty (xu(x))^{1-\lambda+p(A_1-A_2)}(u(x)+xu'(x))^{1-p} f^p(x)dx\right)^{1/p} \times \\ &\times \left(\int_0^\infty (xv(x))^{1-\lambda+q(A_2-A_1)}(v(x)+xv'(x))^{1-q} g^q(x)dx\right)^{1/q}, \end{aligned} \tag{54}$$

where $L = \left(B\left(1 - A_2 p, \lambda - 1 + A_2 p\right)\right)^{\frac{1}{p}} \left(B\left(1 - A_1 q, \lambda - 1 + A_1 q\right)\right)^{\frac{1}{q}}$.

Remark 5. For $u(x) = Aa^x, v(x) = Bb^x, \phi_r = 1 + (1 - \frac{1}{r})(\lambda - 2)\,(r = p, q)$, Theorem 4 gives the following inequality:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, A > 0, B > 0, a > 1, b > 1$ and $f, g \geq 0$ satisfy $0 < \int_{-\infty}^{\infty} a^{(2-\lambda-p)x} f^p(x)dx < \infty$ and $0 < \int_{-\infty}^{\infty} b^{(2-\lambda-q)x} g^q(x)dx < \infty$, then

$$\begin{aligned} &\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(Aa^x + Bb^y)^\lambda} dxdy < \\ &< C\left(\int_{-\infty}^\infty a^{(2-\lambda-p)x} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^\infty b^{(2-\lambda-q)x} g^q(x)dx\right)^{\frac{1}{q}}, \end{aligned} \tag{55}$$

where the constant factor $C = \left(\frac{A^{1-\lambda}}{B \ln b}\right)^{\frac{1}{p}} \left(\frac{B^{1-\lambda}}{A \ln a}\right)^{\frac{1}{q}} B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible. This inequality is an extension of the result of Zhou et.al[22], where they consider the parameter $p \geq q > 1, 1 - \frac{q}{p} < \lambda \leq 2$.

Remark 6. For $u(x) = v(x)$, Theorem 4 gives (52).

For other appropriate values of $\lambda, \phi_p, \phi_q, u(x)$ and $v(x)$ taken in Theorem 2–5, many new inequalities can be obtained.

5 Applications

In this section, we will give the generalizations of Hardy-Littlewood's inequality. Let $f \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)dx, \quad n = 0, 1, 2, 3, \ldots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)dx \tag{56}$$

where the constant factor π is the best possible.

In [20, 21], Gao gave the integral version of Hardy-Littlewood's inequality as follows :

Let $h \in L^2(0,1)$ and $h \neq 0$. If

$$f(x) = \int_0^1 t^x |h(t)| dt, \quad x \in [0, \infty),$$

then

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 h^2(t) dt \tag{57}$$

and

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 t h^2(t) dt. \tag{58}$$

Theorem 11. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h \in L^2(0,1)$ and $h(t) \neq 0$. Define a function $f(x)$ by*

$$f(x) = (u'(x))^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| dt, \quad x \in (a, b).$$

If $0 < \int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx < \infty$, then

$$\left(\int_a^b f^p(x) dx \right)^{1+\frac{1}{p}} < \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx \right)^{\frac{1}{p}} \int_0^1 t h^2(t) dt. \tag{59}$$

Proof. We can write

$$f^p(x) = f^{p-1}(x) u'(x)^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| dt.$$

Now applying, Schwartz inequality and Corollary-1, we have

$$\begin{aligned}
&\left(\int_a^b f^p(x) dx \right)^2 = \\
&= \left\{ \int_0^1 \left(\int_a^b f^{p-1}(x) u'(x)^{\frac{1}{p}} t^{u(x)-\frac{1}{2}} dx \right) t^{\frac{1}{2}} |h(t)| dt \right\}^2 \leq \\
&\leq \int_0^1 \left(\int_a^b f^{p-1}(x) u'(x)^{\frac{1}{p}} t^{u(x)-\frac{1}{2}} dx \right)^2 dt \times \int_0^1 t h^2(t) dt = \\
&= \left(\int_a^b \int_a^b \frac{f^{p-1}(x) u'(x)^{\frac{1}{p}} f^{p-1}(y) u'(y)^{\frac{1}{p}}}{u(x) + u(y)} dx dy \right) \times \int_0^1 t h^2(t) dt \leq \\
&\leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_a^b (u'(x))^{1-p} f^{p(p-1)}(x) u'(x) dx \right)^{\frac{1}{p}} \times \\
&\times \left(\int_a^b (u'(x))^{1-q} f^{q(p-1)}(x) (u'(x))^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \times \int_0^1 t h^2(t) dt = \\
&= \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_a^b (u'(x))^{2-p} f^{p(p-1)}(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f^p(x) dx \right)^{\frac{1}{q}} \times \int_0^1 t h^2(t) dt.
\end{aligned} \tag{60}$$

Since $h(t) \neq 0$, so, $f(x) \neq 0$. Hence it is impossible for equality in (60) and then we get the inequality (59). This completes the theorem. $\square$

Theorem 12. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h \in L^2(0,1)$ *and* $h(t) \neq 0$. *Define a function* $f(x)$ *by*

$$f(x) = (u(x))^{\frac{1}{2} - \frac{1}{p}} (u'(x))^{\frac{1}{p}} \int_0^1 t^{u(x)} |h(t)| dt, \quad x \in (a,b).$$

If $0 < \int_a^b \left(\frac{u'(x)}{u(x)}\right)^{2-p} f^{p(p-1)}(x)dx < \infty$, *then*

$$\left(\int_a^b f^p(x)dx\right)^{1+\frac{1}{p}} < \pi \left(\int_a^b \left(\frac{u'(x)}{u(x)}\right)^{2-p} f^{p(p-1)}(x)dx\right)^{\frac{1}{p}} \int_0^1 th^2(t)dt. \tag{61}$$

Proof. Proceeding as in Theorem refhlthm1 and using Corollary 2, we complete the theorem. $\square$

Remark 7. Taking $p = 2$ in Theorem 11 and Theorem 12, we get

$$\int_0^\infty f^2(x)dx < \pi \int_0^1 th^2(t)dt \tag{62}$$

which is a generalization of Hardy-Littlewood inequality (58).

References

[1] Hardy G.H., Littlewood J.E., Polya G. *Inequalities.* Cambridge University Press, Cambrige, 1952.

[2] Kuang J. *Applied Inequalities.* Shangdong Science and Technology Press, Jinan, 2004.

[3] Wang Z., Guo D. *An Introduction to Special Functions.* Science Press, Beijing, 1979.

[4] Kuang J. *On new extensions of Hilbert's Inequalities.* J. Math. Anal. Appl., 1999, **235**, No. 2, 608–614.

[5] Hong Y. *All-sided Generalization about Hardy-Hilbert's integral inequalities.* Acta Math. Sinica (China), 2001, **44**, 619–626.

[6] Hong Y. *On Hardy-Hilbert's integral inequalities with some parameters.* J. Ineq. Pure and Appl. Math., 2005, **6**, No. 4, Art-92.

[7] Yang B. *On generalization of Hardy-Hilbert's integral inequalities.* Acta Math. Sinica (China),, 1998, **41**, No. 4, 839–844.

[8] Yang B., Debnath L. *On a new generalization of Hilbert's integral inequality and it's application.* RGMIA Research Report Collection, 1999, **2**, No. 5, Article 13.

[9] Yang B. *On new generalizations of Hilbert's inequality.* J. Math. Anal. Appl., 2000, **248**, 29–40.

[10] Yang B. *On a general Hardy-Hilbert's integral inequality with a best value.* Chin. Ann. of Math., 2000, **21A**, 401–408.

[11] Yang B. *On Hardy-Hilbert's integral inequality.* J. Math. Anal. Appl., 2001, **261**, 295–306.

[12] Yang B. *On an extension of Hardy-Hilbert's inequality.* RGMIA Research Report Collection, 2001, **4**, No. 3, Article 7.

[13] Yang B. *On Generalization of the Hardy-Hilbert's integral inequality.* RGMIA Research Report Collection, 2001, **4**, No. 4, Article 1.

[14] Yang B. *On a generalization of Hardy-Hilbert's inequality.* Ann. of Math. (China), 2002, **23**, 247–254.

[15] Yang B., Debnath L. *On the extended Hardy-Hilbert's inequality.* J. Math. Anal. Appl., 2002, **272**, 187–199.

[16] Yang B. *On a new inequality similar to Hardy-Hilbert's inequality.* Math. Ineq. and Appl., 2003, **6**, No. 1, 37–44.

[17] Yang B. *On the extended Hilbert's integral inequality.* J. Ineq. Pure and Appl. Math., 2004, **5**, No. 4, Art-96.

[18] Yang B. *A relation to Hardy-Hilbert's integral inequality and Mulholland's inequality.* J. Ineq. Pure and Appl. Math., 2005, **6**, No. 4, Art-112.

[19] Jia W., Gao M. *An extended Hardy-Hilbert's inequality and it's applications.* J. Ineq. Pure and Appl. Math., 2006, **7**, No. 1, Art-30.

[20] Gao M. *On Hilbert's Inequality and Its Applications.* J. Math. Anal. Appl., 1997, **212**, 316–323.

[21] Gao M., Tan Li, Debnath L. *Some Improvements on Hilbert's Integral Inequality.* J. Math. Anal. Appl., 1999, **229**, 682–689.

[22] Zhou Yu, Gao M. *An Extension on the Hardy-Hilbert's Integral Inequality and its Application.* J. Math. Ineq., 2007, **1**, No. 2, 149–158.

[23] Azar L.E. *On an Extension of Hardy-Hilbert's Integral Inequality with Three Parameters.* Int. Math. Forum, 2007, **2**, No. 48, 2369–2378.

[24] Peachey T.C. *Some integral inequalities related to Hilbert's.* J. Ineq. Pure and Appl. Math., 2003, **4**, No. 1, Art-19.

[25] Brnetic I., Pecaric J. *Generalization of Hilbert's integral inequality.* Math. Ineq. and Appl., 2004, **7**, No. 2, 199–205.

[26] Brnetic I., Pecaric J. *Generalization of inequalities of Hardy-Hilbert type.* Math. Ineq. and Appl., 2004, **7**, No. 2, 217–225.

[27] Mario K., Gao M., Pecaric J., Gao X. *On the Best Constant in Hilbert's inequality.* Math. Ineq. and Appl., 2005, **8**, No. 2, 317–29.

Namita Das
P.G. Department of Mathematics
Utkal University, Vani Vihar
Bhubaneswar, Orissa, India 751004

E-mail: *namitadas440@yahoo.co.in*

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Srinibas Sahoo
Department of Mathematics
Banki Autonomous College, Banki
Orissa, India 754008

E-mail: *sahoosrinibas@yahoo.co.in*