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On preradicals associated to principal functors of module categories, I

A. I. Kashu*

Abstract. The preradicals associated to the functor $Hom_R(U,\text{-}) : R\text{-}Mod \to \mathcal{A}b$ are revealed, their properties and the relations between these preradicals are studied.

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Introduction

The radicals and torsions associated to adjoint situations and Morita contexts were studied in a series of papers, which were totalized in the book [1]. The aim of this article is the generalization, supplement and specification of some results of [1] concerning the preradicals in module categories which are determined by principal functors of module categories:

$$H = H^U = Hom_R(U,\text{-}) : R\text{-}Mod \to \mathcal{A}b \quad ({}_RU \in R\text{-}Mod),$$

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b \quad (U_S \in Mod\text{-}S),$$

$$H' = H_U = Hom_R(\text{-}, U) : R\text{-}Mod \to \mathcal{A}b \quad ({}_RU \in R\text{-}Mod),$$

where $\mathcal{A}b$ is the category of abelian groups. In particular, it will be shown that some results which were proved for adjoint situations and Morita contexts are valid in general case (without supplementary restrictions). The preradicals associated to each of functors H, T and H' will be elucidated, the properties of these preradicals, as well as the relations between them and the conditions of coincidence of some preradicals will be shown.

The part I of this work is dedicated to the study of indicated above questions for the functor $H = Hom_R(U,\text{-})$ for an arbitrary module ${}_RU \in R$-Mod. In the following parts the functors T and H' will be investigated from the same aspect.

1 Preliminary notions and results

The basic notions and results of radical theory in modules can be found in the books [2–5]. For specification of terminology and notations we will remind some of them.

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Let R be a ring with unity and R-Mod is the category of unitary left R-modules. A *preradical* r of R-Mod is a subfunctor of identic functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f : M \to M'$.

Now we remind the principal types of preradicals [2, 4].

A preradical r of R-Mod is called:

– *idempotent preradical* if $r(r(M)) = r(M)$ for every $M \in R$-Mod;

– *radical* if $r(M/r(M)) = 0$ for every $M \in R$-Mod;

– *idempotent radical* if both previous conditions are fulfilled;

– *pretorsion* if $r(N) = N \cap r(M)$ for every $N \subseteq M$;

– *torsion* if r is radical and pretorsion;

– *cohereditary preradical* if $r(M/N) = (r(M) + N)/N$ for every $N \subseteq M$;

– *cotorsion* if r is idempotent and cohereditary.

Every preradical r of R-Mod defines two classes of modules:

1) the class of r-*torsion* modules

$$\mathcal{R}(r) = \{M \in R\text{-}Mod \,|\, r(M) = M\};$$

2) the class of r-*torsionfree* modules

$$\mathcal{P}(r) = \{M \in R\text{-}Mod \,|\, r(M) = 0\}.$$

The special types of preradicals indicated above can be described by associated classes of modules. More exactly:

- every *idempotent preradical* r is described by the class $\mathcal{R}(r)$, which is closed under homomorphic images and direct sums; such classes are called *pretorsion classes*;
- every *radical* r is described by the class $\mathcal{P}(r)$, which is closed under submodules and direct products; the classes with such properties are called *pretorsionfree classes*;
- every *idempotent radical* r can be restored both by the class $\mathcal{R}(r)$ and $\mathcal{P}(r)$; the class $\mathcal{R}(r)$ is pretorsion and closed under extensions – such classes are called *torsion classes*; the class $\mathcal{P}(r)$ is pretorsionfree and closed under extensions – such classes are called *torsionfree classes.*

If r is a torsion then $\mathcal{R}(r)$ is a *hereditary torsion class* and $\mathcal{P}(r)$ is a *stable torsionfree class.* If r is a cotorsion, then $\mathcal{P}(r)$ is simultaneously a torsion class and a torsionfree class; such classes are called *TTF-classes.*

If r is an idempotent preradical of R-Mod, then it can be restored by the class $\mathcal{R}(r)$ in the following way:

$$r(M) = \sum\{N \subseteq M \,|\, N \in \mathcal{R}(r)\}.$$

Dually, if r is a radical of R-Mod, then it can be expressed by the class $\mathcal{P}(r)$ as follows:

$$r(M) = \cap\,\{N \subseteq M \,|\, M/N \in \mathcal{P}(r)\}.$$

In the theory of radicals in modules an essential role is played by the following two *operators of Hom-orthogonality.* For an arbitrary class of modules $\mathcal{K} \subseteq R$-Mod we define:

$$\mathcal{K}^{\uparrow} = \{M \in R\text{-}Mod \,|\, Hom_R(M, N) = 0 \;\text{ for every }\; N \in \mathcal{K}\},$$

$$\mathcal{K}^{\downarrow} = \{N \in R\text{-}Mod \,|\, Hom_R(M,N) = 0 \;\text{ for every }\; M \in \mathcal{K}\}.$$

The following facts are well known. For every class $\mathcal{K} \subseteq R$-Mod we have:

– $\mathcal{K}^{\uparrow}$ is a torsion class;
– $\mathcal{K}^{\downarrow}$ is a torsionfree class;
– $\mathcal{K}^{\downarrow\uparrow}$ is the least torsion class containing $\mathcal{K}$;
– $\mathcal{K}^{\uparrow\downarrow}$ is the least torsionfree class containing $\mathcal{K}$.

If r is an idempotent radical then:

$$\mathcal{R}(r) = \mathcal{P}(r)^{\uparrow}, \quad \mathcal{P}(r) = \mathcal{R}(r)^{\downarrow}.$$

In the family of all preradicals of the category R-Mod the relation of *partial order* can be defined as follows:

$$r_1 \leq r_2 \overset{\text{def}}{\Longleftrightarrow} r_1(M) \subseteq r_2(M) \text{ for every } M \in R\text{-Mod}.$$

For the preradicals of special types this relation can be expressed by associated classes of modules. In particular:

– for idempotent preradicals
$$r_1 \leq r_2 \iff \mathcal{R}(r_1) \subseteq \mathcal{R}(r_2);$$
– for radicals
$$r_1 \leq r_2 \iff \mathcal{P}(r_1) \supseteq \mathcal{P}(r_2);$$
– for idempotent radicals
$$r_1 \leq r_2 \iff \mathcal{R}(r_1) \subseteq \mathcal{R}(r_2) \iff \mathcal{P}(r_1) \supseteq \mathcal{P}(r_2).$$

2 Preradicals associated to functor *H*

Let $U \in R$-Mod be an arbitrary left R-module and consider the functor $H = Hom_R(U, \text{-}) : R\text{-}Mod \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. We denote:

$$Gen\,({}_RU) = \{M \in R\text{-}Mod \,|\, \text{there exists an epi } U^{(\mathfrak{A})} \to M \to 0\},$$

i.e. $Gen\,({}_RU)$ is the class of modules generated by the fixed module ${}_RU$. It is clear that the class $Gen\,({}_RU)$ is closed under homomorphic images and direct sums, so it is a pretorsion class. We define by ${}_RU$ the function r^U as follows:

$$r^U(M) = \sum_{f:U \to M} Im\, f, \quad M \in R\text{-}Mod,$$

i.e. $r^U(M)$ is the *trace* of ${}_RU$ in ${}_RM$ for every $M \in R$-Mod. The following fact is obvious.

Proposition 2.1. *For every module $U \in R$-Mod the function r^U is an idempotent radical of R-Mod, determined by the class of r^U-torsion modules*: $\mathcal{R}(r^U) = Gen\,({}_RU)$. □

For the functor H we denote:

$$Ker\,H = \{M \in R\text{-}Mod \,|\, H(M) = 0\}.$$

From the definition of operator $(\ )^{\downarrow}$ it follows for the class $\mathcal{K} = \{_R U\}$ that $\operatorname{Ker} H = \{_R U\}^{\downarrow}$. From the properties of the functor H we have

Proposition 2.2. *$Ker\, H$ is a torsionfree class, i.e. it is closed under submodules, direct products and extensions.* □

Therefore the class $\operatorname{Ker} H$ defines an *idempotent radical* $\overline{r}^U$ such that $\mathcal{P}(\overline{r}^U) \overset{\text{def}}{=\!=} Ker\, H \ \ (= \{_R U\}^{\downarrow})$. The respective torsion class for $\overline{r}^U$ is:

$$\mathcal{R}(\overline{r}^U) = (Ker\, H)^{\uparrow} = \{_R U\}^{\downarrow\uparrow} = (Gen\,(_R U))^{\downarrow\uparrow}.$$

Since $\mathcal{R}(r^U) = Gen\,(_R U)$, it follows that $\mathcal{R}(\overline{r}^U)$ is the least torsion class containing $\mathcal{R}(r^U)$. In the language of preradicals this means the following.

Proposition 2.3. *For every module $U \in R\text{-}Mod$ we have $r^U \leq \overline{r}^U$ and $\overline{r}^U$ is the least idempotent radical, containing r^U.* □

Now we will investigate the question when these preradicals coincide: $r^U = \overline{r}^U$. For that we introduce the following notion.

Definition 1. A module $_R U$ will be called *weakly projective* if the functor $H = Hom_R(U, \text{-}) : R\text{-}Mod \to \mathcal{A}b$ preserves the exactness of the short exact sequences of the form:

$$0 \to r^U(M) \underset{\subseteq}{\overset{i}{\longrightarrow}} M \underset{\text{nat}}{\overset{\pi}{\longrightarrow}} M\,/\,r^U(M) \to 0$$

for every module $M \in R$-Mod, where i is the inclusion and π is the natural epimorphism.

In other words, $_R U$ is weakly projective if for every $M \in R$-Mod and every R-morphism $f : U \to M\,/\,r^U(M)$ there exists an R-morphism $g : U \to M$ such that $\pi g = f$ (π is natural morphism):

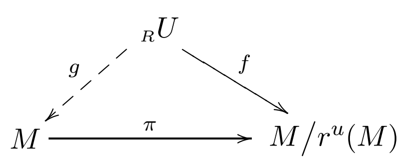

Fig. 1

Proposition 2.4. *For the module $_R U$ the following conditions are equivalent:*
1) $r^U = \overline{r}^U$;
2) r^U *is an (idempotent) radical*;
3) $Gen(_R U) = (Ker\, H)^{\uparrow} \ \ (= \{_R U\}^{\downarrow\uparrow})$;
4) $_R U$ *is a weakly projective module.*

Proof. 1) $\iff$ 2) $\iff$ 3) follows from Proposition 2.3.
2) $\Rightarrow$ 4). If r^U is a radical, then for every $M \in R$-Mod we have:

$$M\,/\,r^U(M) \in \mathcal{P}(r^U) = \mathcal{P}(\overline{r}^U) = Ker\, H,$$

therefore $Hom_R(U, M) / r^U(M) = 0$ and that implies immediately that ${}_RU$ is weakly projective ($f = 0 \Rightarrow g = 0$).
4) $\Rightarrow$ 2). Let ${}_RU$ be weakly projective and we verify that $r^U(M / r^U(M)) = 0$ for every $M \in R$-Mod. Consider an arbitrary R-morphism $f : U \to M / r^U(M)$. From condition 4) it follows that there exists a morphism $g : U \to M$ such that $\pi g = f$. Since $Im\, g \subseteq r^U(M)$ by definition of $r^U(M)$, we have $\pi g = 0$ and $f = 0$. So $Hom_R(U, M / r^U(M)) = 0$, i.e. $r^U(M / r^U(M)) = 0$ and r^U is a radical. □

Examples. 1) If ${}_RU$ is a projective module, then it is weakly projective, therefore $r^U = \overline{r}^U$.
2) If ${}_RU$ is a generator of R-Mod, then $Gen\,({}_RU) = R$-Mod, so $r^U = \overline{r}^U = \mathbf{1}$, where $\mathbf{1}$ is the greatest trivial preradical of R-Mod ($\mathbf{1}(M) = M$ for every $M \in R$-Mod).

More strong than the conditions of Proposition 2.4 is the request that the idempotent preradical r^U must be a cotorsion. To indicate when such situation takes place we need the

Definition 2 [1]. A module ${}_RU$ will be called *cohereditary below* if the class $\{{}_RU\}^{\downarrow}$ is cohereditary (*i.e. a TTF-class*).

This means that if $Hom_R(U, M) = 0$ for a module $M \in R$-Mod, then $Hom_R(U, M/N) = 0$ for every submodule $N \subseteq M$.

From Proposition 2.4 and definitions follows

Proposition 2.5. *For a module ${}_RU$ the following conditions are equivalent:*
1) *r^U is a cotorsion;*
2) *$r^U = \overline{r}^U$ and the class $\mathcal{P}(\overline{r}^U) = Ker\, H$ is cohereditary;*
3) *${}_RU$ is weakly projective and cohereditary below.* □

It is obvious that if the module ${}_RU$ is projective, then r^U is a cotorsion.

3 Preradicals defined by trace-ideal $I = Trace_U({}_RR)$

For a fixed module $U \in R$-Mod we consider its trace in ${}_RR$:

$$I = r^U({}_RR) = \sum_{f : U \to R} Im\, f,$$

which is a two-sided ideal of R. It defines the following three classes of modules (see [6]):

$${}_I\mathcal{T} = \{M \in R\text{-Mod} \mid IM = M\},$$

$${}_I\mathcal{F} = \{M \in R\text{-Mod} \mid m \in M, Im = 0 \Rightarrow m = 0\},$$

$$\mathcal{A}(I) = \{M \in R\text{-Mod} \mid IM = 0\},$$

i.e. ${}_I\mathcal{T}$ is the class of I-accessible modules, ${}_I\mathcal{F}$ is the class of modules without nonzero elements annihilated by I, and $\mathcal{A}(I)$ consists of the modules annihilated by I.

It is easy to verify the following properties of these classes.

Proposition 3.1. 1) ${}_I\mathcal{T}$ *is a torsion class;*
2) ${}_I\mathcal{F}$ *is a torsion free and stable class;*
3) $\mathcal{A}(I)$ *is closed under submodules, homomorphic images and direct products (hence also under direct sums). So the class* $\mathcal{A}(I)$ *is simultaneously a pretorsion and a pretorsionfree class.* □

Therefore the class ${}_I\mathcal{T}$ defines an *idempotent radical* r^I such that:

$$\mathcal{R}(r^I) \stackrel{\text{def}}{=\!=} {}_I\mathcal{T},$$

while the class ${}_I\mathcal{F}$ determines a *torsion* r_I such that:

$$\mathcal{P}(r_I) \stackrel{\text{def}}{=\!=} {}_I\mathcal{F},$$

which is the ideal torsion, defined by I (see [4]).

The class $\mathcal{A}(I)$ as pretorsion (and hereditary) class determines a *pretorsion* $r_{(I)}$ by the rule:

$$\mathcal{R}(r_{(I)}) \stackrel{\text{def}}{=\!=} \mathcal{A}(I).$$

For every $M \in R$-mod we have:

$$r_{(I)}(M) = \{m \in M \mid I \cdot m = 0\}.$$

From the other hand, $\mathcal{A}(I)$ as pretorsionfree (and cohereditary) class defines the *cohereditary radical* $r^{(I)}$ such that:

$$\mathcal{P}(r^{(I)}) \stackrel{\text{def}}{=\!=} \mathcal{A}(I),$$

which acts by the rule:

$$r^{(I)}(M) = IM, \ M \in R\text{-Mod} \ \ \text{(see [2, 4, 6])}.$$

Thus by definitions the idempotent radical r^I has the associated classes:

$$\big({}_I\mathcal{T} = \mathcal{R}(r^I), \ \ {}_I\mathcal{T}^{\downarrow} = \mathcal{P}(r^I)\big),$$

while the torsion r_I is defined by the classes:

$$\big({}_I\mathcal{F}^{\uparrow} = \mathcal{R}(r_I), \ \ {}_I\mathcal{F} = \mathcal{P}(r_I)\big).$$

In continuation we will indicate a series of relations between the classes of modules mentioned above. They imply the respective connexions between the preradicals defined by these classes.

Proposition 3.2. 1) $\mathcal{A}(I)^{\uparrow} = {}_I\mathcal{T}$; 2) $\mathcal{A}(I)^{\downarrow} = {}_I\mathcal{F}$.

Proof. 1) $(\subseteq)$. Let $M \in \mathcal{A}(I)^{\uparrow}$. Since $M/IM \in \mathcal{A}(I)$, we have $Hom_R(M, M/IM) = 0$, hence $M/IM = 0$ and $M = IM$.

$(\supseteq)$. Let $M \in {}_I\mathcal{T}$. Then for every $N \in \mathcal{A}(I)$ and $f : M \to N$ we have $f(M) = f(IM) = I \cdot f(M) \subseteq I \cdot N = 0$, so $f = 0$. Thus $Hom_R(M, N) = 0$ for every $N \in \mathcal{A}(I)$, i.e. $M \in \mathcal{A}(I)^{\uparrow}$.

2) $(\subseteq)$. Let $M \in \mathcal{A}(I)^{\downarrow}$. If $m \in M$ and $I \cdot m = 0$, then since $Rm \in \mathcal{A}(I)$ we have $Hom_R(Rm, M) = 0$, therefore $Rm = 0$ and $m = 0$. This means that $M \in {}_I\mathcal{F}$.

$(\supseteq)$. Let $M \in {}_I\mathcal{F}$. We consider an arbitrary module $N \in \mathcal{A}(I)$ and an R-morphism $f : N \to M$. For every element $n \in N$ we have:

$$I \cdot f(n) = f(I \cdot n) \subseteq f(IN) = f(0) = 0,$$

and from the assumption $M \in {}_I\mathcal{F}$ now follows $f(n) = 0$, thus $f = 0$. In that way $Hom_R(M, N) = 0$ for every $N \in \mathcal{A}(I)$ and so $M \in \mathcal{A}(I)^{\downarrow}$. □

From the relations of Proposition 3.2 the corresponding connexions between the preradicals defined by ideal I follow. Namely, from ${}_I\mathcal{T} = \mathcal{A}(I)^{\uparrow}$ we obtain ${}_I\mathcal{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow}$, therefore the class ${}_I\mathcal{T}^{\downarrow}$ $\big(\stackrel{\text{def}}{=} \mathcal{P}(r^I)\big)$ is the least torsionfree class containing $\mathcal{A}(I)$ $\big(\stackrel{\text{def}}{=} \mathcal{P}(r^{(I)})\big)$.

Similarly, from ${}_I\mathcal{F} = \mathcal{A}(I)^{\downarrow}$ we have ${}_I\mathcal{F}^{\uparrow} = \mathcal{A}(I)^{\downarrow\uparrow}$, therefore ${}_I\mathcal{F}^{\uparrow}$ $\big(\stackrel{\text{def}}{=} \mathcal{R}(r_I)\big)$ is the least torsion class containing $\mathcal{A}(I)$ $\big(\stackrel{\text{def}}{=} \mathcal{R}(r_{(I)})\big)$. Translating this facts in the language of preradicals, associated to these classes, we obtain the following results.

Proposition 3.3. 1) $r^I \leq r^{(I)}$ *and* r^I *is the greatest idempotent radical contained in* $r^{(I)}$.
2) $r_I \geq r_{(I)}$ *and* r_I *is the least idempotent radical containing* $r_{(I)}$. □

Thus we have two pairs of "near" preradicals: $r^I \leq r^{(I)}$ and $r_I \geq r_{(I)}$. It is natural to search the conditions of its coincidence.

Proposition 3.4. *The following conditions are equivalent:*
1) $r^I = r^{(I)}$;
2) $r^{(I)}$ *is idempotent*;
3) $\mathcal{A}(I) = {}_I\mathcal{T}^{\downarrow}$;
4) $r_I = r_{(I)}$;
5) $r_{(I)}$ *is a radical*;
6) $\mathcal{A}(I) = {}_I\mathcal{F}^{\uparrow}$;
7) $I = I^2$.

Proof. Consists in the direct verification (see, for example,[4], p. 22). □

If the equivalent conditions of Proposition 3.4 are fulfilled, then $\mathcal{A}(I)$ is TTF-class, r^I is a cotorsion defined by the classes $\big({}_I\mathcal{T}, \mathcal{A}(I)\big)$ and r_I is a jansian torsion with the associated classes $(\mathcal{A}(I), {}_I\mathcal{F})$.

4 Relations between preradicals defined by *H* and preradicals defined by *I*

In this section we will show that there exists some remarkable connexions between the preradicals $r^U, \overline{r}^U$ of Section 2 and preradicals defined by ideal I (Section 3). For that we clarify firstly the relations between the respective classes of modules. We start by the following remark.

Lemma 4.1. *For every module $M \in R$-Mod we have $IM \subseteq r^U(M)$ (where ${}_RU$ is a fixed module and $I = r^U({}_RR)$).*

Proof. We must verify that $\big(\sum\limits_{f:U\to R} Im\, f\big)\, M \subseteq \sum\limits_{f:U\to M} Im\, g$. For every $f : U \to R$ and $m \in M$ we have the R-morphism

$$g_{(f,m)} : U \to M, \quad g_{(f,m)}(u) \overset{\text{def}}{=\!=} f(u) \cdot m, \ u \in U.$$

Since $Im\, g_{(f,m)} = (Im\, f) \cdot m \subseteq \sum\limits_{f:U\to M} Im\, g = r^U(M)$ for every $f : U \to M$ and $m \in M$, we obtain $IM \subseteq r^U(M)$. □

Lemma 4.2. $Ker\, H \subseteq \mathcal{A}(I)$ *(i.e. $\mathcal{P}(\overline{r}^U) \subseteq \mathcal{P}(r^{(I)})$, hence $\overline{r}^U \geq r^{(I)}$).*

Proof. Let $M \in Ker H$, i.e. $Hom_R(U, M) = 0$. Then $r^U(M) = \sum\limits_{f:U\to M} Im\, g = 0$ and by Lemma 4.1 we have $IM \subseteq r^U(M) = 0$, so $M \in \mathcal{A}(I)$. □

Lemma 4.3. ${}_I\mathcal{T} \subseteq Gen({}_RU)$ *(i.e. $\mathcal{R}(r^I) \subseteq \mathcal{R}(r^U)$, hence $r^I \leq r^U$).*

Proof. Let $M \in {}_I\mathcal{T}$, i.e. $IM = M$. From Lemma 4.1 we have $M = IM \subseteq r^U(M)$, thus $M = r^U(M)$. Therefore, $M \in \mathcal{R}(r^U) = Gen({}_RU)$. □

In a schematic form the relations between the preradicals indicated above can be presented as follows:

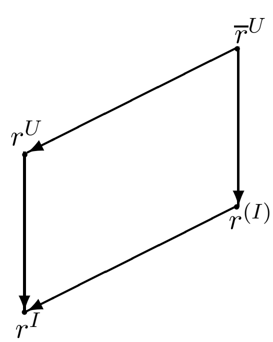

Fig.2

where the arrow $r_1 \leftarrow r_2$ means $r_1 \leq r_2$.

In the Propositions 2.4 and 3.4 the criterions of coincidences $r^U = \overline{r}^U$ and $r^I = \overline{r}^I$ are indicated. Now we will consider the case when all four preradicals of Fig. 2 coincide.

Proposition 4.4. *The following conditions are equivalent:*

1) $r^U = r^I$ (i.e. $Gen\,({}_RU) = {}_I\mathfrak{T}$);

2) $\overline{r}^U = r^I$ (i.e. $Ker\,H)^{\uparrow} = {}_I\mathfrak{T}$);

3) $\overline{r}^U = r^{(I)}$ (i.e. $Ker\,H = \mathcal{A}(I)$);

4) $r^U = r^{(I)}$;

5) $I\,U = U$.

Proof. We will prove that every condition 1)–4) implies the coincidence of all four preradicals.

1) If $r^U = r^I$, then since r^I is a radical we have that r^U is a radical, so $r^U = \overline{r}^U$ (Proposition 2.4). Therefore $r^I = \overline{r}^U$ and $r^I = r^{(I)} = \overline{r}^U$.

2) If $\overline{r}^U = r^I$ then is obvious that all preradicals coincide.

3) If $\overline{r}^U = r^{(I)}$, then since $\overline{r}^U$ is idempotent, follows that $r^{(I)}$ is idempotent, hence $\overline{r}^I = r^{(I)}$ (Proposition 3.4) and then $\overline{r}^U = r^I$.

4) If $r^U = r^{(I)}$, then r^U is a radical and $r^{(I)}$ is idempotent, therefore $r^U = \overline{r}^U$ and $r^I = r^{(I)}$.

From the previous arguments follows that the conditions 1)–4) are equivalent.

1) $\Rightarrow$ 5). If $r^U = r^I$, then $\mathcal{R}(r^U) = \mathcal{R}(r^I)$, i.e. $Gen\,({}_RU) = {}_I\mathfrak{T}$. Since ${}_RU \in Gen\,({}_RU)$, we have ${}_RU \in {}_I\mathfrak{T}$, i.e. $I\,U = U$.

5) $\Rightarrow$ 1). Let $I\,U = U$, i.e. ${}_RU \in {}_I\mathfrak{T}$. Then $Gen\,({}_RU) \subseteq {}_I\mathfrak{T}$ (because ${}_I\mathfrak{T}$ is a torsion class). From Lemma 4.3 we obtain $Gen\,({}_RU) = {}_I\mathfrak{T}$, thus $r^U = r^I$. □

Corollary 4.5. *If $I\,U = U$, then module ${}_RU$ is weakly projective and $I = I^2$.*

Proof. The conditions of Proposition 4.4 implies in particular $r^U = \overline{r}^U$ and $r^{(I)} = r^I$, therefore ${}_RU$ is weakly projective (Proposition 2.4) and $I = I^2$ (Proposition 3.4). □

Remark. In the previous study do not participate the pair of preradicals $(r_I, r_{(I)})$. In general case the relation between preradicals $\overline{r}^U$ and r_I can be expressed by inclusion $\mathcal{P}(\overline{r}^U) \subseteq \mathcal{R}(r_I)$ (i.e. $Ker\,H \subseteq \mathcal{A}(I)^{\downarrow\uparrow}$). In the case when $I\,U = U$ (Proposition 4.4) we have $\mathcal{P}(\overline{r}^U) = \mathcal{R}(r_I)$, since then $Ker\,H = \mathcal{A}(I) = \mathcal{A}(I)^{\downarrow\uparrow}$.

In conclusion we totalize by the following scheme the relations between all classes of modules studied above (Fig. 3).

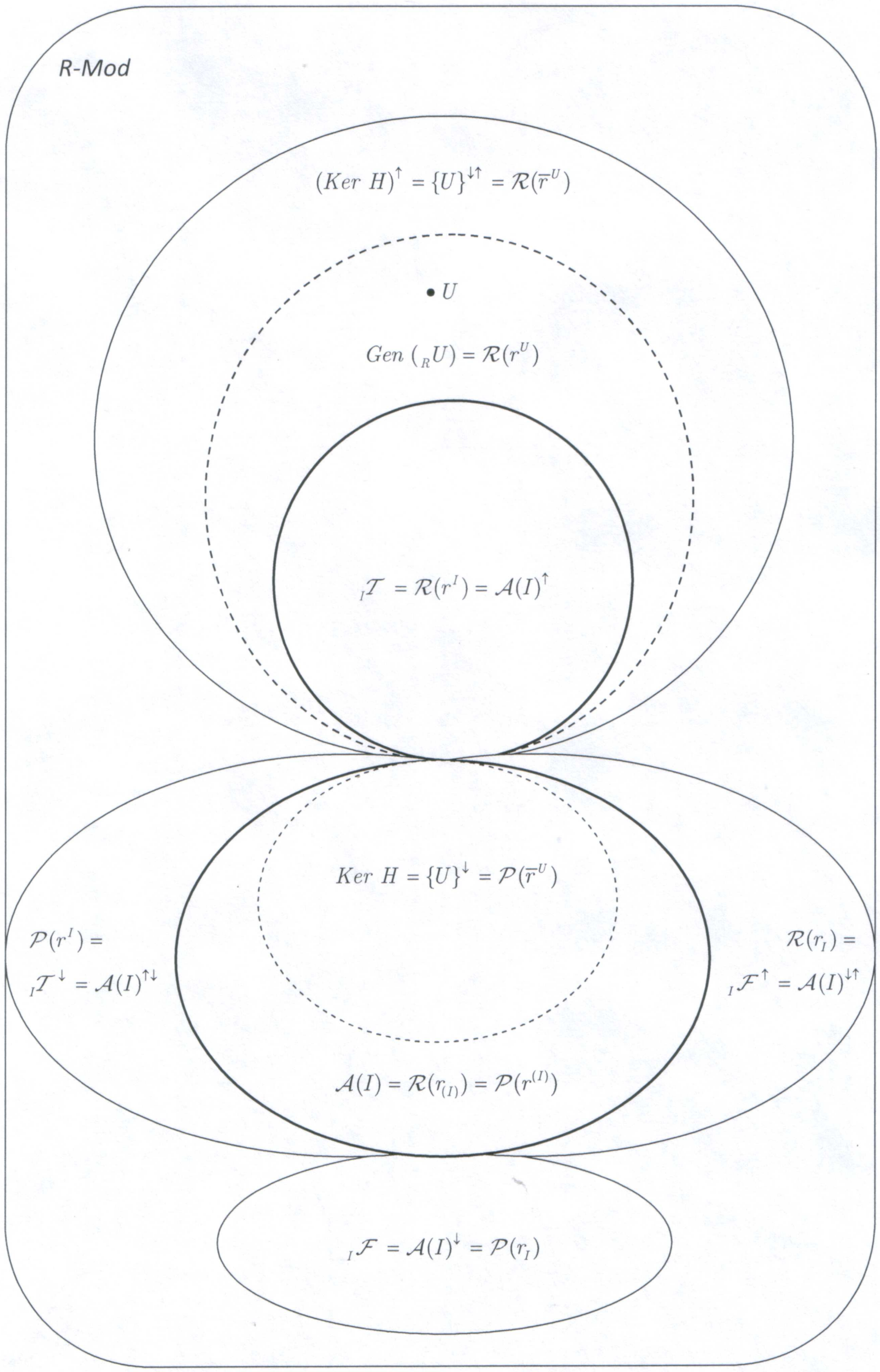
R-Mod
$(Ker\ H)^{\uparrow} = \{U\}^{\downarrow\uparrow} = \mathcal{R}(\overline{r}^{U})$
$\bullet\, U$
$Gen\ ({}_{R}U) = \mathcal{R}(r^{U})$
${}_{I}\mathcal{T} = \mathcal{R}(r^{I}) = \mathcal{A}(I)^{\uparrow}$
$Ker\ H = \{U\}^{\downarrow} = \mathcal{P}(\overline{r}^{U})$
$\mathcal{P}(r^{I}) =$
${}_{I}\mathcal{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow}$
$\mathcal{R}(r_{I}) =$
${}_{I}\mathcal{F}^{\uparrow} = \mathcal{A}(I)^{\downarrow\uparrow}$
$\mathcal{A}(I) = \mathcal{R}(r_{(I)}) = \mathcal{P}(r^{(I)})$
${}_{I}\mathcal{F} = \mathcal{A}(I)^{\downarrow} = \mathcal{P}(r_{I})$

Fig. 3.

This general situation will be completed after the study of preradicals, associated to the functor of tensor product T, adding two preradicals t^V and $\overline{t}^V$ (dual to r^U and $\overline{r}^U$), connected with r_I and $r_{(I)}$ similar as the pairs $(r^U, \overline{r}^U)$ and $(r^I, \overline{r}^{(I)})$ are connected (see Fig. 2).

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chişinău MD−2028
Moldova
E-mail: *kashuai@math.md*

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