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A complete classification of quadratic differential systems according to the dimensions of $\mathbf{\mathit{Aff}\,(2,\mathbb{R})}$−orbits

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Abstract. In this article we consider the action of the group $Aff\,(2,\mathbb{R})$ of affine transformations and time rescaling on real planar quadratic differential systems. Via affine invariant conditions we give a complete stratification of this family of systems according to the dimension $\mathcal{D}$ of affine orbits proving that $3 \le \mathcal{D} \le 6$. Moreover we give a complete topological classification of all the systems located on the orbits of dimension $\mathcal{D} \le 5$ constructing the affine invariant criteria for the realization of each of 49 possible topologically distinct phase portraits.

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We consider here real planar differential systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{1}$$

with

$$\begin{aligned} p_0 = a, \quad p_1(x,y) = cx + dy, \quad p_2(x,y) = gx^2 + 2hxy + ky^2, \\ q_0 = b, \quad q_1(x,y) = ex + fy, \quad q_2(x,y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

We say that these systems are quadratic if $|p_2(x,y)| + |q_2(x,y)| \neq 0$.

Consider also the group $Aff\,(2,\mathbb{R})$ of affine transformations given by the equalities:

$$\bar{x} = \alpha x + \beta y + \nu, \quad \bar{y} = \gamma x + \delta y + \varkappa, \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0, \quad \alpha, \beta, \gamma, \delta, \nu, \varkappa \in \mathbb{R}.$$

According to [1] the operators of the linear representation of the group $Aff\,(2,\mathbb{R})$

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in the space of the coefficients and variables of systems (1) will take the form

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x}+a\frac{\partial}{\partial a}+d\frac{\partial}{\partial d}-e\frac{\partial}{\partial e}-g\frac{\partial}{\partial g}+k\frac{\partial}{\partial k}-2l\frac{\partial}{\partial l}-m\frac{\partial}{\partial m},\\
X_2 &= y\frac{\partial}{\partial x}+b\frac{\partial}{\partial a}+e\frac{\partial}{\partial c}+(f-c)\frac{\partial}{\partial d}-e\frac{\partial}{\partial f}+l\frac{\partial}{\partial g}+\\
&\quad+(m-g)\frac{\partial}{\partial h}+(n-2h)\frac{\partial}{\partial k}-l\frac{\partial}{\partial m}-2m\frac{\partial}{\partial n},\\
X_3 &= x\frac{\partial}{\partial y}+a\frac{\partial}{\partial b}-d\frac{\partial}{\partial c}+(c-f)\frac{\partial}{\partial e}+d\frac{\partial}{\partial f}-2h\frac{\partial}{\partial g}-k\frac{\partial}{\partial h}+\\
&\quad+(g-2m)\frac{\partial}{\partial l}+(h-n)\frac{\partial}{\partial m}+k\frac{\partial}{\partial n},\\
X_4 &= y\frac{\partial}{\partial y}+b\frac{\partial}{\partial b}-d\frac{\partial}{\partial d}+e\frac{\partial}{\partial e}-h\frac{\partial}{\partial h}-2k\frac{\partial}{\partial k}+l\frac{\partial}{\partial l}-n\frac{\partial}{\partial n},\\
X_5 &= \frac{\partial}{\partial x}-c\frac{\partial}{\partial a}-e\frac{\partial}{\partial b}-2g\frac{\partial}{\partial c}-2h\frac{\partial}{\partial d}-2l\frac{\partial}{\partial e}-2m\frac{\partial}{\partial f},\\
X_6 &= \frac{\partial}{\partial y}-d\frac{\partial}{\partial a}-f\frac{\partial}{\partial b}-2h\frac{\partial}{\partial c}-2k\frac{\partial}{\partial d}-2m\frac{\partial}{\partial e}-2n\frac{\partial}{\partial f}.
\end{aligned}\tag{2}$$

These operators form a six-dimensional Lie algebra [1].

Let $\tilde{a}=(a,b,c,d,e,f,g,h,k,\, l,m,n)$ be the 12-tuple of the coefficients of systems (1), i.e. each particular system (1) yields a point in $E^{12}(\tilde{a})$, where $E^{12}(\tilde{a})$ is the Euclidean space of the coefficients of the right-hand sides of systems (1). We denote by $\tilde{a}(q)\in E^{12}(\tilde{a})$ the point which corresponds to the system, obtained from a system (1) with coefficients $\tilde{a}$ via a transformation $q\in \mathit{Aff}(2,\mathbb{R})$.

Definition 1. Consider a system (1) and its corresponding point $\tilde{a}\in E^{12}(\tilde{a})$. We call the set $O(\tilde{a})=\{\tilde{a}(q)|q\in \mathit{Aff}\ (2,\mathbb{R})\}$ the $\mathit{Aff}(2,\mathbb{R})$ - *orbit* of this system.

It is known from [1] that

$$\mathfrak{D}\overset{def}{=}\dim_{\mathbb{R}} O(\tilde{a})=rank\mathcal{M},$$

where $\mathcal{M}$ is the matrix constructed on the coordinate vectors of operators (2):

$$\mathcal{M}=\begin{pmatrix}
a & 0 & 0 & d & -e & 0 & -g & 0 & k & -2l & -m & 0\\
b & 0 & e & -c+f & 0 & -e & l & -g+m & -2h+n & 0 & -l & -2m\\
0 & a & -d & 0 & c-f & d & -2h & -k & 0 & g-2m & h-n & k\\
0 & b & 0 & -d & e & 0 & 0 & -h & -2k & l & 0 & -n\\
-c & -e & -2g & -2h & -2l & -2m & 0 & 0 & 0 & 0 & 0 & 0\\
-d & -f & -2h & -2k & -2m & -2n & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We denote by $\Delta_{i,j,k,l,m,n}$ the minor of the 6th order of the matrix M_1, constructed on the columns i,j,k,l,m,n $(1\le i<j<k<l<m<n\le 12)$ and by $\Delta^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_s}$ the minor of the order s $(s=5,4,3)$ constructed on the lines $i_1,i_2,\ldots,i_s$ $(1\le i_1<i_2<\ldots<i_s\le 6\})$ and on the columns $j_1,j_2,\ldots,j_s$ $(1\le j_1<j_2<\ldots<j_s\le 12\})$.

In [2] a minimal polynomial basis of $GL(2,\mathbb{R})$-invariant polynomials (which are also named center-affine comitants and invariants) is constructed. We shall use here the following elements of this basis, defined in tensorial form (we keep the notations from [2]):

$$\begin{gathered} I_1=a^\alpha_\alpha,\ I_2=a^\alpha_\beta a^\beta_\alpha,\ I_3=a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\gamma}\varepsilon^{pq},\ I_5=a^\alpha_p a^\beta_{\gamma q}a^\gamma_{\alpha\beta}\varepsilon^{pq},\\ I_7=a^\alpha_{pr}a^\beta_{\alpha q}a^\gamma_{\beta s}a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8=a^\alpha_{pr}a^\beta_{\alpha q}a^\gamma_{\delta s}a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\ I_9=a^\alpha_{pr}a^\beta_{\beta q}a^\gamma_{\gamma s}a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\\ I_{18}=a^\alpha a^q a^p_\alpha\varepsilon_{pq},\ I_{21}=a^\alpha a^\beta a^q a^p_{\alpha\beta}\varepsilon_{pq},\ K_1=a^\alpha_{\alpha\beta}x^\beta,\ K_2=a^p_\alpha x^\alpha x^q\varepsilon_{pq},\\ K_3=a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma,\ K_4=a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma,\ K_5=a^p_{\alpha\beta}x^\alpha x^\beta x^q\varepsilon_{pq},\ K_6=a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta,\\ K_7=a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,\ K_{11}=a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q\varepsilon_{pq},\ K_{13}=a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu,\\ K_{21}=a^p x^q\varepsilon_{pq},\ K_{23}=a^p a^q_{\alpha\beta}x^\alpha x^\beta\varepsilon_{pq}. \end{gathered}\tag{3}$$

Here the following notations are used:

$$\begin{gathered} a^1=a,\quad a^2=b,\quad a^1_1=c,\quad a^1_2=d,\quad a^2_1=e,\quad a^2_2=f,\quad a^1_{11}=g,\quad a^1_{12}=h,\\ a^1_{22}=k,\quad a^2_{11}=l,\quad a^2_{12}=m,\quad a^2_{22}=n,\quad x^1=x,\quad x^2=y, \end{gathered}$$

and the unit bi-vectors ε^{pq} and ε_{pq} have the coordinates: $\varepsilon^{11}=\varepsilon^{22}=\varepsilon_{11}=\varepsilon_{22}=0$, $\varepsilon^{12}=-\varepsilon^{21}=\varepsilon_{12}=-\varepsilon_{21}=1$,

We consider the polynomials

$$\begin{aligned} C_i(\tilde a,x,y)&=yp_i(\tilde a,x,y)-xq_i(\tilde a,x,y)\in\mathbb{R}[\tilde a,x,y],\ i=0,1,2,\\ D_i(\tilde a,x,y)&=\frac{\partial}{\partial x}p_i(\tilde a,x,y)+\frac{\partial}{\partial y}q_i(\tilde a,x,y)\in\mathbb{R}[\tilde a,x,y],\ i=1,2, \end{aligned}\tag{4}$$

which are the only GL-comitants of degree one with respect to the coefficients of systems (1) that could exist for these systems. Comparing (3) with (4) we have the following identities: $C_0\equiv K_{21}$, $C_1\equiv K_2$, $C_2\equiv K_5$, $D_1\equiv I_1$, $D_2\equiv 2K_1$.

Using the so-called *transvectant of index k* (see [3]) of two real polynomials f and g:

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}$$

we shall construct the following GL–comitants of the second degree with respect to the coefficients of initial systems:

$$\begin{aligned} &T_1=(C_0,C_1)^{(1)}, && T_2=(C_0,C_2)^{(1)}, && T_3=(C_0,D_2)^{(1)},\\ &T_4=(C_1,C_1)^{(2)}, && T_5=(C_1,C_2)^{(1)}, && T_6=(C_1,C_2)^{(2)},\\ &T_7=(C_1,D_2)^{(1)}, && T_8=(C_2,C_2)^{(2)}, && T_9=(C_2,D_2)^{(1)}. \end{aligned}$$

According to [4] the transvectant $(f,g)^{(k)}$ of two GL–comitants (respectively T–comitants) of systems (1) is a GL–comitant (respectively T–comitant) of these systems too.

In what follows we shall construct the following T–comitants (and CT–comitants, see [5] for detailed definitions), which are responsible for the dimensions of the affine orbits for systems (1):

$$\beta(\tilde{a}) = 27I_8 - I_9 - 18I_7 = -2\text{Discrim}\big(C_2(\tilde{a},x,y)\big),$$
$$M(\tilde{a},x,y) = 2\,\text{Hess}\big(C_2(x,y)\big), \quad \widehat{H}(\tilde{a},x,y) = (-T_8 + 8T_9 + 2D_2^2)/72,$$
$$D(\tilde{a},x,y) = \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6) - (C_1,T_5)^{(1)} +$$
$$+6D_1(C_1D_2 - T_5) - 9D_1^2C_2\big]/36, \quad U_1(\tilde{a},x,y) = (C_2,D)^{(1)},$$
$$U_2(\tilde{a},x,y) = I_1K_1^2(2K_1^2K_2 - 2K_2K_6 - K_1K_{11}) - 2K_1^3(K_2K_4 + 2K_7K_{21}) +$$
$$+4K_1K_6^2K_{21} + K_1^2[2K_4K_{11} + K_2K_{13} + 2K_{23}(K_6 + K_7)] - 4K_6^2K_{23},$$
$$U_3 = K_2^2 - 4K_5K_{21}.$$

However we also need several affine comitants which we shall construct here following [6]. Denote by $\tilde{a}(\tau)$ the point from the space $E^{12}(\tilde{a})$ that corresponds to the system, obtained from a system (1) with coefficients $\tilde{a}$ via a translation $\tau: x = \bar{x} + x_0,\ y = \bar{y} + y_0$. It is evident that $\tilde{a}(\tau) = \tilde{a}(x_0,y_0)$. According to [6] if $I(\tilde{a})$ is a center-affine invariant of systems (1), then the polynomial

$$\bar{K}(\tilde{a},x,y) = I(\tilde{a}(x_0,y_0))|_{\{x_0=x,y_0=y\}}$$

is an affine comitant of these systems. So, considering (3) we obtain the following affine comitants of systems (1):

$$Af_i(\tilde{a},x,y) = I_i(\tilde{a}(x_0,y_0))|_{\{x_0=x,y_0=y\}}, \quad (i = 1,2,5,18,21).$$

We shall use the notations

$$\begin{array}{ll} W_1 = Af_1^2 - Af_2, & W_2 = Af_1Af_{18} - Af_{21}, \\ V_1 = Af_1^2 - 2Af_2, & V_2 = Af_1Af_{18} - 4Af_{21}, \\ \mathcal{V}_1 = \widehat{H}^2 + Af_{21}^2, & \mathcal{U} = Af_5^2 + U_1^2 + U_2^2, \\ \mathcal{V}_2 = D^2 + U_3^2. & \end{array} \tag{5}$$

In what follows by $\bar{U}(\tilde{a},x,y) = 0$ (where $\bar{U}(\tilde{a},x,y)$ is an arbitrary comitant) we shall understand $\bar{U}(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ (i.e. this comitant identically vanishes as a polynomial in x and y).

Taking into consideration Remark 1 (see below) according to [7] we have the next result.

Proposition 1. *A system* (1) *is located on the affine orbit of the dimension six if and only if one of the following three sets of conditions holds:*

$$(i)\ \beta \neq 0; \quad (ii)\ \beta = 0,\ K_5\mathcal{U} \neq 0; \quad (iii)\ \beta = 0,\ K_5 = 0, Af_5 \neq 0.$$

Remark 1. In Proposition 1 we use the set of conditions $\beta = 0,\ K_5 = 0, Af_5 \neq 0$ which is equivalent to the set of conditions $\beta = 0,\ K_5 = 0,\ Af_4(Af_4 - Af_3) \neq 0$ from [7, Theorem, page 126]. We note also that $\mathcal{U}$ here denotes the expression $Af_5^2 + T_{16}^2 + Kom^2$ from [7].

Considering Proposition 1 it remains to construct the affine invariant criteria for a system (1) to be located on the orbit of a given dimension $s \le 5$.

Lemma 1. *The rank of the matrix $\mathcal{M}$ is equal to five if and only if $\beta = 0$, $\mathcal{U} = 0$ and one of the following four sets of conditions holds:*

(i) $M\mathcal{V}_1 \neq 0$; *(ii)* $M = 0,\ K_5 W_1 V_2 \neq 0$;
(iii) $M = W_1 = 0,\ K_5 W_2 \neq 0$; *(iv)* $K_5 = 0,\ W_2 \neq 0$.

Proof. By Proposition 1 a system (1) is located on the affine orbit of the dimension less than six (i.e. the rank of the matrix $\mathcal{M}$ is ≤ 5) if and only if $\beta = 0$ and either

$$(\alpha_1)\ K_5 \neq 0 \text{ and } \mathcal{U} = 0, \quad \text{or} \quad (\alpha_2)\ K_5 = Af_5 = 0. \tag{6}$$

1) Assume first $K_5 \neq 0$. As $\beta = 0$ following [2] we could use a center-affine transformation which brought the binary form $K_5(x, y)$ to the canonical form: $K_5(x, y) = x^2(x + \delta y)$ with $\delta \in \{0, 1\}$. Moreover, the same transformation will bring systems (1) in this case to the form (excluding also the linear term x in the second equation via an additional translation):

$$\begin{aligned} \dot{x} &= a + cx + dy + (2m + \delta)x^2 + 2hxy, \\ \dot{y} &= b + fy - x^2 + 2mxy + 2hy^2, \quad \delta \in \{0, 1\}. \end{aligned} \tag{7}$$

For these systems we calculate $M = -8x^2\delta^2$ and we shall consider two subcases: $M \neq 0$ and $M = 0$.

a) If $M \neq 0$ then $\delta = 1$. Since according to (5) the condition $\mathcal{U} = 0$ implies $Af_5 = U_1 = U_2 = 0$ we have: Coefficient$[Af_5, y] = -2h^2 = 0$. So we obtain $h = 0$ and then $Af_5 = -d(5m^2 + 4m + 1) = 0$ and this evidently yields $d = 0$. Therefore we obtain the systems

$$\dot{x} = a + cx + (2m + 1)x^2, \quad \dot{y} = b + fy - x^2 + 2mxy, \tag{8}$$

for which calculations yield:

$$\begin{aligned} U_1 &= 6m(cf - f^2 - 2am - 2bm)x^4, \\ U_2 &= (1 + 3m)^3(cf - f^2 - 2am - 2bm)x^6. \end{aligned} \tag{9}$$

We note that the GL-invariant U_2 keeps the value, indicated in (9), after any translation $(x, y) \mapsto (\tilde{x}+x_0, \tilde{y}+y_0)$ with arbitrary $(x_0, y_0) \in \mathbb{R}^2$ applied to systems (8). In other words for any system located in the orbit under the translation group action of a system (8), i.e. for systems of the form

$$\begin{aligned} \dot{x} &= a + x_0(c + x_0 + 2mx_0) + (c + 2x_0 + 4mx_0)x + (2m + 1)x^2, \\ \dot{y} &= b - x_0(x_0 - 2my_0) + fy_0 - 2x(x_0 - my_0) + (f + 2mx_0)y - x^2 + 2mxy, \end{aligned}$$

we have $U_2 = (1 + 3m)^3(cf - f^2 - 2am - 2bm)x^6$. This means that the polynomial U_2 is a CT-comitant [5] for the family of systems (8) and hence the condition $U_2 = 0$ is affine invariant.

Clearly the conditions $U_1 = U_2 = 0$ (i.e. $\mathcal{U} = 0$) imply $(cf - f^2 - 2am - 2bm) = 0$ and then we can convince ourself that all the minors of order 6 of the matrix M_1 vanish. We claim that the existence of at least one nonzero minor of order 5 is equivalent to the condition $\mathcal{V}_1 \neq 0$, i.e. considering (5) to the condition $\widehat{H}^2 + Af_{21}^2 \neq 0$.

Indeed, for systems (8) we calculate $\widehat{H} = m^2x^2$. On the other hand we obtain $\Delta^{1,2,4,5,6}_{5,6,10,11,12} = -8m^4$, i.e. if $\widehat{H} \neq 0$ then $rank(M_1) = 5$.

Assume $\widehat{H} = 0$, i.e. $m = 0$. Then for systems (8) we have $Af_{21} = [a + b + cx + fy](a+cx+x^2)^2$. At the same time we calculate $\Delta^{1,2,4,5,6}_{2,5,7,8,10} = 2f$, $\Delta^{1,2,3,4,5}_{3,5,7,8,10} = 2(f-c)$ and $\Delta^{1,2,3,4,5}_{2,3,7,10,11} = 2(a + b)$. As $Af_{21} \neq 0$ is equivalent to $(a + b)^2 + c^2 + f^2 \neq 0$ we conclude that in this case there exist non-zero minors of order 5. It remains to observe that in the case $Af_{21} = 0$ (i.e. $f = c = a + b = 0$) all the minors of order 5 vanish. Thus, our claim is proved.

b) Assume now $M = 0$, i.e. for systems (7) we have $\delta = 0$. In order to examine the condition $\mathcal{U} = 0$ (i.e. $Af_5^2 + U_1^2 + U_2^2 = 0$, see (5)) for these systems we calculate: Coefficient$[Af_5, x] = -6h^2 = 0$ and this yields $h = 0$. Therefore we have $Af_5 = -5dm^2 = 0$, i.e. $dm = 0$ and then we obtain Coefficient$[U_1, x^3y] = -6d^2$. So the condition $U_1 = 0$ yields $d = 0$ and this leads to the systems

$$\dot{x} = a + cx + 2mx^2, \quad \dot{y} = b + fy - x^2 + 2mxy, \tag{10}$$

for which calculations yield:

$$U_1 = 6m(cf - f^2 - 2am)x^4, \quad U_2 = 27m^3(cf - f^2 - 2am)x^6. \tag{11}$$

We note that the GL-invariant U_2 keeps the value, indicated above, after any translation $(x, y) \mapsto (\tilde{x} + x_0, \tilde{y} + y_0)$ with arbitrary $(x_0, y_0) \in \mathbb{R}^2$ applied to systems (10). This means that the polynomial U_2 is a CT-comitant [5] for the family of systems (10) and hence the condition $U_2 = 0$ is invariant under the affine transformation.

Evidently the conditions $U_1 = U_2 = 0$ (i.e. $\mathcal{U} = 0$) imply $m(cf - f^2 - 2am) = 0$ and then all the minors of order 6 for the matrix M_1 vanish. We shall consider two subcases: $m \neq 0$ and $m = 0$.

b_1) If $m \neq 0$ then without loss of generality for systems (10) we may assume $f = 0$ due to the translation $(x, y) \mapsto \left(x - \frac{f}{2m}, y - \frac{f}{2m^2}\right)$. Therefore considering (11) in this case the conditions $U_1 = U_2 = 0$ yield $a = 0$. So we get the systems

$$\dot{x} = cx + 2mx^2, \quad \dot{y} = b - x^2 + 2mxy, \tag{12}$$

for which $W_1 = 4mx(c + 4mx)$. We note that all the minors of order 6 for the matrix M_1 corresponding to these systems vanish. On the other hand we have $\Delta^{1,2,4,5,6}_{5,6,8,10,11} = -4m^4 \neq 0$, i.e. $rank(M_1) = 5$. It remains to observe that for systems (12) Coefficient$[V_2, x^6] = -8m^3$. Therefore the condition $m \neq 0$ implies $V_2 \neq 0$ and hence the conditions *(ii)* of Lemma 1 are valid.

b_2) Assuming $m = 0$ and considering (10) we get the family of systems

$$\dot{x} = a + cx, \quad \dot{y} = b + fy - x^2, \tag{13}$$

for which we calculate

$$W_1 = 2cf, \quad \text{Coefficient}[V_2, xy] = cf(c^2 - f^2). \tag{14}$$

We shall examine two cases: $W_1 \neq 0$ and $W_1 = 0$.

γ_1) Admit first $W_1 \neq 0$, i.e. $cf \neq 0$. If $c^2 - f^2 \neq 0$ (this implies $V_2 \neq 0$) by (14) we obtain $cf(c-f) \neq 0$. Therefore $rank(M_1) = 5$ as we have $\Delta^{2,3,4,5,6}_{1,2,5,10,11} = cf(c-f)$.

Assume $c^2 - f^2 = 0$. If $f = c$ (respectively $f = -c$) for systems (13) we calculate $V_2 = -4a(a+cx)^2$ (respectively $V_2 = -4(a+cx)^3$). On the other hand we have $\Delta^{1,2,3,4,5}_{1,2,5,10,11} = 2a^2$ (respectively $\Delta^{2,3,4,5,6}_{1,2,4,5,10} = 4c^4$) and evidently if $V_2 \neq 0$ then $rank(M_1) = 5$ (we note that in the second case the condition $W_1 \neq 0$ yields $c \neq 0$ and this implies $V_2 \neq 0$). Moreover straightforward calculations show us that the condition $V_2 = 0$ (and this happens only in the first case) implies $a = 0$ and all the minors of order 5 vanish. So, the conditions *(ii)* of Lemma 1 are true.

γ_2) Suppose now $W_1 = 0$, i.e. $cf = 0$. In this case for systems (13) we obtain:

$$\begin{array}{lll} (\beta_1) & W_2 = -a(a^2 + bf^2 - 2afx - f^2x^2 + f^3y) & \text{if} \quad c = 0; \\ (\beta_2) & W_2 = (bc^2 - a^2)(a + cx) & \text{if} \quad f = 0. \end{array} \tag{15}$$

On the other hand in the case (β_1) (respectively (β_2)) we have that $\Delta^{1,2,3,4,5}_{1,2,5,10,11}$ equals $2a^2$ (respectively $2(a^2 - bc^2)$. So if $W_2 \neq 0$ then $rank(M_1) = 5$, and it can be easily verified that in the case (β_1) as well as in the case (β_2) the condition $W_2 = 0$ implies the vanishing of all the minors of order 5. This completes the proof of the conditions *(iii)* of Lemma 1.

Remark 2. It follows from the reasons above that in the case $m \neq 0$ for systems (10) we have $V_2 \neq 0$. Hence we decide that in the case $\mathcal{U} = V_2 = 0$ and $W_1 \neq 0$ for systems (10) the relations $m = 0$, $f = c \neq 0$ and $a = 0$ hold.

2) Assume finally $K_5 = 0$ (see condition (α_2) from (6)). As systems (1) are quadratic (i.e. there exists at least one quadratic term) then via an affine transformation systems (1) can be brought to the systems (see for example, [10])

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + xy. \tag{16}$$

Straightforward calculations show us that for these systems $U_1 = U_2 = 0$. Moreover, the GL-invariant U_2 vanishes after any translation $(x, y) \mapsto (\tilde{x} + x_0, \tilde{y} + y_0)$ with arbitrary $(x_0, y_0) \in \mathbb{R}^2$ applied to systems (16). So according to (5) the condition $\mathcal{U} = 0$ is equivalent to $Af_5 = 0$. Since for systems (16) we have $Af_5 = -5d/4$ then we obtain $d = 0$ and for these systems we calculate:

$$W_2 = (c + 3x)(a + cx + x^2)(bc + bx - ay). \tag{17}$$

On the other hand for the matrix M_1 corresponding to systems (16) with $d = 0$ we have $\Delta^{1,2,4,5,6}_{2,3,5,7,12} = -2b$ and $\Delta^{1,2,3,5,6}_{2,5,6,7,12} = a$. It is clear that if $W_2 \neq 0$ then $rank(M_1) = 5$. Moreover straightforward calculations show that for $a = b = 0$ (i.e. when $W_2 = 0$) all the minors of order 5 vanish and hence the conditions *(iv)* of the lemma are proved. This completes the proof of Lemma 1. □

Lemma 2. *The rank of the matrix $\mathcal{M}$ is equal to four if and only if $\beta = 0$, $\mathcal{U} = 0$ and one of the following four sets of conditions holds:*

(i) $M \neq 0,\ \mathcal{V}_1 = 0$; *(ii)* $M = V_2 = 0,\ K_5 W_1 \neq 0$;
(iii) $M = W_1 = W_2 = 0,\ K_5 \mathcal{V}_2 \neq 0$; *(iv)* $K_5 = W_2 = 0$.

Proof. According to the hypothesis of the lemma we assume $\beta = 0$, $\mathcal{U} = 0$ and we shall consider step by step the sets of conditions *(i)*- *(iv)*.

Conditions (i). As it was proved earlier (see the proof of Lemma 1, page 33) when $\beta = 0$, $M \neq 0$ and $\mathcal{U} = 0$ systems (1) will be brought via an affine transformation to systems (8) for which the conditions (9) hold. Moreover, it was proved that the condition $\mathcal{V}_1 = 0$ (i.e. $\widehat{H} = Af_{21} = 0$) yields for systems (8) $m = f = c = a + b = 0$ (see page 34). So we get the family of systems:

$$\dot{x} = a + x^2, \quad \dot{y} = -a - x^2,$$

for which without loss of generality we may assume $a \in \{0, -1, 1\}$ due to the transformation $(x, y, t) \mapsto \big(|a|^{-1/2}x, |a|^{-1/2}y, |a|^{1/2}t\big)$ if $a \neq 0$.

It remains to observe that for the matrix M_1 corresponding to these systems all the minors of order 6 and 5 vanish and $\Delta^{1,2,3,5}_{5,7,8,10} = -2$. Thus the systems of this family could be located only on the orbit of dimension 4.

Conditions (ii). In this case the condition $V_2 = 0$ holds. Therefore according to Remark 2 when $M = 0$, $\mathcal{U} = 0$ and $K_5 W_1 \neq 0$ systems (1) could be brought via an affine transformation to systems (13), for which $f = c \neq 0$ and $a = 0$. In other words when the conditions *(ii)* of Lemma 2 are satisfied, then we get the family of systems

$$\dot{x} = cx, \quad \dot{y} = b + cy - x^2. \tag{18}$$

As $c \neq 0$ (since $W_1 \neq 0$) we may assume $b = 0$ and $c = 1$ due to the transformation $(x, y, t) \mapsto \left(x, \dfrac{1}{c}(y - b), \dfrac{t}{c}\right)$. It remains to note that all the minors of order 6 and 5 for the matrix M_1 corresponding to these systems vanish. On the other hand $\Delta^{2,4,5,6}_{1,2,7,10} = 1$, i.e. system (18) (with $b = 0$ and $c = 1$) is located on the orbit of dimension 4.

Conditions (iii). In this case the condition $W_2 = 0$. As $M = W_1 = 0$ and $K_5 \neq 0$ it was proved earlier (see the proof of Lemma 1, page 34) that in this case systems (1) will be brought via an affine transformation to systems (13) with $cf = 0$ (i.e. $W_1 = 0$, see (14)). We shall examine two subcases: $c = 0$ and $c \neq 0$.

a) Assume first $c = 0$. Then considering (15) the condition $W_2 = 0$ yields $a = 0$ and we get the systems

$$\dot{x} = 0, \quad \dot{y} = b + fy - x^2, \tag{19}$$

for which $D = -f^2x^3$, $U_3 = x^2(4bx^2 + f^2y^2)$. Moreover, for a system located in the orbit under the translation group action of a system (19), i.e. for systems of the form

$$\dot{x} = 0, \quad \dot{y} = b - x_0^2 + fy_0 - 2xx_0 + fy - x^2, \tag{20}$$

for the GL-comitant U_3 we have $U_3 = x^2(4bx^2 + f^2y^2) + 4fx^3(xy_0 - yx_0)$. We recall that by (5) the condition $\mathcal{V}_2 \neq 0$ is equivalent to $D^2 + U_3^2 \neq 0$.

We note that all the minors of order 5 for the matrix M_1 corresponding to these systems vanish. On the other hand $\Delta^{1,2,3,6}_{2,5,7,10} = -2f^2$. Hence, if $D \neq 0$ (i.e. $f \neq 0$) we obtain $rank(M_1) = 4$.

Assume $D = 0$. Then $f = 0$ and then for any point (x_0, y_0) for system (20) we have $U_3 = 4bx^4$, i.e. for these systems the GL-comitant U_3 is a CT-comitant.

On the other hand we calculate $\Delta^{1,2,4,5}_{2,5,7,10} = 4b$. It is clear that if $U_3 \neq 0$ (this implies $\mathcal{V}_2 \neq 0$) then $rank(M_1) = 4$. Moreover straightforward calculations show that for $f = b = 0$ (i.e. when $\mathcal{V}_2 = 0$) all the minors of order 4 vanish and hence the conditions *(iii)* of Lemma 2 are proved in this case.

It remains to note that without loss of generality we may assume either $f = 1$ and $b = 0$ if $f \neq 0$ (due to the change $(x, y, t) \mapsto (x, (y - b)/f, t/f)$) or $f = 0$ and $b \in \{0, \pm 1\}$ (due to the change $(x, y, t) \mapsto (|b|^{1/2}x, |b|y, t)$).

b) Supposing $c \neq 0$ the condition $W_1 = 0$ yields $f = 0$. Moreover we may assume $c = 1$ due to the change $(x, y, t) \mapsto (x, y/c, t/c)$. Then the condition $W_2 = 0$ (see case (β_2) from (15)) gives $b - a^2 = 0$, i.e. $b = a^2$. Therefore we get the family of systems

$$\dot{x} = a + x, \quad \dot{y} = a^2 - x^2, \tag{21}$$

for the respective matrix M_1 of which we have $\Delta^{2,3,4,5}_{1,5,7,10} = -1$, i.e. $rank(M_1) = 4$. It remains to note that we may assume $a = 0$ due to the affine transformation $\bar{x} = x + a$, $\bar{y} = -2ax + y$. We also observe that for the obtained system as well as for any system located on its orbit under the translation group action we have $U_3 = x^2y^2 \neq 0$.

Conditions (iv). As it was shown in the proof of Lemma 1 (see page 35) when $K_5 = \mathcal{U} = 0$ systems (1) will be brought via an affine transformation to systems (16) with $d = 0$. Moreover, if $W_2 = 0$ according to (17) we obtain $a = b = 0$. So we arrive at the systems

$$\dot{x} = cx + x^2, \quad \dot{y} = xy,$$

for which all the minors of order 5 of the corresponding matrix M_1 vanish. But for this matrix we have $\Delta^{1,2,5,6}_{3,5,7,8} = 1$. Thus the systems of this family could be located only on the orbit of dimension 4. It remains to note that we may assume $c \in \{0, 1\}$ due to the change $(x, y, t) \mapsto (cx, y, t/c)$ if $c \neq 0$. ☐

Lemma 3. *The rank of the matrix $\mathcal{M}$ is equal to three if and only if the following conditions hold:* $M = W_2 = \mathcal{V}_2 = 0$, $K_5 \neq 0$.

Proof. Necessity. Assume that a system (1) is located on the orbit of dimension 3. As it follows from the proof of Lemma 2 this system could be located on the orbit of dimension less than or equal to 3 if and only if $\mathcal{U} = 0$ and the conditions *(iii)* of Lemma 2 with $\mathcal{V}_2 = 0$ instead of $\mathcal{V}_2 \neq 0$ are fulfilled. Moreover in this case via an affine transformation we arrive at a system of the form (19) with $b = f = 0$. So we get the system

$$\dot{x} = 0, \quad \dot{y} = -x^2, \tag{22}$$

for which the conditions provided by Lemma 3 are verified.

Sufficiency. Assume that the hypothesis of Lemma 3 is fulfilled. As $M = 0$ and $K_5 \neq 0$ then there exists an affine transformation which will brought systems (1) to the form (7) with $\delta = 0$, i.e. to the systems:

$$\begin{aligned} \dot{x} &= a + cx + dy + 2mx^2 + 2hxy, \\ \dot{y} &= b + fy - x^2 + 2mxy + 2hy^2, \end{aligned} \tag{23}$$

for which $\beta = 0$. We claim that for these systems the condition $W_2 = 0$ implies $W_1 = \mathcal{U} = 0$. Indeed, for systems (23) calculations yield:

$$\text{Coefficient}[W_2, x^3y^3] = 16h^3, \quad \text{Coefficient}[W_2\big|_{\{h=0\}}, x^6] = 16m^3,$$
$$\text{Coefficient}[W_2\big|_{\{h=m=0\}}, y^3] = -d^3, \ \text{Coefficient}[W_2\big|_{\{h=m=d=0\}}, x^3] = cf(2c+f).$$

We remark that if $h = m = d = 0$ then for systems above we obtain

$$Af_5 = U_1 = U_2 = 0, \quad W_1 = 2cf$$

and considering (5) this leads to the identity $\mathcal{U} = 0$. We also observe, that $W_2 = 0$ yields $cf(2c + f) = 0$. If $cf = 0$ then evidently $W_1 = 0$. In the case $f = -2c$ (considering also the conditions $h = m = d = 0$) we calculate $\text{Coefficient}[W_2, xy] = 6c^4$. Thus we get $c = 0$ and again we obtain $W_1 = 0$. Our claim is proved.

So the hypothesis of Lemma 2 corresponding to the conditions *(iii)* is verified except the condition $\mathcal{V}_2 \neq 0$. According to Lemma 2 in this case we have $rank(M_1) \leq 4$ and we obtain the equality if and only if $\mathcal{V}_2 \neq 0$.

Suppose now $\mathcal{V}_2 = 0$. As it was proved above the condition $W_2 = 0$ implies $h = m = d = cf = 0$ and we get two possibilities: $c = 0$ and $c \neq 0$. As it was shown in the proof of Lemma 2 (see page 36) in the case $c \neq 0$ the CT-comitant U_3 (and hence, $\mathcal{V}_2$) could not vanish. So the condition $c = 0$ has to be fulfilled and then we arrive at the systems (19) for which the condition $\mathcal{V}_2 = 0$ yields $b = f = 0$. Thus we get the system (22) for the corresponding matrix $\mathcal{M}$ of which we have found $rank(\mathcal{M}) = 3$ (see above). This completes the proof of Lemma 3. □

In order to formulate and prove the Main Theorem we need some more invariant polynomials constructed in [11] as follows (we keep the respective notations).

We consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [13], where

$$\begin{aligned} \mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}. \end{aligned}$$

Then setting $\mu_0(a) = \text{Res}_x(p_2, q_2)/y^4$ we construct the following polynomials:

$$
\begin{aligned}
\mu_i(a,x,y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4; \quad \kappa(a)=(M,K)^{(2)}/4; \quad \kappa_1(a)=(M,C_1)^{(2)};\\
L(a,x,y) &= 4K(a,x,y)+8H(a,x,y)-M(a,x,y);\\
R(a,x,y) &= L(a,x,y)+8K(a,x,y);\\
K_1(a,x,y) &= p_1(x,y)q_2(x,y)-p_2(x,y)q_1(x,y);\\
K_2(a,x,y) &= 4\,Jacob(J_2,\xi)+3\,Jacob(C_1,\xi)D_1-\xi(16J_1+3J_3+3D_1^2);\\
K_3(a,x,y) &= 2C_2^2(2J_1-3J_3)+C_2(3C_0K-2C_1J_4)+2K_1(3K_1-C_1D_2),
\end{aligned}
$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and

$$
J_1 = Jacob(C_0,D_2),\ J_2 = Jacob(C_0,C_2),\ J_3 = Discrim(C_1),\\
J_4 = Jacob(C_1,D_2),\ \xi = M-2K.
$$

To distinguish topologically different phase portraits we also need the following invariant polynomials (constructed also in [11]):

$$
\begin{aligned}
B_3(\tilde{a},x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,,\\
B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)},\\
B_1(\tilde{a}) &= \text{Res}_x\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,,\\
H(\tilde{a},x,y) &= (T_8-8T_9-2D_2^2)/18\ (=-4\widehat{H}(\tilde{a},x,y));\\
N(\tilde{a},x,y) &= K(\tilde{a},x,y)+H(\tilde{a},x,y);\\
\theta(\tilde{a}) &= \text{Discrim}\big(N(\tilde{a},x,y)\big);\\
H_1(\tilde{a}) &= -\big((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)};\\
H_2(\tilde{a},x,y) &= (C_1,\ 2H-N)^{(1)} - 2D_1N;\\
H_3(\tilde{a},x,y) &= (C_2,D)^{(2)};\\
H_4(\tilde{a}) &= \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};\\
H_5(\tilde{a}) &= \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)} + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};\\
H_6(\tilde{a},x,y) &= 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)};\\
H_7(\tilde{a}) &= (N,C_1)^{(2)};\\
H_8(\tilde{a}) &= 9\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2,D)^{(3)}\Big]^2;\\
H_9(\tilde{a}) &= -\Big(\big((D,D)^{(2)},D\big)^{(1)}D\Big)^{(3)};\\
H_{10}(\tilde{a}) &= \big((N,D)^{(2)},\ D_2\big)^{(1)};\\
H_{11}(\tilde{a},x,y) &= 8H\big[(C_2,D)^{(2)}+8(D,D_2)^{(1)}\big] + 3H_2^2;
\end{aligned}
$$

$$N_1(\tilde{a},x,y) = C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},$$
$$N_2(\tilde{a},x,y) = D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)},$$
$$N_3(\tilde{a},x,y) = (C_2,C_1)^{(1)},$$
$$N_4(\tilde{a},x,y) = 4\,(C_2,C_0)^{(1)} - 3C_1D_1,$$
$$N_5(\tilde{a},x,y) = \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},$$
$$N_6(\tilde{a},x,y) = 8D + C_2\left[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\right].$$

Some important geometric propriety of the constructed above polynomials $\mu_i(\tilde{a},x,y)$ ($i = 0,1,\ldots,4$ is revealed by the next lemma proved in [13].

Lemma 4 ([13]). *A system* (1) *is degenerate (i.e.* $\gcd(P,Q) \neq 1$*) if and only if* $\mu_i = 0$ *for all* $i = 0,1,..,4$.

Main Theorem *(i) A system* (1) *is located on an affine orbit of the given above dimension if and only if one of the respective sets of the conditions holds:*

$$6 \;\Leftrightarrow\; \beta \neq 0 \quad or \quad \beta = 0 \quad and \quad \mathcal{U} \neq 0;$$

$$5 \;\Leftrightarrow\; \beta = 0,\ \mathcal{U} = 0 \text{ and either } \begin{cases} M\mathcal{V}_1 \neq 0, \ \text{or} \\ M = 0,\ K_5W_1V_2 \neq 0, \ \text{or} \\ M = W_1 = 0,\ K_5W_2 \neq 0, \ \text{or} \\ K_5 = 0,\ W_2 \neq 0; \end{cases}$$

$$4 \;\Leftrightarrow\; \beta = 0,\ \mathcal{U} = 0 \text{ and either } \begin{cases} M \neq 0,\ \mathcal{V}_1 = 0, \ \text{or} \\ M = V_2 = 0,\ K_5W_1 \neq 0, \ \text{or} \\ M = W_1 = W_2 = 0,\ K_5\mathcal{V}_2 \neq 0, \ \text{or} \\ K_5 = W_2 = 0; \end{cases}$$

$$3 \;\Leftrightarrow\; M = W_2 = \mathcal{V}_2 = 0,\ K_5 \neq 0.$$

(ii) Assume that a quadratic system is located on the affine orbit of the dimension less than or equal to 5. Then the phase portrait of this system is topologically equivalent to one of the 49 topologically distinct phase portraits given in Fig. 1. Moreover in Table 1 we give necessary and sufficient conditions, invariant with respect to the action of the affine group and time rescaling, for the realization of each one of the phase portraits corresponding to a system located on an orbit of the given dimension (≤ 5*). The first column of Table 1 contains dimension of the orbit. In the second column we list the necessary and sufficient affine invariant conditions for a system to be located on the orbit of the respective dimension. In the third column the additional conditions needed for the realization of the corresponding phase portrait in the last column are listed.*

Proof. The proof of the statement *(i)* of Main Theorem follows immediately from Proposition 1 and Lemmas 1 – 3. So we shall concentrate our attention on the proof of the statement *(ii)*.

Table 1

$\mathfrak{D}$	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*				*Phase portrait*
5	$\beta = 0,\ \mathcal{U} = 0,\ M\mathcal{V}_1 \neq 0$	$H \neq 0$	$H_6 \neq 0$	$K \neq 0$	$H_{11} > 0, \mu_2 > 0, L > 0$	$\mathcal{P}_1$
					$H_{11} > 0, \mu_2 > 0, L < 0$	$\mathcal{P}_2$
					$H_{11} > 0, \mu_2 < 0, K < 0$	$\mathcal{P}_3$
					$H_{11} > 0,\ \mu_2 < 0,\ K > 0,\ L > 0$	$\mathcal{P}_4$
					$H_{11} > 0,\ \mu_2 < 0,\ K > 0,\ L < 0$	$\mathcal{P}_5$
					$H_{11} > 0, \mu_2 = 0, K < 0$	$\mathcal{P}_6$
					$H_{11} > 0,\ \mu_2 = 0,\ K > 0,\ L < 0$	$\mathcal{P}_7$
					$H_{11} > 0,\ \mu_2 = 0,\ K > 0,\ L > 0$	$\mathcal{P}_8$
					$H_{11} < 0,\ L > 0$	$\mathcal{P}_9$
					$H_{11} < 0,\ L < 0$	$\mathcal{P}_{10}$
					$H_{11} = 0,\ L > 0$	$\mathcal{P}_{11}$
					$H_{11} = 0,\ L < 0$	$\mathcal{P}_{12}$
				$K = 0$	$H_{11} \neq 0,\ K_1\mu_3 < 0$	$\mathcal{P}_{13}$
					$H_{11} \neq 0,\ K_1\mu_3 > 0$	$\mathcal{P}_{14}$
					$H_{11} \neq 0,\ \mu_3 = 0$	$\mathcal{P}_{15}$
					$H_{11} = 0,\ \varkappa_2 < 0$	$\mathcal{P}_9$
					$H_{11} = 0,\ \varkappa_2 > 0$	$\mathcal{P}_{10}$
			$H_6 = 0$	$N \neq 0$	$H_{11} \neq 0,\ L > 0$	$\mathcal{P}_1$
					$H_{11} \neq 0,\ L < 0$	$\mathcal{P}_5$
					$H_{11} \neq 0,\ L = 0$	$\mathcal{P}_{14}$
					$H_{11} = 0,\ L < 0$	$\mathcal{P}_{16}$
					$H_{11} = 0, L > 0, K < 0$	$\mathcal{P}_{17}$
					$H_{11} = 0, L > 0, K > 0$	$\mathcal{P}_{18}$
					$H_{11} = 0,\ L = 0$	$\mathcal{P}_{19}$
				$N = 0$	$H_3 > 0$	$\mathcal{P}_5$
					$H_3 < 0$	$\mathcal{P}_{10}$
					$H_3 = 0$	$\mathcal{P}_{16}$

Table 1 *(continued)*

$\mathfrak{D}$	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
5	$\beta=0,\ \mathcal{U}=0,$ $M\mathcal{V}_1\neq 0$	$H=0$	$D\neq 0$	$N_5>0$	$\mathcal{P}_1$
				$N_5<0$	$\mathcal{P}_9$
				$N_5=0$	$\mathcal{P}_{11}$
			$D=0$	$N_5<0$	$\mathcal{P}_9$
				$N_5>0,\ \mu_4>0$	$\mathcal{P}_{20}$
				$N_5>0,\ \mu_4<0$	$\mathcal{P}_{21}$
				$N_5>0,\ \mu_4=0,\ N_1\neq 0$	$\mathcal{P}_{22}$
				$N_5>0,\ \mu_4=0,\ N_1=0$	$\mathcal{P}_{23}$
				$N_5=0,\ \mu_4\neq 0$	$\mathcal{P}_{24}$
				$N_5=0,\ \mu_4=0$	$\mathcal{P}_{25}$
	$M=\mathcal{U}=0,$ $K_5W_1V_2\neq 0$	$H\neq 0$	$H_{11}\neq 0$	$K_3>0$	$\mathcal{P}_{26}$
				$K_3<0$	$\mathcal{P}_{27}$
				$K_3=0$	$\mathcal{P}_{28}$
			$H_{11}=0$	$K_3>0$	$\mathcal{P}_{29}$
				$K_3<0$	$\mathcal{P}_{30}$
				$K_3=0$	$\mathcal{P}_{31}$
		$H=0$	$\mu_3K_1>0,\ K_3\geq 0$		$\mathcal{P}_{32}$
			$\mu_3K_1>0,\ K_3<0$		$\mathcal{P}_{33}$
			$\mu_3K_1<0$		$\mathcal{P}_{34}$
	$M=\mathcal{U}=0,$ $W_1=0, K_5W_2\neq 0$	$D\neq 0$			$\mathcal{P}_{35}$
		$D=0$	$\mu_4<0$		$\mathcal{P}_{36}$
			$\mu_4>0$		$\mathcal{P}_{29}$
	$K_5=0,\ W_2\neq 0$	$H_{11}<0$			$\mathcal{P}_{37}$
		$H_{11}>0$	$\mu_2>0$		$\mathcal{P}_{38}$
			$\mu_2<0$		$\mathcal{P}_{39}$
			$\mu_2=0$		$\mathcal{P}_{40}$
		$H_{11}=0$	$\mu_2\neq 0$		$\mathcal{P}_{41}$
			$\mu_2=0$		$\mathcal{P}_{42}$

Table 1 *(continued)*

<table>
<tr><th>$\mathfrak{D}$</th><th>Necessary and sufficient conditions</th><th colspan="2">Additional conditions for phase portraits</th><th>Phase portrait</th></tr>
<tr><td rowspan="11">4</td><td rowspan="3">$\beta=0,\ \mathcal{U}=0,$ $M\neq 0, \mathcal{V}_1=0$</td><td colspan="2">$N_5>0$</td><td>$\mathcal{P}_{23}$</td></tr>
<tr><td colspan="2">$N_5<0$</td><td>$\mathcal{P}_{9}$</td></tr>
<tr><td colspan="2">$N_5=0$</td><td>$\mathcal{P}_{43}$</td></tr>
<tr><td>$M=\mathcal{U}=V_2=0, K_5W_1\neq 0$</td><td colspan="2">–</td><td>$\mathcal{P}_{32}$</td></tr>
<tr><td rowspan="4">$M=\mathcal{U}=W_1=0,$ $W_2=0, K_5\mathcal{V}_2\neq 0$</td><td colspan="2">$D\neq 0$</td><td>$\mathcal{P}_{44}$</td></tr>
<tr><td rowspan="3">$D=0$</td><td>$N_6\neq 0$</td><td>$\mathcal{P}_{45}$</td></tr>
<tr><td>$N_6=0,\ U_3<0$</td><td>$\mathcal{P}_{29}$</td></tr>
<tr><td>$N_6=0,\ U_3>0$</td><td>$\mathcal{P}_{46}$</td></tr>
<tr><td rowspan="2">$K_5=0,\ W_2=0$</td><td colspan="2">$H_{11}\neq 0$</td><td>$\mathcal{P}_{47}$</td></tr>
<tr><td colspan="2">$H_{11}=0$</td><td>$\mathcal{P}_{48}$</td></tr>
<tr><td>3</td><td>$M=W_2=\mathcal{V}_2=0, K_5\neq 0$</td><td colspan="2">–</td><td>$\mathcal{P}_{49}$</td></tr>
</table>

In other words we assume that a quadratic system is located on an affine orbit of dimension ≤ 5 and we shall determine the phase portrait of this system as well as the respective affine invariant conditions for its realization.

According to Lemmas 1 – 3 for a system located on an orbit of the dimension ≤ 5 the conditions $\beta=0$ and $\mathcal{U}=0$ have to be fulfilled and in what follows we assume that these conditions hold.

1) The case $M\neq 0$. In this case via an affine transformation a quadratic system (1) could be brought to the form (8) (see page 33), i.e.

$$\dot{x}=a+cx+(2m+1)x^2,\quad \dot{y}=b+fy-x^2+2mxy, \tag{24}$$

for which the condition $cf-f^2-2am-2bm=0$ holds. For these systems we have $H=-4m^2x^2$.

a) The subcase $H\neq 0$. Then $m\neq 0$ and we may assume $f=0$ due to the translation $(x,y)\mapsto\left(x-\frac{f}{2m}, y-\frac{f}{2m^2}\right)$. Then for these systems the condition above yields $m(a+b)=0$ and as $m\neq 0$ we get $b=-a$. Thus we obtain the systems

$$\dot{x}=a+cx+(2m+1)x^2,\quad \dot{y}=-a-x^2+2mxy, \tag{25}$$

for which we calculate the needed invariant polynomials applied in [11] (keeping the respective notations):

$$\begin{gathered}B_3=\theta=\eta=\mu_0=H_7=0,\quad H_6=-2048m^4(-4a+c^2-8am-4am^2)x^6,\\ H_{11}=768m^4(-4a+c^2-8am)x^4,\quad H=-4m^2x^2,\quad K=4m(1+2m)x^2,\\ L=8(1+2m)x^2,\quad \mu_2=4am^2(1+2m)x^2,\quad N=4m(1+m)x^2.\end{gathered} \tag{26}$$

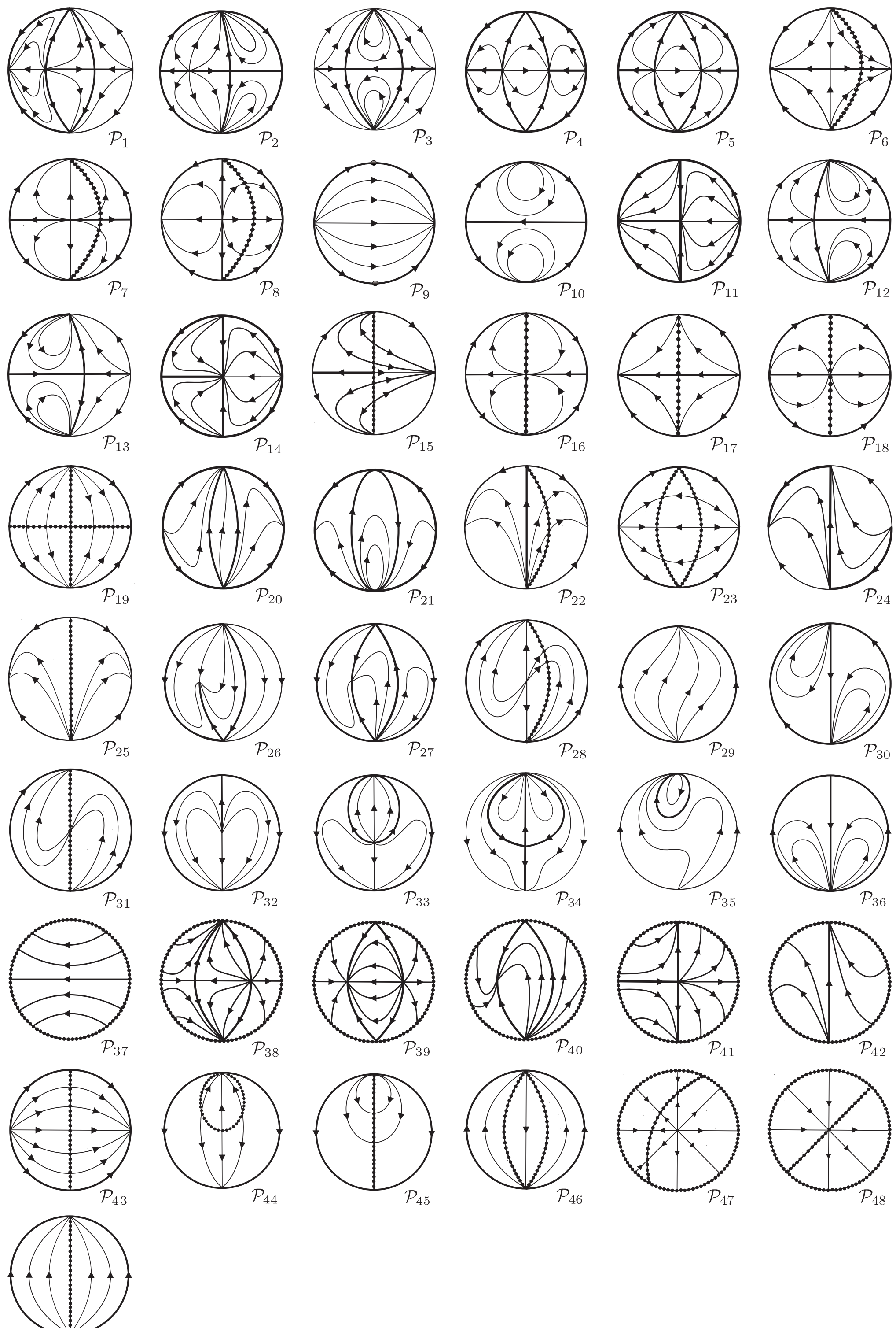

Table 1. Topologically distinct phase portraits

As $H = -4\widehat{H}$ by (5) the condition $H \neq 0$ implies $\mathcal{V}_1 \neq 0$, i.e. for $m \neq 0$ a system (25) is located on the orbit of dimension 5.

On the other hand as $\beta = 0$ (which is equivalent to $\eta = 0$ from [11]), $M \neq 0$ and $B_3 = \theta = \mu_0 = H_7 = 0$, according to [8] and [9] the family of non-degenerate systems (25) possesses invariant lines (considered with multiplicity and including the line at infinity) of total multiplicity four if $H_6 \neq 0$ and at least five if $H_6 = 0$.

a_1) *The possibility* $H_6 \neq 0$. According to [11] for the non-degenerate systems (25) we get the following phase portraits (we keep the respective notations from [11]):

Picture 4.12(a) $[\mathcal{P}_1]$ if $K \neq 0,\ H_{11} > 0,\ \mu_2 > 0,\ L > 0$;
Picture 4.12(b) $[\mathcal{P}_2]$ if $K \neq 0,\ H_{11} > 0,\ \mu_2 > 0,\ L < 0$;
Picture 4.12(c) $[\mathcal{P}_3]$ if $K \neq 0,\ H_{11} > 0,\ \mu_2 < 0,\ K < 0$;
Picture 4.12(d) $[\mathcal{P}_4]$ if $K \neq 0,\ H_{11} > 0,\ \mu_2 < 0,\ K > 0,\ L > 0$;
Picture 4.12(e) $[\mathcal{P}_5]$ if $K \neq 0,\ H_{11} > 0,\ \mu_2 < 0,\ K > 0,\ L < 0$;
Picture 4.15(a) $[\mathcal{P}_9]$ if $K \neq 0,\ H_{11} < 0,\ L > 0$;
Picture 4.15(b) $[\mathcal{P}_{10}]$ if $K \neq 0,\ H_{11} < 0,\ L < 0$;
Picture 4.24(a) $[\mathcal{P}_{11}]$ if $K \neq 0,\ H_{11} = 0,\ L > 0$;
Picture 4.24(b) $[\mathcal{P}_{12}]$ if $K \neq 0,\ H_{11} = 0,\ L < 0$;
Picture 4.19(a) $[\mathcal{P}_{13}]$ if $K = 0,\ N \neq 0,\ H_{11} \neq 0,\ K_1\mu_3 < 0$;
Picture 4.19(b) $[\mathcal{P}_{14}]$ if $K = 0,\ N \neq 0,\ H_{11} \neq 0,\ K_1\mu_3 > 0$;
Picture 4.36(a) $[\mathcal{P}_9]$ if $K = 0,\ N \neq 0,\ H_{11} = 0,\ \varkappa_2 < 0$;
Picture 4.36(b) $[\mathcal{P}_{10}]$ if $K = 0,\ N \neq 0,\ H_{11} = 0,\ \varkappa_2 > 0$.

Here in square brackets the respective phase portraits from Figure 1 which are topologically equivalent to those from [11], respectively are indicated. As by (26) the conditions $H \neq 0$ and $K = 0$ imply $N \neq 0$ we arrive at the respective conditions from Table 1.

It remains to look for degenerate systems which could belong to the family (25) when the conditions $H \neq 0$ (i.e. $m \neq 0$) and $H_6 \neq 0$ (i.e. $c^2 - 4a(1+m)^2 \neq 0$) hold. According to Lemma 4 we calculate the polynomials μ_i for this family:

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = 4am^2(1+2m)x^2, \ \mu_3 = 4acmx^2(x + mx - my),$$
$$\mu_4 = ax^2[(c^2 + 4am^2)x^2 - 2m(c^2 - 4am)xy + 4am^2y^2].$$

Evidently the conditions $\mu_i = 0$ $(i = 0, 1, .., 4)$ (see Lemma 4) are equivalent to $a = 0$.

Assume first $K \neq 0$. Considering (26) we have $m(1 + 2m) \neq 0$ and hence the condition $a = 0$ is equivalent to $\mu_2 = 0$. So we get the family of degenerate systems

$$\dot{x} = x[c + (2m+1)x], \quad \dot{y} = x(-x + 2my), \tag{27}$$

for which $H_6 = -2048c^2m^4x^6 \neq 0$. For the respective family of linear systems

$$\dot{x} = c + (2m+1)x, \quad \dot{y} = -x + 2my$$

we have $\lambda_1 = 2m + 1$, $\lambda_2 = 2m$ and therefore $\operatorname{sign}(\lambda_1\lambda_2) = \operatorname{sign}(K)$. Moreover by (26) we have $\operatorname{sign}(\lambda_1) = \operatorname{sign}(L)$.

We observe that due to $H_6 \neq 0$ (i.e. $c \neq 0$) the invariant line $c + (2m+1)x = 0$ does not coincide with line $x = 0$ (filled with singularities).

Thus after some standard investigations we decide that the phase portrait of a degenerate system (27) corresponds to picture $\mathcal{P}_6$ if $K < 0$; $\mathcal{P}_7$ if $K > 0$ and $L < 0$ and to picture $\mathcal{P}_8$ if $K > 0$ and $L > 0$.

Suppose now $K = 0$. As $H \neq 0$ (i.e. $m \neq 0$) considering (26) we have $m = -1/2$ and then systems (25) become

$$\dot{x} = a + cx, \quad \dot{y} = -a - x^2 - xy, \tag{28}$$

for which we calculate:

$$\mu_2 = 0, \quad \mu_3 = -acx^2(x+y), \quad H_{11} = 48c^2x^4, \quad H_6 = 128(a - c^2)x^6.$$

We observe that the systems above could be degenerate (i.e. $a = 0$) only if $H_{11} \neq 0$ as $H_{11} = 0$ gives $c = 0$ and then $H_6 = 128ax^6 \neq 0$. On the other hand if $H_{11} \neq 0$ then the condition $a = 0$ is equivalent to $\mu_3 = 0$.

So, setting $a = 0$ in systems (28) we may assume $c = 1$ (due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$) and we easily get the phase portrait $\mathcal{P}_{15}$.

a_2) *The possibility* $H_6 = 0$. Then we obtain $c^2 - 4a(1+m)^2 = 0$ and we need to distinguish 2 cases: $N \neq 0$ and $N = 0$.

If $N \neq 0$ then by (26) we have $m+1 \neq 0$ and therefore we get $a = c^2/(4(1+m)^2)$. We observe that due to $H \neq 0$ by (26) the condition $K = 0$ is equivalent to $L = 0$. So, according to [12] the phase portrait of such a system corresponds to *Picture 5.14(a)* [$\mathcal{P}_1$] if $L > 0$; *Picture 5.14(b)* [$\mathcal{P}_5$] if $L < 0$ and to *Picture 5.18* [$\mathcal{P}_{14}$] if $L = 0$.

On the other hand for systems (25) we have

$$\mu_3 = c^3mx^2[(1+m)x - my]/(1+m)^2, \quad H_{11} = 768c^2m^6x^4/(1+m)^2.$$

Therefore according to Lemma 4 the necessary condition $\mu_3 = 0$ for a system to be degenerate yields $c = 0$ and this condition is given by $H_{11} = 0$. In this case evidently systems (25) (with $c = a = 0$) become degenerate systems

$$\dot{x} = (2m+1)x^2, \quad \dot{y} = x(-x + 2my),$$

which are a subfamily of systems (27) (corresponding to $c = 0$). So in the case $2m+1 \neq 0$ (i.e. $L \neq 0$) the singular invariant line $x = 0$ coincides with the invariant line of the respective linear systems and hence we get the phase portrait $\mathcal{P}_{16}$ if $L < 0$; $\mathcal{P}_{17}$ if $L > 0$ and $K < 0$ and $\mathcal{P}_{18}$ if $L > 0$ and $K > 0$. If $L = 0$ we have $m = -1/2$ and we get two singular lines. This evidently leads to picture $\mathcal{P}_{19}$.

Assume $N = 0$. As $H \neq 0$ and $H_6 = 0$ by (26) we get $m + 1 = c = 0$. In this case for systems (25) we have

$$H_6 = H_2 = 0, \quad H_3 = 32ax^2$$

and according to [8] systems (25) possess invariant line of total multiplicity 6.

On the other hand in the case $H_3 \neq 0$ (i.e. $a \neq 0$ and systems are non-degenerate) according to [12] the phase portrait corresponds to *Picture 6.8* $[\mathcal{P}_5]$ if $H_3 > 0$ and to *Picture 6.9* $[\mathcal{P}_{10}]$ if $H_3 < 0$.

Assuming $H_3 = 0$ (i.e. $a = 0$) we get the degenerate system

$$\dot{x} = -x^2, \quad \dot{y} = -x(x + 2y),$$

the phase portrait of which corresponds to $[\mathcal{P}_{16}]$.

b) The subcase $H = 0$. Then $m = 0$ and considering (24) we get the family of systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + fy - x^2, \tag{29}$$

for which the condition $(c - f)f = 0$ must hold. For these systems we calculate:

$$B_3 = \theta = \mu_0 = N = K = H = 0, \quad D = -f^2x^2(x + y),$$
$$N_1 = 8(c - f)x^4, \quad N_2 = 4(4a - c^2 + f^2)x, \quad N_5 = -16(4a - c^2)x^2,$$
$$Af_{21} = (a + cx + x^2)^2(a + b + cx + fy).$$

So according to [9] and [8] these systems possess invariant line of total multiplicity at least 4.

As $H = 0$ the condition $\mathcal{V}_1 = 0$ is equivalent to $Af_{21} = 0$, i.e. a system (29) is located on the orbit of dimension four if and only if $a + b = c = f = 0$. In this case we obtain the degenerate system

$$\dot{x} = a + x^2, \quad \dot{y} = -(a + x^2),$$

where $a \in \{-1, 0, 1\}$ due to the rescaling $(x, y, t) \mapsto (|a|^{-1/2}x, |a|^{-1/2}y, |a|^{1/2}t)$ if $a \neq 0$. For these systems $N_5 = -64ax^2$ and we obviously obtain the phase portrait $\mathcal{P}_{23}$ (respectively $\mathcal{P}_9$; $\mathcal{P}_{43}$) if $N_5 > 0$ (respectively $N_5 < 0$; $N_5 = 0$).

Assuming $\mathcal{V}_1 \neq 0$ we shall consider two possibilities: $D \neq 0$ and $D = 0$.

b_1) The possibility $D \neq 0$. In this case $f \neq 0$ and then for systems (29) we obtain $f = c \neq 0$. Then we may consider $b = 0$ and $c = 1$ due to the transformation $(x, y, t) \mapsto (cx, (c^2y - b)/c, t/c)$. So we arrive at the family of systems

$$\dot{x} = a + x + x^2, \quad \dot{y} = y - x^2,$$

for which we calculate: $N_1 = 0$, $N_2 = 16ax$, $N_5 = 16(1 - 4a)x^2$. So according to [8] these systems possess invariant line of total multiplicity at least 5. Moreover following [12] for non-degenerate systems we get the phase portraits

- *Picture 5.13* $[\mathcal{P}_1]$ if $N_2 \neq 0$, $N_5 > 0$;
- *Picture 5.15* $[\mathcal{P}_9]$ if $N_2 \neq 0$, $N_5 < 0$;
- *Picture 5.17* $[\mathcal{P}_{11}]$ if $N_2 \neq 0$, $N_5 = 0$;
- *Picture 6.7* $[\mathcal{P}_1]$ if $N_2 = 0$.

We observe that the condition $N_2 = 0$ implies $N_5 > 0$ and that the systems above could not be degenerate due to $\mu_2 = x^2 \neq 0$ (see Lemma 4).

b_2) *The possibility* $D = 0$ In this case $f = 0$ and we arrive at the family of systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b - x^2, \tag{30}$$

for which we calculate:

$$\begin{gathered} N_1 = 8cx^4,\ N_2 = 4(4a - c^2)x,\ N_5 = 16(c^2 - 4a)x^2,\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \\ \mu_4 = [(a+b)^2 - bc^2]x^4,\ Af_{21} = (a + b + cx)(a + cx + x^2)^2. \end{gathered} \tag{31}$$

So according to [9] and [8] these systems possess invariant line of total multiplicity at least 4. On the other hand by [11] and [12] for non-degenerate systems we get the following phase portraits:

- *Picture 4.29(a)* $[\mathcal{P}_{20}]$ if $N_1 \neq 0,\ N_2 \neq 0,\ N_5 > 0,\ \mu_4 > 0$;
- *Picture 4.29(b)* $[\mathcal{P}_{21}]$ if $N_1 \neq 0,\ N_2 \neq 0,\ N_5 > 0,\ \mu_4 < 0$;
- *Picture 4.33* $[\mathcal{P}_9]$ if $N_1 \neq 0,\ N_2 \neq 0,\ N_5 < 0$;
- *Picture 5.28* $[\mathcal{P}_{24}]$ if $N_1 \neq 0,\ N_2 = 0$;
- *Picture 5.20* $[\mathcal{P}_{20}]$ if $N_1 = 0,\ N_2 \neq 0,\ N_5 > 0$;
- *Picture 5.24* $[\mathcal{P}_9]$ if $N_1 = 0,\ N_2 \neq 0,\ N_5 < 0$;
- *Picture 6.10* $[\mathcal{P}_{24}]$ if $N_1 = 0,\ N_2 = 0$.

It remains to determine the phase portraits for degenerate systems (30), i.e. by Lemma 4 the condition $(a+b)^2 - bc^2 = 0$ must hold. On the other hand the condition $Af_{21} \neq 0$ gives $(a + b)^2 + c^2 \neq 0$ and this implies $b \geq 0$.

If $b > 0$ then we may assume $b = 1$ due to the rescaling $(x, y, t) \mapsto (b^{-1/2}x, b^{-1/2}y, b^{1/2}t)$. Therefore we get $c = \pm(a + 1)$ and it is sufficient to consider only the case $c = a + 1$ due to the change $(x, y, t, c) \mapsto (-x, -y, -t, -c)$ which keeps systems (30). Thus we obtain the family of degenerate systems

$$\dot{x} = (1 + x)(a + x), \quad \dot{y} = 1 - x^2. \tag{32}$$

Taking into consideration the two invariant lines $x = -1$ (singular) and $x = a$ which could coincide if $a = 1$ as well as the critical value $a = -1$ (when the respective linear system is also degenerate) we arrive at phase portrait $\mathcal{P}_{22}$ if $a^2 - 1 \neq 0$; $\mathcal{P}_{25}$ if $a = 1$ and $\mathcal{P}_{23}$ if $a = -1$.

It remains to note that for systems (32) we have $\mu_4 = 0$, $N_1 = 8(1 + a)x^4$, $N_2 = -4(a - 1)^2 x$ and $N_5 = 16(a - 1)^2 x^2$.

If $b = 0$ then the condition $\mu_4 = 0$ gives $a = 0$ and then $Af_{21} = cx^3(c + x)^2 \neq 0$. Hence we can assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. So we get the degenerate system

$$\dot{x} = x(1 + x), \quad \dot{y} = -x^2,$$

for which $N_1 = 8x^4$, $N_2 = -4x$ and $N_5 = 16x^2$.

Considering (31) and the case $\mu_4 = 0$ examined above we observe that the condition $N_5 < 0$ implies $\mu_4 \neq 0$ and $N_5 = 0$ if and only if $N_2 = 0$. Moreover the condition $N_1 = 0$ implies $\mu_4 \geq 0$.

Considering this observation we could unite the conditions for the realization of topologically distinct phase portraits in the considered case (including the degenerate systems) as follows:

- $\mathcal{P}_9$ if $N_5 < 0$;
- $\mathcal{P}_{20}$ if $N_5 > 0$, $\mu_4 > 0$;
- $\mathcal{P}_{21}$ if $N_5 > 0$, $\mu_4 < 0$;
- $\mathcal{P}_{22}$ if $N_5 > 0$, $\mu_4 = 0$, $N_1 \neq 0$;
- $\mathcal{P}_{23}$ if $N_5 > 0$, $\mu_4 = 0$, $N_1 = 0$;
- $\mathcal{P}_{24}$ if $N_5 = 0$, $\mu_4 \neq 0$;
- $\mathcal{P}_{25}$ if $N_5 = 0$, $\mu_4 = 0$.

Thus we arrive at the respective conditions from Table 1.

2) The case $M = 0$ and $K_5 \neq 0$. As it was shown in the proof of Lemma 1 (see page 34) in this case via an affine transformation a quadratic system (1) could be brought to the form (10), for which the condition $m(cf - f^2 - 2am) = 0$ holds. For these systems we have $H = -4m^2x^2$.

a) The subcase $H \neq 0$. Then $m \neq 0$ and we may assume $f = 0$ due to the translation $(x, y) \mapsto \left(x - \frac{f}{2m}, y - \frac{f}{2m^2}\right)$. Then for these systems the condition above gives $am = 0$ and as $m \neq 0$ we get $a = 0$. Then we arrive at the family of systems

$$\dot{x} = cx + 2mx^2, \quad \dot{y} = b - x^2 + 2mxy, \tag{33}$$

for which calculations yield:

$$\begin{gathered} B_3 = \theta = 0,\ N = 4m^2x^2,\ K_5 = x^3 \neq 0,\ H_{11} = 768c^2m^4x^4, \\ K_3 = -24bm^2x^6,\ D = 4bm^2x^3, \quad N_6 = 8(c^2 + 4bm^2)x^3, \\ \text{Coefficient}[V_2, x^6] = -8m^3, \quad W_1 = 4mx(c + 4mx). \end{gathered} \tag{34}$$

So according to [8] and [9] the family of non-degenerate systems (33) possesses invariant lines of total multiplicity at least four.

The condition $H \neq 0$ implies $V_2W_1 \neq 0$, i.e. for $m \neq 0$ a system (33) is located on the orbit of dimension 5. As for these systems we have

$$\mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = -4bcm^2x^3, \quad \mu_4 = bx^3[(4bm^2 - c^2)x + 2c^2my]$$

according to Lemma 4 a system (33) becomes degenerate if and only if $b = 0$ and this is equivalent to $K_3 = 0$.

As $M = 0$, $B_3 = \theta = 0$ and $N \neq 0$, according to [11] and [12] for a non-degenerate system (33) we obtain the following phase portraits:

Picture 4.31(a) $[\mathcal{P}_{26}]$ $\Leftrightarrow$ $H_{11} \neq 0,\ N_6 \neq 0,\ K_3 > 0$;
Picture 4.31(b) $[\mathcal{P}_{27}]$ $\Leftrightarrow$ $H_{11} \neq 0,\ N_6 \neq 0,\ K_3 < 0$;
Picture 5.23 $[\mathcal{P}_{26}]$ $\Leftrightarrow$ $H_{11} \neq 0,\ N_6 = 0$;
Picture 4.44(a) $[\mathcal{P}_{29}]$ $\Leftrightarrow$ $H_{11} = 0,\ K_3 > 0$;
Picture 4.44(b) $[\mathcal{P}_{30}]$ $\Leftrightarrow$ $H_{11} = 0,\ K_3 < 0$;

Assume now that systems (33) are degenerate, i.e. $K_3 = 0$ (that implies $b = 0$). As $x = 0$ is an invariant line filled with singularities and the condition $c = 0$ is equivalent to $H_{11} = 0$, we obviously obtain the phase portrait $\mathcal{P}_{28}$ if $H_{11} \neq 0$ and $\mathcal{P}_{31}$ if $H_{11} = 0$. Taking into account that the conditions $H_6 = 0$ and $H_{11} \neq 0$ by (34) imply $K_3 > 0$ an that the condition $K_3 < 0$ implies $H_6 \neq 0$ we obviously arrive at the respective conditions from Table 1.

b) The subcase $H = 0$. Then $m = 0$ and systems (10) become

$$\dot{x} = a + cx, \quad \dot{y} = b + fy - x^2. \tag{35}$$

For these systems we calculate

$$\begin{gathered} M = B_3 = N = 0, \quad K_5 = x^3 \neq 0, \quad N_3 = 3(c-f)x^3, \\ N_6 = 8c(c-f)x^3, \quad K_1 = -cx^3, \quad K_3 = 6(2c-f)fx^6, \\ D_1 = c + f, \quad D = -f^2x^3, \quad \mu_3 = -c^2fx^3, \quad W_1 = 2cf, \quad V_2 = (a+cx)\times \\ \big[b(c^2-f^2) - 4a^2 - 2a(3c-f)x - (c-f)(3c+f)x^2 + f(c^2-f^2)y\big] \end{gathered} \tag{36}$$

So according to [9] and [8] non-degenerate systems (35) possess invariant straight lines of total multiplicity at least four.

b_1) Assume first $W_1 \neq 0$, i.e. $cf \neq 0$. Then $\mu_3 \neq 0$ and according to Lemma 4 the family of systems (35) does not contain degenerate systems.

If $V_2 \neq 0$ then by statement (i) of Main Theorem any system (35) is located on an orbit of dimension 5. Moreover from (36) it follows that the condition $W_1N_6 \neq 0$ implies $K_1DN_3 \neq 0$ and $N_6 = 0$ gives $N_3 = 0$ (due to $W_1 \neq 0$). So as $M = 0$ and $B_3 = N = 0$, according to [11] and [12] a non-degenerate system could have one of the following phase portraits:

Picture 4.37(a) $[\mathcal{P}_{32}]$ $\Leftrightarrow$ $N_6 \neq 0,\ \mu_3K_1 > 0,\ K_3 \geq 0$;
Picture 4.37(b) $[\mathcal{P}_{33}]$ $\Leftrightarrow$ $N_6 \neq 0,\ \mu_3K_1 > 0,\ K_3 < 0$;
Picture 4.37(c) $[\mathcal{P}_{34}]$ $\Leftrightarrow$ $N_6 \neq 0,\ \mu_3K_1 < 0$;
Picture 5.27 $[\mathcal{P}_{32}]$ $\Leftrightarrow$ $N_6 = 0$.

We remark that when $N_6 = 0$ (i.e. $f = c$) we have $N_4 = 12ax^2 \neq 0$ due to $V_2 \neq 0$. We observe also that *Picture 5.27* is topologically equivalent to *Picture 4.37(a)* and the condition $N_6 = 0$ implies $\mu_3K_1 > 0$ and $K_3 > 0$. So we get the respective conditions given in Table 1.

Suppose $V_2 = 0$. By (36) due to $cf \neq 0$ we obtain

$$b(c^2-f^2) - 4a^2 = a(3c-f) = (c-f)(3c+f) = (c^2-f^2) = 0.$$

This implies $c = f$ (otherwise $c + f = 3c + f = 0$ gives $c = f = 0$). So if $W_1 \neq 0$ and $V_2 = 0$ we obtain $a = c - f = 0$ and $cf \neq 0$ and then $W_2 = -3c^3x^3 \neq 0$. Then by statement (i) of Main Theorem any system (35) in this case is located on an orbit of dimension 4.

Thus for $W_1 \neq 0$ and $V_2 = 0$ we arrive at systems

$$\dot{x} = cx, \quad \dot{y} = b + cy - x^2,$$

for which we have

$$M = B_3 = N = N_3 = N_4 = 0, \quad \mu_3 = -c^3x^3, \quad W_1 = 2c^2 \neq 0.$$

According to [8] these systems possess invariant straight lines of total multiplicity six and by [12] we get the phase portrait *Picture 6.11* which is topologically equivalent to $\mathcal{P}_{32}$.

b_2) Assume now $W_1 = 0$, i.e. $cf = 0$. If $D \neq 0$ by (36) we have $f \neq 0$ and this implies $c = 0$. Moreover we may assume $f = 1$ and $b = 0$ due to the change $(x, y, t) \mapsto (x, (y - b)/f, t/f)$. So we get the family of systems

$$\dot{x} = a, \quad \dot{y} = y - x^2, \tag{37}$$

for which we calculate:

$$W_2 = -a(a^2 - 2ax - x^2 + y), \quad M = B_3 = N = N_6 = 0, N_3 = -3x^3, \ D = -x^3.$$

If $W_2 \neq 0$ then $a \neq 0$ and we may assume $a = 1$ due to the rescaling $(x, y, t) \mapsto (ax, a^2y)$. By statement ($i$) of Main Theorem this system is located on an orbit of dimension 5. According to [11] it possesses invariant lines of total multiplicity 4 (more exactly the infinite line is of multiplicity 4) and its phase portrait corresponds to *Picture 4.46* which is topologically equivalent to $\mathcal{P}_{35}$.

Assume $W_2 = 0$. Then $a = 0$ and system (37) is degenerate having the parabola $y = x^2$ filled with singularities. Obviously we get the phase portrait $\mathcal{P}_{44}$. On the other hand as $D \neq 0$ we have $\mathcal{V}_2 \neq 0$ and by statement (i) of Main Theorem this system is located on an orbit of dimension 4.

Suppose now $D = 0$, i.e. $f = 0$. Then systems (35) become

$$\dot{x} = a + cx, \quad \dot{y} = b - x^2, \tag{38}$$

for which calculations yield

$$M = 0, \quad K_5 = x^3, \quad W_1 = D = 0, \ U_3 = x^2(4bx^2 - 4axy + c^2y^2),$$
$$W_2 = -(a^2 - bc^2)(a + cx).$$

As $D = 0$ the condition $\mathcal{V}_2 = 0$ is equivalent to $U_3 = 0$. So by statement (i) of Main Theorem a system of this family is located on an orbit of dimension 5 (respectively 4; 3) if $W_2 \neq 0$ (respectively $W_2 = 0$, $U_3 \neq 0$; $W_2 = U_3 = 0$).

On the other hand for systems (38) we have

$$M = B_3 = N = D = 0, \quad N_3 = 3cx^3, \quad N_6 = 8c^2x^3,$$
$$D_1 = c, \quad \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = (a^2 - bc^2)x^4$$

and according to [9] and [8] non-degenerate systems (38) possess invariant straight lines of total multiplicity at least four.

α) Assume first $W_2 \neq 0$. Then $\mu_4 \neq 0$ and by Lemma 4 systems (38) are non-degenerate. According to [11, Table 2] in the case $N_3 \neq 0$ (then $D_1 \neq 0$) we get the phase portrait *Picture 4.38(a)* if $\mu_4 > 0$ and *Picture 4.38(b)* if $\mu_4 < 0$. However there is a missprint in [11, Table 2].

Remark 3. Assume that a quadratic systems has a configuration of invariant lines given by *Config. 4.38*. Then its phase portrait corresponds to *Picture 4.38(a)* [$\mathcal{P}_{36}$] if $\mu_4 < 0$ and *Picture 4.38(b)* [$\mathcal{P}_{29}$] if $\mu_4 > 0$.

If $N_6 = 0$ (i.e. $c = 0$) we have $N_3 = D_1 = 0$ and $N_4 = 12ax^2 \neq 0$ due to $W_2 \neq 0$. According to [12] in this case the phase portrait corresponds to *Picture 5.30* which is topologically equivalent to $\mathcal{P}_{29}$. As the condition $c = 0$ implies $\mu_4 = a^2x^4 > 0$ we could unite the cases $N \neq 0$ and $N = 0$ as it is given in Table 1.

β) Suppose now $W_2 = 0$, i.e. $bc^2 - a^2 = 0$. Then $\mu_4 = 0$ and by Lemma 4 systems (38) become degenerate.

If $N_6 \neq 0$ then $c \neq 0$ (this implies $U_3 \neq 0$) and we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. So we obtain $b = a^2$ and this leads to the degenerate systems

$$\dot{x} = a + x, \quad \dot{y} = (a + x)(a - x),$$

with $a \in \{0, 1\}$ due to the rescaling $(x, y) \mapsto (ax, a^2y)$ in the case $a \neq 0$. Obviously in both cases we obtain the same phase portrait $\mathcal{P}_{45}$.

If $N_6 = 0$ then $c = 0$ and the condition $W_2 = 0$ gives $a = 0$. So we get the systems

$$\dot{x} = 0, \quad \dot{y} = b - x^2,$$

where $b \in \{0, \pm 1\}$ due to the rescaling $(x, y) \mapsto (|b|^{-1/2}x, |b|^{-1}y)$, in the case $b \neq 0$. Evidently we obtain the phase portrait $\mathcal{P}_{29}$ if $b < 0$, $\mathcal{P}_{46}$ if $b > 0$ and $\mathcal{P}_{49}$ if $b = 0$. We observe that for the systems above we have $U_3 = 4bx^4$ and hence $\text{sign}\,(b) = \text{sign}\,(U_3)$ (if $U_3 \neq 0$). We recall also that for $U_3 = 0$ (i.e. $b = 0$) the respective system is located on the orbit of dimension 3.

It remains to note that for systems (10) we have $\text{Coefficient}[W_1, x^2] = 16m^2$ and hence the condition $W_1 = 0$ implies $H = 0$.

3) The case $K_5 = 0$**.** It was shown earlier in the proof of Lemma 1 (see page 35) that in this case a system can be brought via an affine transformation to form (16), for which the condition $\mathcal{U} = 0$ gives $d = 0$. So we get the family

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + xy, \tag{39}$$

for which we calculate:

$$W_2 = (c+3x)(a+cx+x^2)(bc+bx-ay), \quad H_{10}=0, \quad H_{12}=-8a^2x^2,$$
$$H_{11}=48(c^2-4a)x^4, \quad \mu_2 = ax^2.$$

By statement (i) of Main Theorem a system of this family is located on an orbit of dimension 5 if $W_2 \neq 0$ and of dimension 4 if $W_2 = 0$. We observe that the condition $W_2 = 0$ is equivalent to $a = b = 0$ and then systems (39) become degenerate.

Assume $W_2 \neq 0$. According to [10] a non-degenerate system (39) could possess one of the following phase portraits:

Picture $C_2.5(a)$	$[\mathcal{P}_{39}]$	$\Leftrightarrow$	$H_{12}\neq 0,\ H_{11}>0,\ \mu_2<0$;
Picture $C_2.5(b)$	$[\mathcal{P}_{38}]$	$\Leftrightarrow$	$H_{12}\neq 0,\ H_{11}>0,\ \mu_2>0$;
Picture $C_2.6$	$[\mathcal{P}_{37}]$	$\Leftrightarrow$	$H_{12}\neq 0,\ H_{11}<0$;
Picture $C_2.7$	$[\mathcal{P}_{41}]$	$\Leftrightarrow$	$H_{12}\neq 0,\ H_{11}=0$;
Picture $C_2.8$	$[\mathcal{P}_{40}]$	$\Leftrightarrow$	$H_{12}=0,\ H_{11}\neq 0$;
Picture $C_2.9$	$[\mathcal{P}_{42}]$	$\Leftrightarrow$	$H_{12}=0,\ H_{11}=0$;

We observe that the condition $\mu_2 \neq 0$ implies $H_{12} \neq 0$. Moreover if $H_{11} < 0$ then $\mu_2 > 0$. So we arrive at the respective conditions given in Table 1.

Suppose now $W_2 = 0$, i.e. $a = b = 0$. In this case we get the family of degenerate systems

$$\dot{x} = cx + x^2, \quad \dot{y} = xy.$$

Obviously we obtain the phase portrait given by picture $\mathcal{P}_{47}$ if $c \neq 0$ and $\mathcal{P}_{48}$ if $c = 0$. It remains to observe that for $a = b = 0$ we have $H_{11} = 48c^2x^4$ and this polynomial gives the condition $c = 0$. □

References

[1] Popa M.N. *Algebraic methods for differential systems.* Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, **15** (in Romanian).

[2] Sibirsky K.S. *Introduction to algebraic theory of invariants of differential equations*, Chisinau, Stiinta, 1982 (in Russian, published in English in 1988).

[3] Olver P.J. *Classical Invariant Theory*, London Mathematical Society student texts: **44**, Cambridge University Press, 1999.

[4] Boularas D., Calin Iu., Timochouk L., Vulpe N. *T-comitants of quadratic systems: a study via the translation invariants*, Delft University of Technology, Report **96-90**, Delft 1996

[5] Schlomiuk D., Vulpe N. *Geometry of quadratic differential systems in the neighborhood of infinity*, J. Differential Equations, 2005, **215**, 357–400.

[6] Boularas D. *Computation of affine covariants of quadratic bivariate differential systems*, J. Complexity, 2000, **16**, No. 4, 691–715.

[7] Gherstega N., Orlov V. *Classification of Aff*$(2,\mathbb{R})-$*orbit's dimensions for quadratic differential system*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2008, No. 2(57), 122-126.

[8] Schlomiuk D., Vulpe N. *Planar quadratic differential systems with invariant straight lines of at least five total multiplicity*, Qualitative Theory of Dynamical Systems, 2004, **5**, 135–194.

[9] Schlomiuk D., Vulpe N. *Planar quadratic differential systems with invariant straight lines of total multiplicity four*, Nonlinear Anal., 2008, **68**, No. 4, 681–715.

[10] Schlomiuk D., Vulpe N. *The full study of planar quadratic differential systems possessing a line of singularities at infinity*, J. Dynam. Differential Equations, 2008, **20**, 737–775.

[11] Schlomiuk D., Vulpe N. *Integrals and phase portraits of planar quadratic systems with invariant lines of total multiplicity four*, Bul. Acad. Ştiinţe Repub. Mold. Mat., 2008, No. 1(56), 27–83.

[12] Schlomiuk D., Vulpe N. *Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity*, Rocky Mountain J. Math., 2008, **38**, 2015–2076.

[13] Baltag V. A., Vulpe N. I. *Total multiplicity of all finite critical points of the polynomial differential system,* Planar nonlinear dynamical systems (Delft, 1995). Differential Equations & Dynam. Systems, 1997, **5**, No. 3-4, 455–471.

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