BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(60), 2009, Pages 19–28
ISSN 1024–7696

# About group topologies of the primary Abelian group of finite period which coincide on a subgroup and on the factor group *

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**Abstract.** Let $G$ be any Abelian group of the period $p^n$ and $G_1 = \{g \in G | pg = 0\}$, $G_2 = \{g \in G | p^{n-1}g = 0\}$. If $\tau$ and $\tau'$ are a metrizable, linear group topologies such that $G_2$ is a closed subgroup in each of topological groups $(G, \tau)$ and $(G, \tau')$, then $\tau|_{G_2} = \tau'|_{G_2}$ and $(G, \tau)/G_1 = (G, \tau')/G_1$ if and only if there exists a group isomorphism $\varphi : G \to G$ such that the following conditions are true:
1. $\varphi(G_2) = G_2$;
2. $g - \varphi(g) \in G_1$ for any $g \in G$;
3. $\varphi : (G, \tau) \to (G, \tau')$ is a topological isomorphism.

**Mathematics subject classification:** 22A05.
**Keywords and phrases:** Topological group, natural homomorphism, topological isomorphism, subgroup of topological group, factor group of topological group, basis of neighborhoods of zero..

Wyen studying properties of lattices of all group topologies[1] on Abelians groups or their sublattices there is a need to establish the interconnections between group topologies which coincide on some subgroups and on some factor groups.

A partial answer to this question is given in the present article.

The main result of this article is Theorem 9.

**1. Notations.** During all this work, if it is not stipulated opposite, we shall adhere to the following notations;

1.1. $p$ is some fixed prime number;

1.2. $n$ is some fixed natural number;

1.3. $\mathbb{N}$ is the set of all natural numbers;

1.4. $G$ is an Abelian group of the period $p^n$;

1.5. $G'$ is a subgroup of the group $G$;

1.6. $\omega : G \to G/G'$ is the natural homomorphism (i.e. $\omega(g) = g + G'$ for any $g \in G$);

1.7. If $A \subseteq G$ then we denote by $< A >$ the subgroup in $G$, generated by the subset $A$. In particular we denote by $< g >$ the subgroup in $G$ generated by the element $g$;

1.8. If $\{A_\gamma | \gamma \in \Gamma\}$ is some set of groups, then we denote by $\bigoplus\limits_{\gamma\in\Gamma} A_\gamma$ the direct sum of these groups;

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*The author was partially supported by the grant 08.820.08.12 RF.
[1]The considered topologies are not necessarily Hausdorff

1.9. If $\tau$ is a group topology on $G$, then we denote by $\tau|_{G'}$ the induced topology on $G'$, i.e. $\tau|_{G'} = \{U \bigcap G' | U \in \tau\}$;

1.10. If $(G, \tau)$ is a topological group, then we denote by $(G, \tau)/G'$ the topological group $(G/G', \bar{\tau})$, where $\bar{\tau} = \{\omega(U) | U \in \tau\}$.

**2. Proposition.** *If $\tau$ and $\tau'$ are group topologies on $G$ then the following statements are true:*

**2.1**. *If $\tau|_{G'} = \tau'|_{G'}$ then topological groups $(G, \tau)$ and $(G, \tau')$ possess such bases $\{W_\gamma | \gamma \in \Gamma\}$ and $\{W'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero respectively, that $W_\gamma \bigcap G' = W_\gamma \bigcap G'$ for any $\gamma \in \Gamma$. Moreover if topologies $\tau$ and $\tau'$ are linear, then both $W_\gamma$ and $W'_\gamma$ are subgroups of the group $G$;*

**2.2**. *If $(G, \tau)/G' = (G, \tau')/G'$, then topological groups $(G, \tau)$ and $(G, \tau')$ possess such bases $\{W_\gamma | \gamma \in \Gamma\}$ and $\{W'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero, respectively, that $\omega(W_\gamma) = \omega(W'_\gamma)$ for any $\gamma \in \Gamma$. Moreover if topologies $\tau$ and $\tau'$ are linear, then both $W_\gamma$ and $W'_\gamma$ are subgroups of the group $G$;*

**2.3**. *Let $G_1$ and $G_2$ be such subgroups of group $G$ that $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$ and $\tau|_{G_1} = \tau'|_{G_1}$. If $(G, \tau)/G_2 = (G, \tau')/G_2$, then topological groups $(G, \tau)$ and $(G, \tau')$ possess such bases $\{U_\gamma | \gamma \in \Gamma\}$ and $\{U'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero, respectively, that $U_\gamma \bigcap G_1 = U'_\gamma \bigcap G_1$ and $G_2 + U_\gamma = G_2 + U'_\gamma$ for any $\gamma \in \Gamma$. Moreover if topologies $\tau$ and $\tau'$ are linear, then $U_\gamma$ and $U'_\gamma$ are subgroups of the group $G$.*

**Proof**. Let $\{V_\alpha | \alpha \in \Omega\}$ and $\{V'_\beta | \beta \in \Delta\}$ be some bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, respectively, moreover, if topological groups $(G, \tau)$ and $(G, \tau')$ are linear, then $V_\alpha$ and $V'_\beta$ are subgroups of the group $G$.

**Proof of the statement 2.1.** For any $\alpha \in \Omega$ and $\beta \in \Delta$ we shall consider sets $W_{\alpha,\beta} = V_\alpha + (V'_\beta \bigcap G')$ and $W'_{\alpha,\beta} = V'_\beta + (V_\alpha \bigcap G')$ and we shall show that sets $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ and $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ are required bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, respectively.

As $G'$ is a subgroup and $V'_\beta \bigcap G' \subseteq G'$ and $V_\alpha \bigcap G' \subseteq G'$, then

$$W_{\alpha,\beta} \bigcap G' = \big(V_\alpha + (V'_\beta \bigcap G')\big) \bigcap G' = (V_\alpha \bigcap G') + (V'_\beta \bigcap G') =$$

$$\big(V'_\beta + (V_\alpha \bigcap G')\big) \bigcap G' = W'_{\alpha,\beta} \bigcap G'.$$

Let's check up now that the sets $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ and $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ are bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, accordingly.

As $V_\alpha = V_\alpha + 0 \subseteq V_\alpha + (V'_\beta \bigcap G') = W_{\alpha,\beta}$, then the set $W_{\alpha,\beta}$ is a neighborhood of zero in the topological group $(G, \tau)$.

If $U$ is an arbitrary neighborhood of zero in $(G, \tau)$, then $V_{\alpha_0} \subseteq U$ for some $\alpha_0 \in \Omega$. As $(G, \tau)$ is a topological group, then there exists such $\alpha_1 \in \Omega$ that $V_{\alpha_1} + V_{\alpha_1} \subseteq V_{\alpha_0}$, and as $\tau|_{G'} = \tau'|_{G'}$ there exists such $\beta_1 \in \Delta$ that $V'_{\beta_1} \bigcap G' \subseteq V_{\alpha_1} \bigcap G'$. Then

$$W_{\alpha_1,\beta_1} = V_{\alpha_1} + (V'_{\beta_1} \bigcap G') \subseteq V_{\alpha_1} + V_{\alpha_1} \subseteq V_{\alpha_0} \subseteq U.$$

Hence $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ is a basis of the neighborhoods of zero in the topological group $(G, \tau)$.

It is similarly checked that the set $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ is a basis of the neighborhoods of zero in the topological group $(G, \tau')$.

It is easy to see that if $V_\alpha$ and $V'_\beta$ are subgroups of the group $G$, then $W_{\alpha,\beta}$ and $W'_{\alpha,\beta}$ will be subgroups in the group $G$.

The statement 2.1 is completely proved.

**Proof of the statement 2.2.** For any $\alpha \in \Omega$ and $\beta \in \Delta$ we shall consider sets $W_{\alpha,\beta} = V_\alpha \bigcap (\omega)^{-1}(\omega(V'_\beta))$ and $W'_{\alpha,\beta} = V'_\beta \bigcap \omega^{-1}(\omega(V_\alpha))$. Also we shall show that sets $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ and $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ are required bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, accordingly.

Let's check up in the beginning that $\omega(W_{\alpha,\beta}) = \omega(W'_{\alpha,\beta})$.

If $\overline{g} \in \omega(W_{\alpha,\beta})$, then $\overline{g} = \omega(g)$ for some $g \in W_{\alpha,\beta} = V_\alpha \bigcap \omega^{-1}(\omega(V'_\beta))$ and hence there exists such $g' \in V'_\beta$ that $g - g' \in G'$. Then $g' \in V_\alpha + G' = \omega^{-1}(\omega(V_\alpha))$ and hence $g' \in \omega^{-1}(\omega(V_\alpha)) \bigcap V'_\beta = W'_{\alpha,\beta}$, and $\overline{g} = \omega(g) = \omega(g') \in \omega(W'_{\alpha,\beta})$.

From the arbitrarity of the element $\overline{g}$ it follows that $\omega(W_{\alpha,\beta}) \subseteq \omega(W'_{\alpha,\beta})$.

It is similarly proved that $\omega(W'_{\alpha,\beta}) \subseteq \omega(W_{\alpha,\beta})$, and hence $\omega(W_{\alpha,\beta}) = \omega(W'_{\alpha,\beta})$.

Let's check up now that the sets $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ and $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ are bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, accordingly.

Let $\alpha \in \Omega$ and $\beta \in \Delta$. As $\omega : (G, \tau') \to (G, \tau')/G' = (G, \tau)/G'$ is an open homomorphism, then for any $\beta \in \Delta$ the set $\omega(V_\beta)$ is a neighborhood of zero in the topological group $(G, \tau)/G'$, and hence $\omega^{-1}(\omega(V_\beta))$ will be a neighborhood of zero in topological group $(G, \tau)$. Then the set $W_{\alpha,\beta} = V_\alpha \bigcap \omega^{-1}(\omega(V'_\beta))$ will also be a neighborhood of zero in the topological group $(G, \tau)$.

Besides, if $U$ is a neighborhood of zero in the topological group $(G, \tau)$, then $V_\alpha \subseteq U$ for some $\alpha \in \Omega$, and hence $W_{\alpha,\beta} = V_\alpha \bigcap \omega^{-1}(\omega(V'_\beta) \subseteq V_\alpha \subseteq U$.

Hence the set $\{W_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ is a basis of a neighborhoods of zero in topological group $(G, \tau)$.

It is similarly checked that the set $\{W'_{\alpha,\beta} | \alpha \in \Omega,\ \beta \in \Delta\}$ is a basis of the neighborhoods of zero in topological group $(G, \tau')$.

It is easy to see that if $V_\alpha$ and $V'_\beta$ are subgroups of the group $G$, then $W_{\alpha,\beta}$ and $W'_{\alpha,\beta}$ will be subgroups in the group $G$.

The statement 2.2 is completely proved.

**Proof of the statement 2.3.** Let $\psi : G \to G/G_2$ be the natural homomorphism.

If $G_2 \subseteq G_1$, then with accordance to the statement 2.1 topological groups $(G, \tau)$ and $(G, \tau')$ possess such bases $\{W_\gamma | \gamma \in \Gamma\}$ and $\{W'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero, respectively, that $G_1 \bigcap W_\gamma = G_1 \bigcap W'_\gamma$ for any $\gamma \in \Gamma$, moreover if topologies $\tau$ and $\tau'$ are linear, then $W_\gamma$ and $W'_\gamma|$ will be subgroups of the group $G$.

For every $\gamma \in \Gamma$ we shall consider sets $U_\gamma = W_\gamma \bigcap (W'_\gamma + G_2)$ and $U'_\gamma = W'_\gamma \bigcap (W_\gamma + G_2))$. In the proof of the statement 2.2 it is demonstrated that sets $\{U_\gamma | \gamma \in \Gamma\}$ and $\{U'_\gamma | \gamma \in \Gamma\}$ are bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, respectively, and $\psi(U_\gamma) = \psi(U'_\gamma)$ for any $\gamma \in \Gamma$. Moreover if topological groups $(G, \tau)$ and $(G, \tau')$ are linear, then $U_\gamma$ and $U'_\gamma$ will be subgroups of group $G$.

As $G_2 \subseteq G_1$, then $(W'_\gamma + G_2) \bigcap G_1 = (W'_\gamma \bigcap G_1) + G_2 = (W_\gamma \bigcap G_1) + G_2 = (W_\gamma + G_2) \bigcap G_1$. Then $U_\gamma \bigcap G_1 = W_\gamma \bigcap (W'_\gamma + G_2) \bigcap G_1 =$

$$W_\gamma \bigcap G_1 \bigcap (W'_\gamma + G_2) = W'_\gamma \bigcap G_1 \bigcap (W_\gamma + G_2) = U'_\gamma \bigcap G_1.$$

The statement 2.3 in this case is proved.

Let now $G_1 \subseteq G_2$. Then in accordance with the statement 2.2 topological groups $(G, \tau)$ and $(G, \tau')$ possess such bases $\{W_\gamma | \gamma \in \Gamma\}$ and $\{W'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero, respectively, that $\psi(W_\gamma) = \psi(W'_\gamma)$ for any $\gamma \in \Gamma$, moreover if topologies $\tau$ and $\tau'$ are linear, then $W_\gamma$ and $W'_\gamma|$ will be subgroups of the group $G$.

For every $\gamma \in \Gamma$ we shall consider sets $U_\gamma = W_\gamma + (W'_\gamma \bigcap G_1)$ and $U'_\gamma = W_\gamma + (W_\gamma \bigcap G_1)$.

In the proof of the statement 2.2 it is demonstrated that sets $\{U_\gamma | \gamma \in \Gamma\}$ and $\{U'_\gamma | \gamma \in \Gamma\}$ are bases of the neighborhoods of zero in topological groups $(G, \tau)$ and $(G, \tau')$, respectively, and $U_\gamma \bigcap G_1 = U_\gamma \bigcap G_1$.

As $G_1 \subseteq G_2$ and $\psi(G_2) = \{0\}$, then

$$\psi(U_\gamma) = \psi(W_\gamma + (W'_\gamma \bigcap G_1)) = \psi(W_\gamma) = \psi(W'_\gamma) =$$

$$\psi(W'_\gamma + (W_\gamma \bigcap G_1)) = \psi(U'_\gamma),$$

moreover if topological groups $(G, \tau)$ and $(G, \tau')$ are linear then $U_\gamma$ and $U'_\gamma$ will be subgroups of the group $G$ for any $\gamma \in \Gamma$..

So, the proposition is completely proved.

**3. Definition.** *As usual, we shall name a subgroup $A$ of the Abelian groups $G$ a serving subgroup in $G$ if for any natural number $k$ and any element $a \in A$ from the resolvability of the equation $kx = a$ in the group $G$ its resolvability in $A$ follows.*

**4. Remark.** From the definition of the serving subgroup the following statements follow:

4.1. If $g \in G$ is such an element of group $G$ that $p^{n-1} \cdot g \neq 0$ then the subgroup $< g >= \{kg | k \in \mathbb{N}\}$ is a serving subgroup in the group $G$;

4.2. The direct sum of any number of serving subgroups of the group $G$ is a serving subgroup in the group $G$.

**5. Theorem** (Priufer–Kulikov, see [2, p. 154]). *Every serving subgroup $A$ of a group $G$ is a direct summand in the group $G$.*

**6. Proposition.** *Let $C$ be a serving subgroup of the group $G$. If $C$ is the direct sum of cyclic subgroups of the period $p^n$ and $B$ is such subgroup of the group $G$ that $C \bigcap B = \{0\}$, then there exists such subgroup $A$ of the group $G$ that $B \subseteq A$ and $G$ is the direct sum of subgroups $C$ and $A$.*

**Proof**. We shall consider the set $\Delta$ of all such subgroups $D$ of the group $G$ that $B \subseteq D$ and $D \bigcap C = \{0\}$. As the sum of ascendent chain of subgroups from $\Delta$ belongs to $\Delta$, then $\Delta$ contains maximal elements. If $A$ is some of these maximal element, then $B \subseteq A$ and $A \bigcap C = \{0\}$.

For finishing the proof of the proposition it is necessary to check up that $G = C + A$.

We assume the contrary, i.e. that $G \neq C + A$, and let $g \notin C + A$. As the period of the group $G$ is equal to $p^n$, then there exists such natural number $1 \leq s \leq n$ that $p^s \cdot g \in C + A$ and $p^{s-1} \cdot g \notin C + A$. Let $p^s \cdot g = c + a$, where $c \in C$ and $a \in A$. As $A \bigcap C = \{0\}$ and

$$0 = p^n \cdot g = p^{n-s} \cdot (p^s \cdot g) = p^{n-s} \cdot c + p^{n-s} \cdot a,$$

then $p^{n-s} \cdot c = 0$ and as $C$ is the direct sum of cyclic subgroups of the period $p^n$, then $c = p \cdot c_1$ for some element $c_1 \in C$. Then $a_1 = p^{s-1} \cdot g - c_1 \in G$. As $p^{s-1} \cdot g = a_1 + c_1 \notin A + C$, then $a_1 \notin A$, and $p \cdot a_1 = p \cdot (p^{s-1} \cdot g - c_1) = p^s \cdot g - p \cdot c_1 = p^s \cdot g - c = a \in A$. Then $A_1 = \{0, a_1, 2 \cdot a_1, \ldots, (p-1) \cdot a_1\} + A$ is a subgroup of the group $G$, and $B \subsetneq A \subsetneq A_1$.

From the definition of the subgroup $A$ it follows that $A_1 \bigcap C \neq \{0\}$, and hence $0 \neq k \cdot g + a_1 \in C$ for some natural number $k \leq p - 1$ and some element $a_2 \in A$.

As numbers $k$ and $p^n$ are coprime numbers, then there exist such integers $l$ and $m$ that $l \cdot k + m \cdot p^n = 1$. Then $g = (l \cdot k + m \cdot p^n) \cdot g = l \cdot k \cdot g + p^n \cdot g = l \cdot k \cdot g \in l \cdot (a_2 + C) \subseteq A + C$. We arrived at the contradiction with the choice of the element $g$.

So the proposition is completely proved.

**7. Proposition.** *Let $\{g_\gamma | \gamma \in \Gamma\}$ be a set of elements of the group $G$ of order $p^n$ and $G' = \{g \in G | p^{n-1} \cdot g = 0\}$. If the set $\{\omega(g_\gamma) | \gamma \in \Gamma\}$ is linear independent in the linear space $G/G'$, then $A =< \{g_\gamma | \gamma \in \Gamma\}$ is a serving subgroup in the group $G$ and $A = \bigoplus\limits_{\gamma \in \Gamma} < g_\gamma >$.*

**Proof**. From the Remark 4 it follows that for the proof of the proposition it is enough to prove that $A = \bigoplus\limits_{\gamma \in \Gamma} < g_\gamma >$.

We assume the contrary, i.e. that $A \neq \bigoplus\limits_{\gamma \in \Gamma} < g_\gamma >$. As $\sum\limits_{\gamma \in \Gamma} < g_\gamma >= A$, then there exist such subsets $\{g_{\gamma_1}, \ldots, g_{\gamma_k}\} \subseteq \{g_\gamma | \gamma \in \Gamma\}$ and $\{t_1, \ldots, t_k\} \subseteq \mathbb{N}$ that $\sum\limits_{i=1}^{k} t_i \cdot g_{\gamma_i} = 0$ and $t_i \cdot g_{\gamma_i} \neq 0$ for $i = 1, \ldots, k$.

Let $t_i = s_i \cdot p^{j_i}$, where $0 < s_i$ and $s_i$ are not divisible by $p$ for $i = 1, \ldots, k$.

If $j = min\{j_1, \ldots, j_k\}$ and $S = \{i | j_i = j\}$, then $p^{n-1-j} \cdot t_i$ are divisible by $p^n$ for $i \notin S$. Then

$$0 = p^{n-1-j} \cdot 0 = p^{n-1-j} \cdot (\sum_{i=1}^{k} t_i \cdot g_{\gamma_i}) = \sum_{i=1}^{k} (s_i \cdot p^{j_i - j}) \cdot p^{n-1} g_{\gamma_i} =$$

$$\sum_{i=1}^{k} (s_i \cdot p^{j_i - j}) \cdot \omega(g_{\gamma_i}) = \sum_{i \in S} s_i \cdot \omega(g_{\gamma_i}).$$

We arrived at the contradiction with the fact that the set $\{\omega(g_\gamma) | \gamma \in \Gamma\}$ is linear independent in the linear space $G/G'$.

**8. Proposition.** *Let $G' = \{g \in G | p^{n-1} \cdot g = 0\}$ and $\{g_\gamma | \gamma \in \Gamma\}$ and $\{g'_\gamma | \gamma \in \Gamma\}$ be such sets of elements of the group $G$ of the order $p^n$ that $\omega(g_\gamma) = \omega(g'_\gamma)$ for any $\gamma \in \Gamma$ and the set $\{\omega(g_\gamma) | \gamma \in \Gamma\}$ is linear independent in the linear space $G/G'$. If $A = \bigoplus_{\gamma | \gamma \in \Gamma} < g_\gamma >$ and $A' = \bigoplus_{\gamma | \gamma \in \Gamma} < g'_\gamma >$, then for any subgroup $B$ of the group $G'$ are true the following statements:*

**8.1**. *If $A \bigcap B = \{0\}$, then $A' \bigcap B = \{0\}$;*

**8.2**. *If $G = A \bigoplus B$, then $G = A' \bigoplus B$.*

**Proof 8.1.** Assume the contrary, and let $0 \neq b \in A' \bigcap B$, i.e. $b = \sum_{i=1}^{k} r_i \cdot g'_{\gamma_i}$. As $\omega(g_\gamma) = \omega(g'_\gamma)$ for any $\gamma \in \Gamma$, then $h_{\gamma_i} = g_{\gamma_i} - g'_{\gamma_i} \in G'$.

If $r_i = p^{s_i} \cdot q_i$, where $q_i$ are not divisible by $p$ and $s = min\{s_1, \ldots, s_k\}$, then $p^{n-1-s} \cdot r_i \cdot g_{\gamma_i} \neq 0$ for some number $1 \leq i \leq k$. As $A = \bigoplus_{\gamma | \gamma \in \Gamma} < g_\gamma >$, then $\sum_{i=1}^{k} p^{n-1-s} \cdot r_i \cdot g_{\gamma_i} \neq 0$.

Subsequently

$$p^{n-1-s} \cdot b = p^{n-1-s} \cdot (\sum_{i=1}^{k} r_i \cdot \gamma'_i) = p^{n-1-s} \cdot (\sum_{i=1}^{k} r_i \cdot g_{\gamma_i} - h_{\gamma_i}) =$$

$$\sum_{i=1}^{k} p^{n-1-s} \cdot r_i \cdot g_{\gamma_i} - \sum_{i=1}^{k} p^{n-1-s} \cdot r_i \cdot h_{\gamma_i} = \sum_{i=1}^{k} p^{n-1-s} \cdot r_i \cdot g_{\gamma_i} \neq 0.$$

But this contradicts the equality $A \bigcap B = \{0\}$.

The statement 8.1 is proved.

**Proof 8.2.** As $G = A \bigoplus B$, then $A \bigcap B = \{0\}$. Then, according to the statement 8.1, $A' \bigcap B = \{0\}$ and according to Proposition 6, there exists such subgroup $B'$ that $B \subseteq B'$ and $G = A' \bigoplus B'$. And according to the statement 8.1, $A \bigcap B' = \{0\}$.

So, we have obtained that $B \subseteq B'$ and $A \bigcap B' = \{0\}$. As $G = A \bigoplus B$, then $B = B'$.

The statement 8.2 is proved.

**9. Theorem** *Let $G$ be any Abelian group of the period $p^n$ and $G_2 = \{g \in G | p \cdot g = 0\}$. If $\tau$ and $\tau'$ are such metrizable, linear, group topologies that the subgroup $G_1 = \{g \in G | p^{n-1} \cdot g = 0\}$ is a closed subgroup in each of topological groups $(G, \tau)$ and $(G, \tau')$, then $\tau|_{G_1} = \tau'|_{G_1}$ and $(G, \tau)/G_2 = (G, \tau')/G_2$ if and only if there exist such group isomorphism $\varphi : G \to G$ that the following conditions are satisfied:*

*1. $\varphi(G_1) = G_1$;*

*2. $g - \varphi(g) \in G_2$ for any $g \in G$;*

*3. $\varphi : (G, \tau) \to (G, \tau')$ is a topological isomorphism (i.e. open and continuous isomorphism).*

**Proof. Sufficiency.** Let $\varphi : G \to G$ be a group isomorphism such that conditions 1 - 3 are executed.

If $V \in \tau|_{G_1}$, then there exists such $U \in \tau$ that $U \bigcap G_1 = V$. As $\varphi : (G, \tau) \to (G, \tau')$ is a topological isomorphism, then $U' = \varphi(U) \in \tau'$. Because $\varphi : G \to G$ is a bijection mapping and $\varphi(G_1) = G_1$, it follows

$$\varphi(V) = \varphi(U \bigcap G_1) = \varphi(U) \bigcap \varphi(G_1) = U' \bigcap G_1 \in \tau'|_{G_1}.$$

From the arbitrarity of the set $V$ it follows that $\tau|_{G_1} \subseteq \tau'|_{G_1}$.

It is similarly proved that $\tau'|_{G_1} \subseteq \tau|_{G_1}$, and hence $\tau|_{G_1} = \tau'|_{G_1}$.

Now we consider the following commutative diagram:

$$\begin{tikzcd}
(G,\tau) \arrow[r, "\varphi"] \arrow[d, "\omega"'] & (G,\tau') \arrow[d, "\omega"'] \\
(G,\tau)/G_2 \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{\omega}"'] & (G,\tau')/G_2, \arrow[d, "\bar{\omega}"'] \\
(G,\tau)/G_1 \arrow[r, "\widetilde{\varphi}"] & (G,\tau')/G_1
\end{tikzcd}$$

here $\omega$ and $\bar{\omega}$ are natural homomorphisms, and $\bar{\varphi}$ and $\widetilde{\varphi}$ are such isomorphisms that $\bar{\varphi}(g + G_2) = \varphi(g) + G_2$ and $\widetilde{\varphi}(g + G_1) = \widetilde{\varphi}(g) + G_1$.

As $g - \varphi(g) \in G_2$, then $g + G_2 = \varphi(g) + G_2$. Hence $\bar{\varphi}(g + G_2) = \varphi(g) + G_2 = g + G_2$ and $\widetilde{\varphi}(g + G_1) = \widetilde{\varphi}(g) + G_1$, i.e. $\bar{\varphi} : G/G_2) = G/G_2$ and $\widetilde{\varphi} : G/G_1) = G/G_1$ are identical mappings.

From the fact that $\omega : (G, \tau) \to (G, \tau)/G_2$ and $\omega : (G, \tau') \to (G, \tau')/G_2$ are open and continuous homomorphisms it follows that $\bar{\varphi} : (G, \tau)/G_2 \to (G, \tau')/G_2$ is an open and continuous isomorphism, i.e. $(G, \tau)/G_2 = (G, \tau')/G_2)$.

Sufficiency is completely proved.

**Necessity.** Let $\tau$ and $\tau'$ be such metrizable, linear, group topologies that $\tau|_{G_1} = \tau'|_{G_1}$ and $(G,\tau)/G_2 = (G,\tau')/G_2$. If $\omega : G \to G/G_2$ and $\bar{\omega} : G/G_2 \to G/G_1 = (G/G_2)/(G_1/G_2)$ are natural homomorphisms, then according to the statement 2.3, there exist sets $\{V_i | i \in \mathbb{N}\bigcup\{0\}\}$ and $\{V'_i | i \in \mathbb{N}\bigcup\{0\}\}$ of subgroups which are bases of the neighborhoods of zero in topological groups $(G,\tau)$ and $(G,\tau')$, respectively, and $V_i \bigcap G_1 = V'_i \bigcap G_1$ and $\omega(V_i) = \omega(V'_i)$ for any $i \in \mathbb{N}\bigcup\{0\}$. Without loss of generality, we can consider that $V_0 = V'_0 = G$.

For every $i \in \mathbb{N}$ let $\bar{V}_i = \omega(V_i) = \omega(V'_i)$ and $\widetilde{V}_i = \bar{\omega}(\bar{V}_i)$.

As $\bar{G} = G/G_1$ is a linear space over the field $F_p = \mathbb{Z}/p\cdot\mathbb{Z}$ and $\widetilde{V}_i$ is a subspace of the linear space $\bar{G}$, then for every $i \in \mathbb{N}\bigcup\{0\}$ there exists a set $\{\widetilde{U}_i | i \in \mathbb{N}\bigcup\{0\}\}$ of subspaces of the linear space $\bar{G}$ such that $\widetilde{V}_i = \widetilde{U}_i \bigoplus \widetilde{V}_{i+1}$ for any $i \in \mathbb{N}\bigcup\{0\}$. Then $\widetilde{V}_k = (\bigoplus\limits_{i=k}^{n} \widetilde{U}_i)\bigoplus(\widetilde{V}_{n+1})$ for any $k \leq n \in \mathbb{N}\bigcup\{0\}$. As $G_1$ is a closed subgroup in the topological groups $(G,\tau)$ and $(G,\tau')$, then (se [1], theorem 1.3.2) $\bigcap\limits_{k\in\mathbb{N}} \widetilde{V}_k = \{0\}$ and hence $\widetilde{V}_k = \bigoplus\limits_{i=k}^{\infty} \widetilde{U}_i$.

For every $k \in \mathbb{N}\bigcup\{0\}$ we shall consider a basis $\{\widetilde{x}_{k,\gamma} | \gamma \in \Gamma_k\}$ of the linear space $\widetilde{U}_k$.

As $\widetilde{U}_i \subseteq \widetilde{V}_i = \bar{\omega}(\omega(V_i)))$ for any $i \in \mathbb{N}\bigcup\{0\}$, then for any $k \in \mathbb{N}\bigcup\{0\}$ and any $\gamma \in \Gamma_k$ there exists an element $x_{k,\gamma} \in V_k$ such that $\bar{\omega}(\omega(x_{k,\gamma})) = \widetilde{x}_{k,\gamma}$.

As $\omega(V_i) = \omega(V'_i)$ for any $i \in \mathbb{N}\bigcup\{0\}$, then for any $i \in \mathbb{N}\bigcup\{0\}$ and any $\gamma \in \Gamma$ there exists an element $x'_{i,\gamma} \in V'_i$ such that $\omega(x_{i,\gamma}) = \omega(x'_{i,\gamma})$.

According to Proposition 7, the subgroups $A =< \{x_{k,\gamma} | k \in \mathbb{N}\bigcup\{0\}, \gamma \in \Gamma\} >$ and $A' =< \{x'_{k,\gamma} | k \in \mathbb{N}\bigcup\{0\}, \gamma \in \Gamma\} >$ are serving subgroups of the group $G$ and they are direct sums of cyclic groups of the order $p^n$.

According to the Prufer-Kulikov theorem (see Theorem 5) there exists a subgroup $B$ of the group $G$ such that $G = B \bigoplus A$. Then, according to the statement 8.2, $G = B \bigoplus A'$. As $\bar{\omega}(\omega(A)) = \bar{\omega}(\omega(V_0)) = G/G_1$, then $B \subseteq G_1$.

If $f : \{x_{k,\gamma} | k \in \mathbb{N}\bigcup\{0\}, \gamma \in \Gamma\} \to \{x'_{k,\gamma} | k \in \mathbb{N}\bigcup\{0\}, \gamma \in \Gamma\}$ is a mapping such that $f(x_{k,\gamma}) = x'_{k,\gamma}$ for any $k \in \mathbb{N}\bigcup\{0\}$ and $\gamma \in \Gamma$ then it can be extended to a group isomorphism $\widehat{f} : A \to A'$.

We suppose $\varphi(a + b) = \widehat{f}(a) + b$ for any $a \in A$ and any $b \in B$. Then $\varphi : G \to G$ is a group isomorphism.

As $\omega(x_{k,\gamma}) = \omega(x'_{k,\gamma}) = \bar{x}_{k,\gamma}$ for any $k \in \mathbb{N}\bigcup\{0\}$ and any $\gamma \in \Gamma$, then $h_{k,\gamma} = x_{k,\gamma} - x'_{k,\gamma} \in G_2$.

Let now $g \in G_1$. Then $g = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{s} t_{i,\gamma_j} \cdot x_{i,\gamma_j} + b$, where $b \in B \subseteq G_1$. As

$$0 = \bar{\omega}(\omega(g)) = \sum_{i=1}^{k}\sum_{j=1}^{s} t_{i,\gamma_j} \cdot \bar{\omega}(\omega(x_{i,\gamma_j})) + \bar{\omega}(\omega(b)) = \sum_{i=1}^{k}\sum_{j=1}^{s} t_{i,\gamma_j} \cdot \bar{x}_{i,\gamma_j},$$

then all $t_{i,\gamma_j}$ are divisible by $p$, and hence

$$\varphi(g)=\sum_{i=1}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x'_{i,\gamma_j}+\varphi(b)=\sum_{i=1}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot(x_{i,\gamma_j}-h_{i,\gamma_j})+\varphi(b)=$$

$$\sum_{i=1}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}-\sum_{i=1}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot h_{i,\gamma_j}+b=\sum_{i=1}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b=g.$$

So we have proved that $\varphi(g)=g$ for any $g\in G_1$. Then $\varphi(G_1)=G_1$, i.e. the first statement of the theorems is true.

Let now $g\in G$. Then $g=\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b$, where $b\in B\subseteq G_1$, and hence

$$g-\varphi(g)=\sum_{i=0}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b-(\sum_{i=0}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x'_{i,\gamma_j}+b)=$$

$$\sum_{i=0}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot(x_{i,\gamma_j}-x'_{i,\gamma_j})=\sum_{i=0}^{k}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot h_{i,\gamma_j}\in G_2,$$

i.e. the second statement of the theorem is also true.

For finishing the proof of the theorem it remained to check up that the isomorphism $\varphi:(G.\tau)\to(G.\tau')$ is a topological isomorphism. For this purpose it is enough to verify that $\varphi(V_{k,\gamma})=V'_{k,\gamma}$ for any $k\in\mathbb{N}$ and any $\gamma\in\Gamma$.

So, let $g\in V_{k,\gamma}$. Then $g=\sum\limits_{i=0}^{m}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b$, where $b\in B\subseteq G_1$.

As (see definition of elements $x_{i,\gamma}$) $\sum\limits_{i=k}^{m}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}\in V_k$, then

$$\sum_{i=0}^{k-1}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b=g-\sum_{i=k}^{m}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}\in V_k.$$

Besides that as

$$\sum_{i=0}^{m}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot\bar{x}_{i,\gamma_j}=\omega(\sum_{i=0}^{m}\sum_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b)=$$

$\omega(g)\in\omega(V_k)=\bar{V}_k=\bigoplus\limits_{i=k}^{\infty}\bar{U}_i$, then for any $i<k$ all numbers $t_{i,\gamma_j}$ are divided by $p$, and hence $\sum\limits_{i=0}^{k-1}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}\in G_1$. Then $\sum\limits_{i=0}^{k-1}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b\in G_1\bigcap V_k=G_1\bigcap V'_k$, and hence, $\varphi(\sum\limits_{i=0}^{k-1}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b)=\sum\limits_{i=0}^{k-1}\sum\limits_{j=1}^{s}t_{i,\gamma_j}\cdot x_{i,\gamma_j}+b\in V'_k$. Then

$$\varphi(g) = \varphi(\sum_{i=0}^{k-1}\sum_{j=1}^{s} t_{i,\gamma_j} \cdot x_{i,\gamma_j} + b) + \varphi(\sum_{i=k}^{m}\sum_{j=1}^{s} t_{i,\gamma_j} \cdot x_{i,\gamma_j}) \in$$

$$V'_k + \sum_{i=k}^{m}\sum_{j=1}^{s} t_{i,\gamma_j} \cdot x'_{i,\gamma_j} \subseteq V'_k + V'_k = V'_k.$$

From the arbitrarity of the element $g$ it follows that $\varphi(V_k) \subseteq V'_k$.

In a similar way it can be proved that $\varphi^{-1}(V'_k) \subseteq V_k$, and hence $\varphi(V_k) = V'_k$.

The theorem is completely proved.

## References

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*Received May 12, 2009*