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A note on a subclass of analytic functions defined by a differential operator

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Abstract. By means of the Sălăgean differential operator we define a new class $\mathcal{BS}(m,\mu,\alpha)$ involving functions $f \in \mathcal{A}_n$. Parallel results for some related classes including the class of starlike and convex functions respectively are also obtained.

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1 Introduction and definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathcal{H}(U)$ be the space of holomorphic functions in U, $n \in \mathbb{N} = \{1, 2, ...\}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0 \le \alpha < 1$, which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{3}$$

for some α $(0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$\mathrm{Re} f'(z) > \alpha, \quad z \in U. \tag{4}$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let D^m be the Sălăgean differential operator [3], $D^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, defined as

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^m f(z) &= D(D^{m-1} f(z)), \quad z \in U. \end{aligned}$$

We note that if $f \in \mathcal{A}_n$, then

$$D^m f(z) = z + \sum_{j=n+1}^{\infty} j^m a_j z^j, \quad z \in U.$$

To prove our main theorem we shall need the following lemma.

Lemma 1 (see [2])**.** *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $Rep(z) > \alpha$ for $z \in U$ and $1/2 \le \alpha < 1$.

2 Main results

Definition 1. We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BS}(m, \mu, \alpha)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \ge 0$, $\alpha \in [0, 1)$ if

$$\left| \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^{\mu} - 1 \right| < 1 - \alpha, \qquad z \in U. \tag{6}$$

Remark 1. The family $\mathcal{BS}(m, \mu, \alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BS}(0, 1, \alpha) \equiv \mathcal{S}^*(\alpha)$, $\mathcal{BS}(1, 1, \alpha) \equiv \mathcal{K}(\alpha)$ and $\mathcal{BS}(0, 0, \alpha) \equiv \mathcal{R}(\alpha)$. Another interesting subclass is the special case $\mathcal{BS}(0, 2, \alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathcal{BS}(0, \mu, \alpha) \equiv \mathcal{B}(\mu, \alpha)$ which has been introduced by Frasin and Jahangiri [2].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BS}(m, \mu, \alpha)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 1. *For the function $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \ge 0$, $1/2 \le \alpha < 1$ if*

$$\frac{D^{m+2} f(z)}{D^{m+1} f(z)} - \mu \frac{D^{m+1} f(z)}{D^m f(z)} + \mu \prec 1 + \beta z, \quad z \in U, \tag{7}$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha},$$

then $f \in \mathcal{BS}(m, \mu, \alpha)$.

Proof. If we consider

$$p(z) = \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^{\mu} \tag{8}$$

then $p(z)$ is analytic in U with $p(0) = 1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)} = \frac{D^{m+2} f(z)}{D^{m+1} f(z)} - \mu \frac{D^{m+1} f(z)}{D^m f(z)} + \mu - 1. \tag{9}$$

Using (7) we get

$$\mathrm{Re} \left(1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1 we deduce that

$$\mathrm{Re} \left\{ \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^{\mu} \right\} > \alpha.$$

Therefore, $f \in \mathcal{BS}(m, \mu, \alpha)$, by Definition 1. □

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 1. *If* $f \in \mathcal{A}_n$ *and*

$$Re \left\{ \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2}, \quad z \in U, \tag{10}$$

then

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U. \tag{11}$$

That is, f is convex of order $\frac{1}{2}$.

Corollary 2. *If* $f \in \mathcal{A}_n$ *and*

$$Re \left\{ \frac{2z^2 f''(z) + z^3 f'''(z)}{zf'(z) + z^2 f''(z)} \right\} > -\frac{1}{2}, \quad z \in U, \tag{12}$$

then

$$Re \left\{ f'(z) + zf''(z) \right\} > \frac{1}{2}, \quad z \in U. \tag{13}$$

Corollary 3. *If* $f \in \mathcal{A}_n$ *and*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{14}$$

then

$$Ref'(z) > \frac{1}{2}, \quad z \in U. \tag{15}$$

In another words, if the function f *is convex of order* $\frac{1}{2}$ *then* $f \in \mathcal{BS}(0,0,\frac{1}{2}) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 4. *If* $f \in \mathcal{A}_n$ *and*

$$Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U, \tag{16}$$

then f *is starlike of order* $\frac{1}{2}$.

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