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The cubic differential system with six real invariant straight lines along three directions

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Abstract. We classify all cubic systems possessing exactly six real invariant straight lines along three directions taking into account their degree of invariance. We prove that there are 6 affine different classes of such systems. For every class we carried out the qualitative investigation in the Poincaré disc.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, invariant line.

1 Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1}$$

and the vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

associated to system (1).

Denote $n = max\{deg(P),\, deg(Q)\}$. If $n = 2$ ($n = 3$) then system (1) is called quadratic (cubic).

An algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ (a function $f = exp(g/h)$; $g,h \in \mathbb{C}[x,y]$) is called invariant algebraic curve (invariant exponential function) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \le n-1$ such that the identity holds

$$X(f) \equiv f(x,y)K_f(x,y). \tag{2}$$

It should be observed that if in (2) for invariant algebraic curve $f(x,y) = 0$ we have $K_f(x,y) \equiv f^m(x,y)K(x,y)$ for any natural number $m \in \mathbb{N}$ and polynomial $K(x,y)$, then $exp(1/f), ..., exp(1/f^m)$ are invariant exponential functions. If, in addition, the polynomial $f(x,y)$ does not divide $K(x,y)$, then we say that the invariant algebraic curve $f(x,y) = 0$ has the degree of invariance equal to $m+1$.

Let $f \in \mathbb{C}[x,y]$ and $f = f_1^{n_1} \cdots f_s^{n_s}$ be its factorization in irreducible factors over $\mathbb{C}[x,y]$. Then $f(x,y) = 0$ is an invariant algebraic curve for (1) if and only if each of the algebraic curves $f_j(x,y) = 0$, $j = \overline{1,s}$, has this property.

It is easy to see that there is no correlation between the degree of invariance of the invariant algebraic curve $f(x,y)=0$ and the degree of invariance of its factors $f_j(x,y)=0$, $j=\overline{1,s}$, in general case. For example, for a system $\dot{x}=x^3$, $\dot{y}=y(2x^2+y^2)$, we have that $x^2+y^2=0$ is an algebraic curve with the degree of invariance equal to two, while for each of its factors $x\pm iy=0$, $i^2=-1$, the degree of invariance is equal to one. For the system [5]: $\dot{x}=2x^3$, $\dot{y}=y(3x^2+y^2)$, each of the invariant straight lines $x\pm iy=0$ has the degree of invariance equal to two, and their product $x^2+y^2=0$ has the degree of invariance equal to one.

Let $f_1(x,y)=0,...,f_k(x,y)=0$ be some irreducibles invariant algebraic curves; $f_{k+1}(x,y)=exp(g_{k+1}/h_{k+1}),...,f_s(x,y)=exp(g_s/h_s)$ be some invariant exponential functions of the system (1) and let $\lambda_1,...,\lambda_s$ be some real or complex numbers. We compose the function

$$F=f_1^{\lambda_1}\cdots f_s^{\lambda_s}. \tag{3}$$

If $F\not\equiv const$ and $X(F)\equiv 0$ $(X(F)\equiv -F(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}))$, i.e. $F(x,y)=const$ is a first integral (F is an integrating factor) for (1), then we say that the system (1) is Darboux integrable. In order that (3) be a first integral (an integrating factor) for (1), it is necessary and sufficient that cofactors $K_{f_1},...,K_{f_s}$ and numbers $\lambda_1,...,\lambda_s$ verify the identity

$$\lambda_1K_{f_1}(x,y)+\cdots+\lambda_sK_{f_s}(x,y)\equiv 0$$

$$\Big(\lambda_1K_{f_1}(x,y)+\cdots+\lambda_sK_{f_s}(x,y)\equiv -\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\Big).$$

Later on, we will be interested in invariant algebraic curve of degree one, that is invariant straight lines $\alpha x+\beta y+\gamma=0$.

A set of invariant straight lines can be infinite, finite or empty. Systems with infinite number of invariant straight lines will not be considered.

At present a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines. Here we indicate some problems and works concerning the polynomial differential system with invariant straight lines.The problem of estimation for the number of invariant straight lines which can have a polynomial differential system was considered in [2]; the problem of coexistence of the invariant straight lines and limit cycles in {[9]: $n=2$}; {[4]: $n=3$}; [10]; the problem of coexistence of the invariant straight lines and the singular points of a center type for the cubic system in [3, 11] An interesting relation between the number of invariant straight lines and the possible number of directions for them is established in [1].

The classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities is given in [5].

The cubic system with exactly eight and exactly seven invariant straight lines has been studied in [5-7] and with six invariant straight lines along two directions in [8].

In this paper a qualitative investigation of cubic systems with exactly six real invariant straight lines along three direction is given.

The main obtained results are shown in the following theorem:

Theorem. *Any cubic system having real invariant straight lines along three directions with total degree of invariance six via affine transformation and time rescaling can be written as one of the following six systems. The bifurcation diagrams in the space of parameters and the phase portraits in the Poincaré disc are presented in the figures for each system.*

$$\begin{cases} \dot{x} = x(x+1)(x-a), \;\; a>0, \\ \dot{y} = y(y+1)(-a+dx+(1-d)y), & Fig.1 \\ d(d-1)(a+d-1)(a-d+2) \neq 0; & (Fig.4.1; Tab.4.1, 4.2) \end{cases} \tag{4}$$

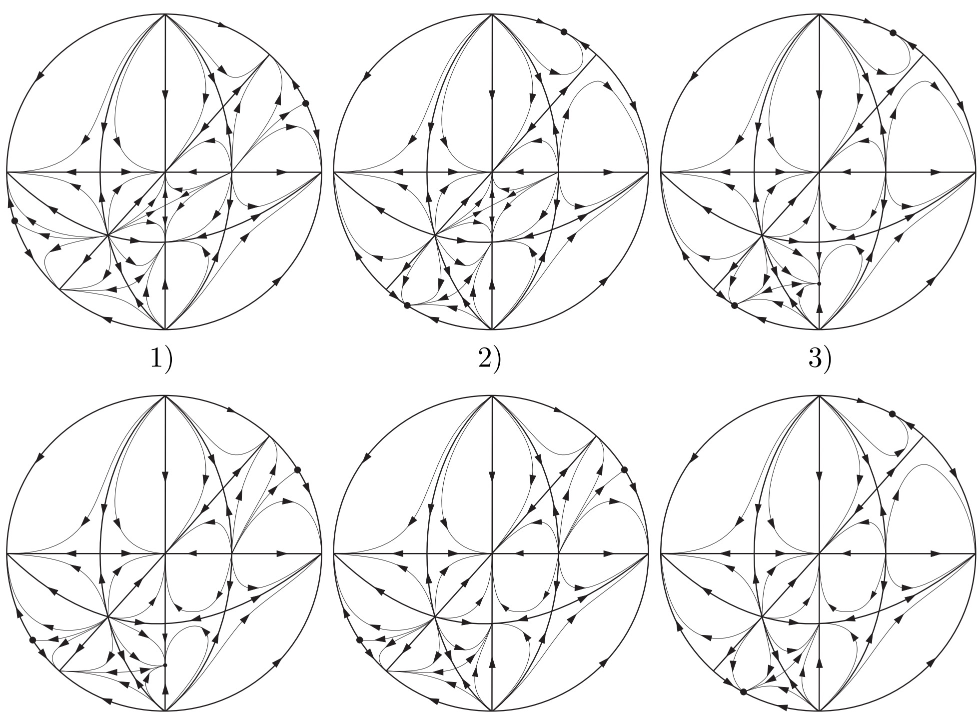
$$\begin{cases} \dot{x} = x(x+1)(x-a), \; a>0, \\ \dot{y} = y(y+1)(a(a-d+1)+dx+(a+1)(a-d+1)y), & Fig.2 \\ d(a-d+1)(a-d+2)(2a-d+1) \neq 0; & (Fig.4.2; Tab.4.3, 4.4) \end{cases} \tag{5}$$

$$\begin{cases} \dot{x} = x^2(x+1), \; d(d-1) \neq 0, \\ \dot{y} = y(y+1)(dx+(1-d)y); & Fig.3\,(Tab.4.5) \end{cases} \tag{6}$$

$$\begin{cases} \dot{x} = x^2(x+1), \; d(d-1) \neq 0, \\ \dot{y} = y^2(1+dx+(1-d)y); & Fig.4\,(Tab.4.6) \end{cases} \tag{7}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y^2(2x-y); & Fig.5 \end{cases} \tag{8}$$

$$\begin{cases} \dot{x} = x(x+1)(a-ax+y), \\ \dot{y} = y(y+1)(a+(2-3a)x+(2a-1)y), & Fig.6 \\ a(3a-1)(2a-1)(2-3a)(a-1) \neq 0. & (Tab.4.7, 4.8) \end{cases} \tag{9}$$



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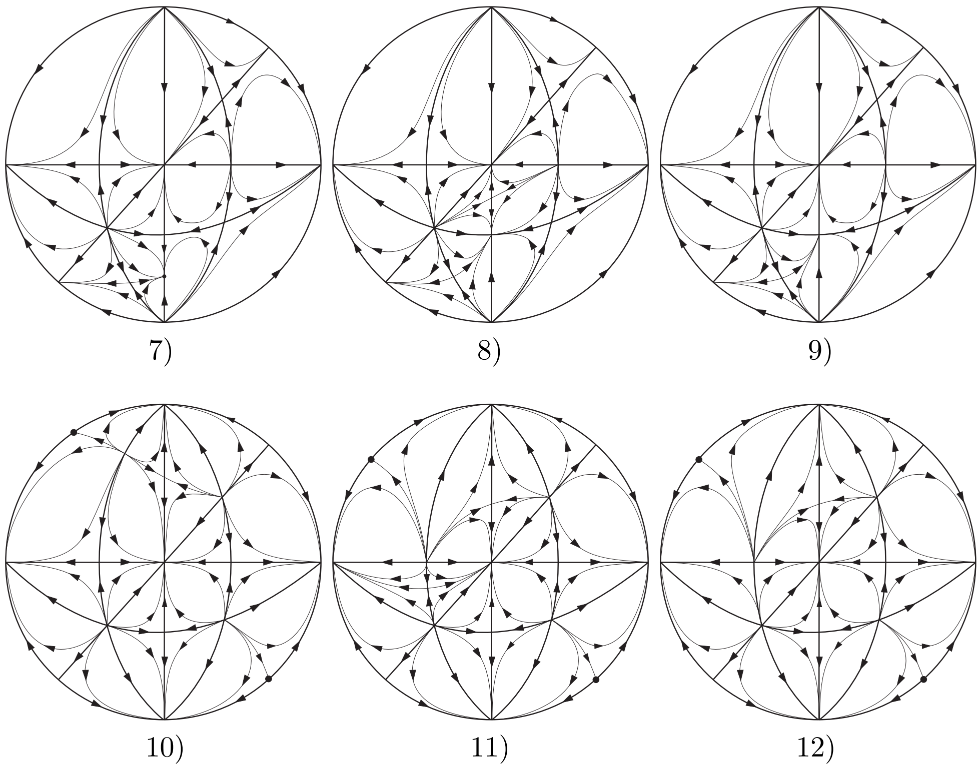
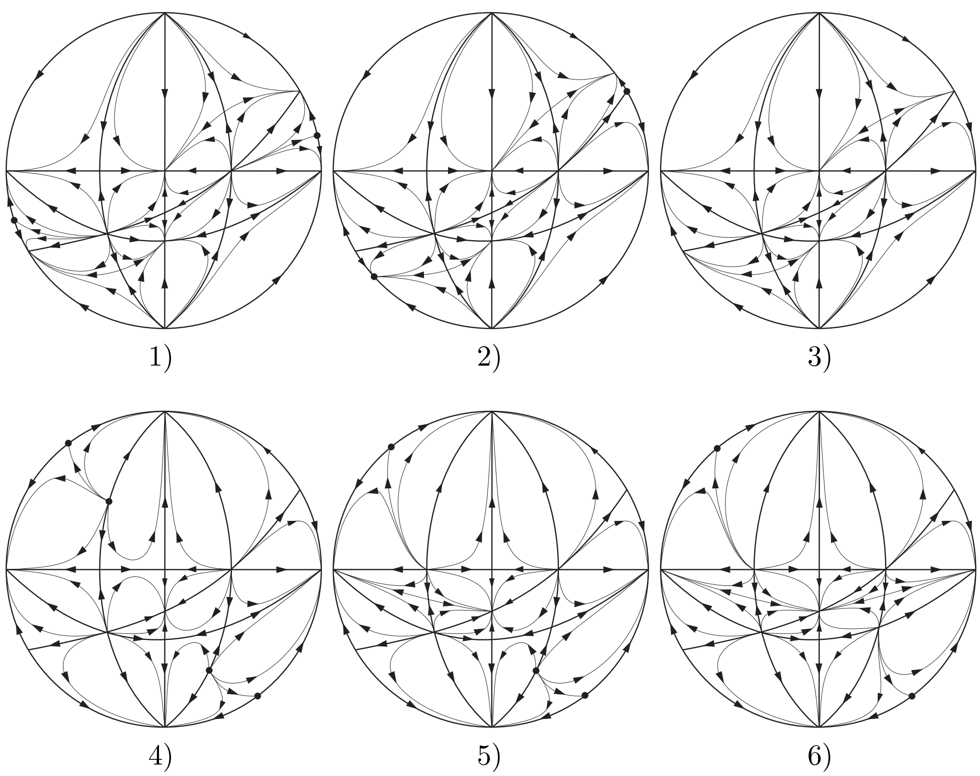


Fig. 1.



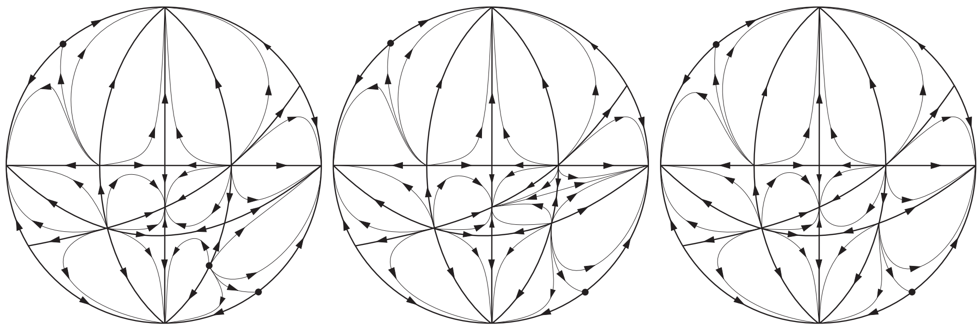


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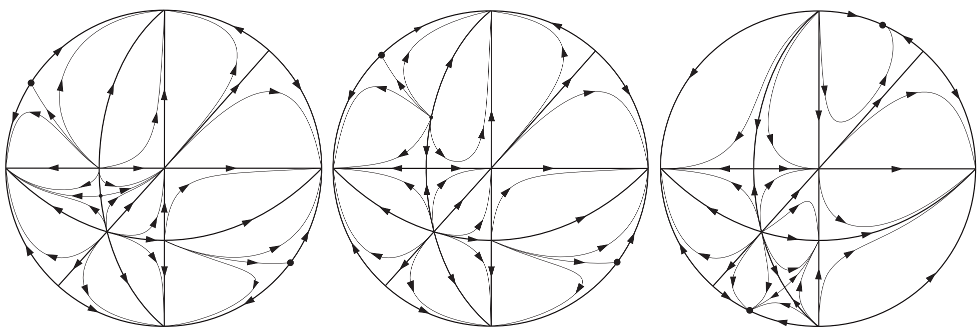
Fig. 2.



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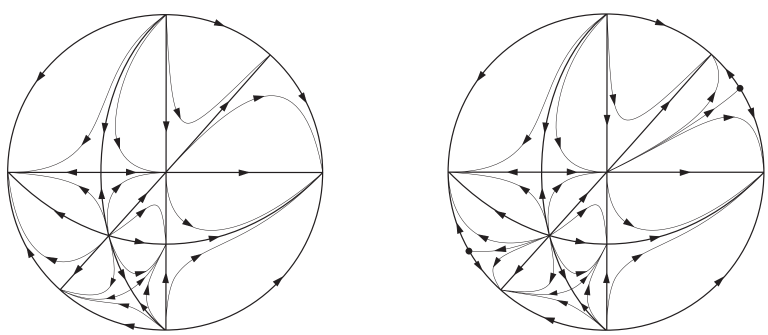
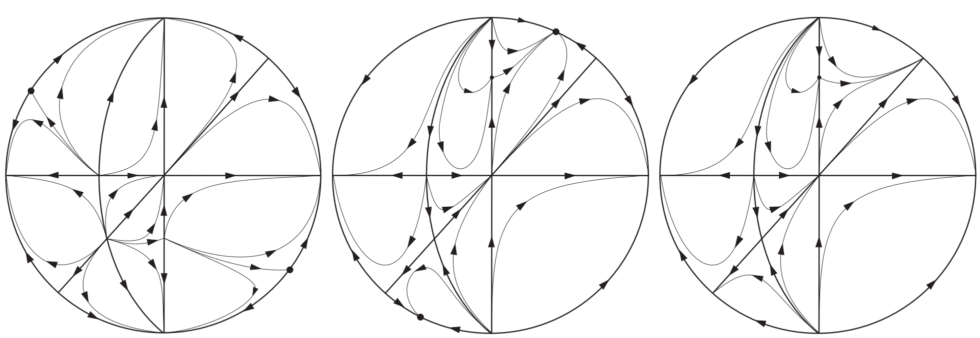


4) 5)

Fig. 3.



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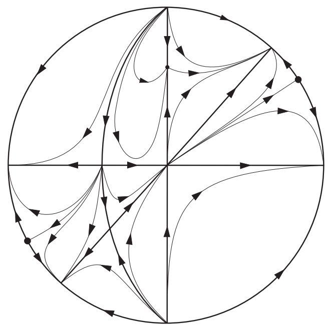


4)
Fig. 4

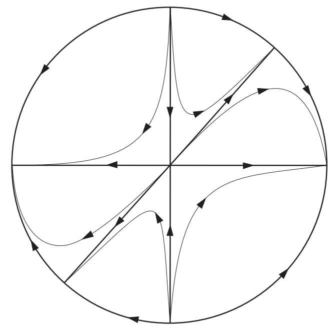
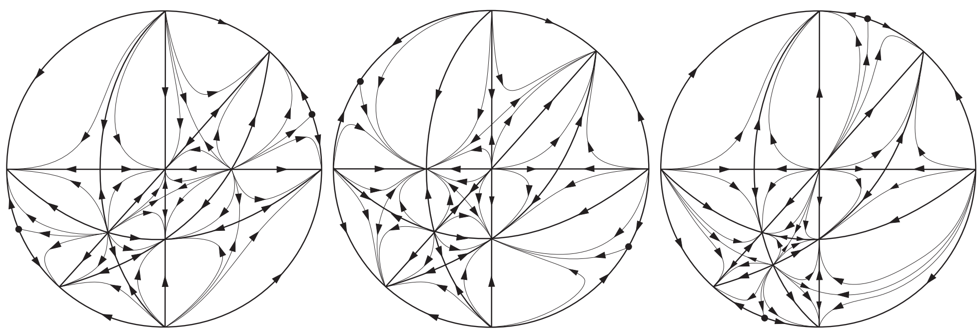


Fig. 5.



1) ⇄ 2) 3)
Fig. 6.

2 Preliminaries

We consider the real cubic differential system

$$\begin{cases} \dfrac{dx}{dt} = \sum_{r=0}^{3} P_r(x,y) \equiv P(x,y), \\ \dfrac{dy}{dt} = \sum_{r=0}^{3} Q_r(x,y) \equiv Q(x,y), \ GCD(P,Q)=1, \end{cases} \tag{10}$$

where $P_r(x,y) = \sum\limits_{j+l=r} a_{jl}x^j y^l$, $Q_r(x,y) = \sum\limits_{j+l=r} b_{jl}x^j y^l$. It is assumed that the right-hand sides of the system (10) have not a non-constant common factor.

We will mention some properties of the system (10):

2.1) in the finite part of the phase plane the system (10) has at most nine singular points;

2.2) at infinity the system (10) has at most four singular points if $yP_3(x,y)$ $-xQ_3(x,y) \not\equiv 0$. In the case $yP_3(x,y) - xQ_3(x,y) \equiv 0$ the infinity is degenerate, i.e. consists only of singular points;

2.3) in the finite part of the phase plane the system (10) can not have more than three colinear singular points;

2.4) in the finite part of the phase plane the system (10) has no more than eight invariant straight lines [5, 6];

2.5) the infinity for (10) represents an invariant straight line;

2.6) the system (10) has invariant straight lines along at most six different directions [1, 12];

2.7) the system (10) can not have more than three invariant straight lines parallel among themselves.

Let $a_jx + b_jy + c_j = 0,\ j = 1, 2,\ a_1b_2 - a_2b_1 \neq 0$ be two real invariant straight lines of the system (10). The transformation $X = a_1x + b_1y + c_1$, $Y = a_2x + b_2y + c_2$ reduces (10) to a system of the Lotka-Volterra form

$$\begin{cases} \dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2) \end{cases} \tag{11}$$

(we preserved the old notations).

The property **2.7)** says that every cubic system with at least four real invariant straight lines can be written in the form (11).

For system (11) a straight line $y = Ax + B,\ A \neq 0$ is invariant if and only if A and B are the solutions of the system:

$$\begin{aligned} &B(b_{01} + b_{02}B + b_{03}B^2) = 0, \\ &b_{11}B + b_{12}B^2 + [b_{01} - a_{10} + (2b_{02} - a_{11})B + (3b_{03} - a_{12})B^2] \cdot A = 0, \\ &b_{21}B + [b_{11} - a_{20} + (2b_{12} - a_{21})B] \cdot A + [b_{02} - a_{11} + (3b_{03} - 2a_{12})B] \cdot A^2 = 0, \\ &b_{21} - a_{30} + (b_{12} - a_{21}) \cdot A + (b_{03} - a_{12}) \cdot A^2 = 0. \end{aligned} \tag{12}$$

Its cofactor is

$$K(x, y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2,$$

where

$$c_{00} = b_{01} + b_{02}B + b_{03}B^2,\ c_{01} = b_{02} + b_{03}B,$$

$$c_{10} = b_{11} + b_{12}B + (b_{02} - a_{11})A + (2b_{03} - a_{12})AB,$$

$$c_{20} = b_{21} + (b_{12} - a_{21})A + (b_{03} - a_{12})A^2,\ c_{11} = b_{12} + (b_{03} - a_{12})A,\ c_{02} = b_{03}.$$

The invariant straight line $Ax - y + B = 0,\ A \neq 0$, of (11) has the degree of invariance not less than two if and only if A and B verify the following seven relations:

$$\begin{aligned} &B(b_{02} + 2b_{03}B) = 0, \quad b_{01} + 2b_{02}B + 3b_{03}B^2 = 0, \\ &a_{10}A + 2b_{02}AB + (b_{12} + 6b_{03}A - a_{12}A)B^2 = 0, \\ &a_{20} + b_{02}A + 2(b_{12} + 3b_{03}A - a_{12}A)B = 0, \\ &b_{11} - a_{20} + (b_{02} - a_{11})A = 0,\ a_{30} + b_{12}A + (2b_{03} - a_{12})A^2 = 0, \\ &b_{21} + (2b_{12} - a_{21})A + (3b_{03} - 2a_{12})A^2 = 0. \end{aligned} \tag{13}$$

In this case, the cofactor of invariant straight line is $K(x, y) = c_{00} + c_{10}x + c_{01}y$, where

$$c_{00} = -b_{02} - 2b_{03}B, \quad c_{10} = -b_{12} + (a_{12} - 2b_{03})A, \quad c_{01} = -b_{03}.$$

Proposition 1. *Let the cubic system have two real not parallel invariant straight lines l_1 and l_2, of which l_1 has the degree of invariance equal to m, $1 \leq m \leq 3$. Then the number of singular points lying on $l_2 \setminus l_1$ is at most $3-m$.*

Proof. In hypothesis of Proposition 1 via affine transformation, system (10) can be written in the form:

$$\dot{x} = x^m \tilde{P}_{3-m}(x,y), \qquad \dot{y} = y\tilde{Q}_2(x,y), \tag{14}$$

where $\tilde{P}_i$, $\tilde{Q}_i$ are polynomials of degree at most i and $\tilde{P}_{3-m}(x,0) \not\equiv 0$, $\tilde{P}_{3-m}(0,y) \not\equiv 0$. The system (14) has the invariant straight lines $l_1 : x = 0$ and $l_2 : y = 0$ of which l_1 has the degree of invariance equal to m. The assertion of Proposition 1 follows from the fact that the equation $\tilde{P}_{3-m}(x,0) = 0$ can not have more than $3-m$ roots. □

We say that the straight lines l_1, l_2 and l_3 are of generic position ("triangle" position) if $l_i \cap l_j \neq \varnothing$ and $l_1 \cap l_2 \cap l_3 = \varnothing$.

Proposition 2. *If cubic system (10) has three real invariant straight lines of generic position, then the sum of their degrees of invariance is at most four.*

Proof. We mention that any invariant straight line of the cubic system (10) can not have the degree of invariance more than three.

As the point of intersection of two invariant straight lines is a singular point for (10), Proposition 1 does not allow that any of these three straight lines l_1, l_2 and l_3 to have the degree of invariance equal to three.

Let each of the invariant straight lines l_1 and l_2 has the degree of invariance equal to two. By affine transformation and time rescaling the system (10) can be written in the form:

$$\begin{cases} \dot{x} = x^2(a + bx + y) \equiv P(x,y), \\ \dot{y} = y^2(c + dx + ey) \equiv Q(x,y), \ GCD(P,Q) = 1, \end{cases} \tag{15}$$

for which $l_1 = x$ and $l_2 = y$, and equalities (12) have the form

$$\begin{gathered} B^2(c + eB) = 0, \ 2cA + dB + 3eAB = 0, \\ a + (1-2d)B - cA - 3eAB = 0, \ eA^2 + (d-1)A - b = 0. \end{gathered} \tag{16}$$

Let $l_3 = y - Ax - B$, $AB \neq 0$. The points $(0,B) = l_1 \cap l_3$ and $(-B/A, 0) = l_2 \cap l_3$ are singular points for (15). Therefore, $P(-B/A,0) = Q(0,B) = 0$, yielding $A = -c/a$ and $B = -c/b$. Substituting these values of A and B in the first three equalities of (16), we get that $c = ab$, $d = b^2$ and $e = b$. In this case, $GCD(P,Q) = a + bx + y$. So, the assumption that system (15) can have invariant straight lines not passing through the origin of coordinates is false. □

3 Canonical forms and Darboux integrability

There are the following possible configurations of six invariant straight lines along three directions:

$$1)\,(3,2,1);\quad 2)\,(3(2),2,1);\quad 3)\,(3(3),2,1);\quad 4)\,(3,2(2),1);$$

$$5)\,(3(2),2(2),1);\quad 6)\,(3(3),2(2),1);\quad 7)\,(2,2,2);\quad 8)\,(2(2),2,2);$$

$$9)\,(2(2),2(2),2);\quad 10)\,(2(2),2(2),2(2)).$$

Notation $(3,2,1)$ means that along one direction there are three distinct straight lines, along the second direction there are two distinct invariant straight lines and along the third direction there is one invariant straight line; $(3(2),2,1)$ means that along one direction the differential system has two distinct straight lines from which one is double (i.e. has the degree of invariance equal to two), along the second direction there are two distinct invariant straight lines and along the third direction there is one invariant straight line and so on.

3.1) Configuration $(3,2,1)$. We note that the point of intersection of two real invariant straight lines of the system (10) is a singular point for this system.

Assume that the cubic system (10) has six distinct invariant straight lines, including one couple Then, taking into account the property **2.3)** from Section 2, the given straight lines can have (up to some affine transformation) one of the following 2 geometric positions given in Fig. 3.1.

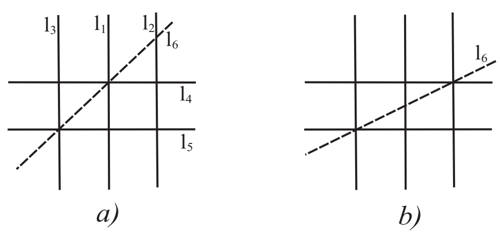


Fig. 3.1

The cubic system which includes both configurations, via affine transformation and time rescaling can be written in the form

$$\begin{cases} \dot{x} = x(x+1)(x-a), \ a > 0, \\ \dot{y} = y(y+1)(c+dx+ey), \ d(|e| + |c(c-d)(c+ad)|) \neq 0. \end{cases} \tag{17}$$

The system (17) has the invariant straight lines

$$l_1 \equiv x = 0, \ l_2 \equiv y = 0, \ l_3 \equiv x+1 = 0, \ l_4 \equiv y+1 = 0, \ l_5 \equiv x - a = 0.$$

We have to determine the conditions on parameters c, d and e such that (17) has only one invariant straight line of the form $l_6 \equiv y - Ax - B = 0, \ A \neq 0$.

For (17) the equalities (12) look as:

$$\begin{array}{l} B(B+1)(eB+c)=0,\ dB+dB^2+[a+c+2(c+e)B+3eB^2]\cdot A=0,\\ A\cdot[a+d-1+(c+e)A+2dB+3eAB]=0,\ eA^2+dA-1=0. \end{array} \tag{18}$$

Otherwise, we observe that the fourth equation of (18) doesn't allow for cubic system of $\dot{x}=x(x+1)(x-a)$, $\dot{y}=cy(y+1)$, $a|c|>0$ the configuration $(3,2,1)$ to be realized.

In the cases $a)$ the straight line l_6 has the equation $y=x$. Putting in (18) $A=1$ and $B=0$, we obtain

$$c=-a,\ e=1-d. \tag{19}$$

In conditions (19) the equalities (18) show that the straight line $y=-x/a$ ($y=(x-a)/(a+1)$) is invariant for (17) if $a+d-1=0$ $(a-d+2=0)$.

Equalities (19) and inequality $(a+d-1)(a-d+2)\neq 0$ show that for (17) the case $a)$ is realized, excluding, at the same time, the cases when (17) can has more than 6 invariant straight lines. In these conditions, (17) can be written in the form (4).

In the cases $b)$ the straight line $l_6:\ y=(x-a)/(a+1)$ is invariant for (17) if

$$c=a(1+a-d),\ e=(a+1)(1+a-d). \tag{20}$$

If $a-d+2=0$ $(2a-d+1=0)$ then (17) has the invariant straight line $l_7=x-y$ ($l_7=x-ay-a$).

The conditions (20) and $(a-d+2)(2a-d+1)\neq 0$ reduce (17) to the system (5).

The systems (4) and (5) are Darboux integrable and have respectively the integrating factors:

$$\mu(x,y)=x^{a/\delta}(x+1)^{-(a+1)/\delta}(x-a)^{-2}y^{(d-a-2)/\delta}(y+1)^{(d+a-1)/\delta}(y-x)^{d/\delta},$$

$$\mu(x,y)=x^{-2}(x+1)^{-\sigma}(x-a)^{-a\sigma}y^{-(1+\sigma)}(y+1)^{-(1+a\sigma)}\left(y-\frac{x-a}{a+1}\right)^{d\sigma},$$

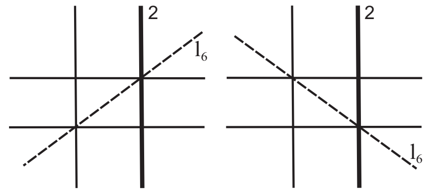
where $\delta=1-d,\ \sigma=1/(a-d+1)$.

3.2) Configuration $(3(2),2,1)$. The cubic system (10), with invariant straight lines of configuration $(3(2),2)$, via affine transformation and time rescaling, can be written in the form

$$\begin{cases} \dot{x}=x^2(x+1),\\ \dot{y}=y(y+1)(c+dx+ey),\ d(|e|+|c(c-d)|)\neq 0. \end{cases} \tag{21}$$

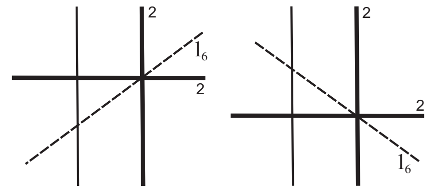
For this system the conditions (12) for the existence of invariant straight lines are of the form (18) with $a=0$.

For (21), the invariant straight line $x=0$ has the degree of invariance equal to two. Taking into account the propriety **2.3)** and Proposition 1, the system (21) can have invariant straight lines along three directions only of one of the following two geometric positions indicated in Fig. 3.2.



a) b)

Fig. 3.2



a) b)

Fig. 3.3

It is obvious that geometrical position of the straight lines in $a)$ and $b)$ are affine equivalent. We will examine only the case $a)$. In order the straight line which passes through singular points $(-1,-1)$ and $(0,0)$, i.e. the straight line $y = x$, to be invariant for (21), it is necessary that $c = 0$ and $e = 1-d$. In this conditions, (21) is reduced to the form (6). This system is Darboux integrable and has an integrating factor

$$\mu(x,y) = x^{-2}(x+1)^{-1/\delta}y^{-1-1/\delta}(y+1)^{-1}(y-x)^{d/\delta},$$

where $\delta = 1-d$.

3.3) Configuration $(3(3),2,1)$ **and** $(3.2(2),1)$. The property **2.3)** and Proposition 1 do not allow the realization of these configurations.

3.4) Configuration $(3(2),2(2),1)$. Considering the configuration $(3(2),2(2))$ of invariant straight lines we obtain the system

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y^2(c+dx+ey), \ d(|e|+|c(c-d)|) \neq 0, \end{cases} \tag{22}$$

which has the invariant straight lines $l_1 = x$, $l_2 = x+1$, $l_3 = y$ and the invariant exponential functions $l_4 = \exp(1/x)$, $l_5 = \exp(1/y)$. The straight lines l_1 and l_3 have the degree of invariance equal to two.

Proposition 2 allows only the positions from Fig.3.3 of the straight lines l_1, l_2, l_3 and $l_6 = y - Ax - B$, $A \neq 0$.

For (22) the equations (12) with condition $A \neq 0$ can be written as:

$$\begin{array}{l} B^2(c+eB) = 0, \ (dB + (2c+3eB)A)B = 0, \\ cA + (2d+3eA)B - 1 = 0, \ eA^2 + dA - 1 = 0. \end{array} \tag{23}$$

On the straight line $l_3 = x+1$ the system (22) can have only the singular points $(-1,0)$ and $(-1,(d-c)/e)$. The straight line which passes through the points $(0,0)$ and $(-1,(d-c)/e)$ is described by the equation $y = (c-d)x/e$. Putting in (23) $A = (c-d)/e$ and $B = 0$, we obtain that $e = cd(c-d)$. This leads to the system

$$\dot{x} = x^2(x+1), \ \dot{y} = y^2(c+dx+c(c-d)y), \ c(c-d) \neq 0,$$

which by substitutions $d \to cd, x \to x, y \to y/c$ can be reduced to a system (7).

The system (7) is Darboux integrable and has an integrating factor

$$\mu(x,y) = x^{-1/\delta}\exp(\delta/x)(x+1)^{-2}y^{(2d-3)/\delta}\exp(-\delta/y)(y-x)^{d/\delta},$$

where $\delta = 1 - d$.

3.5) Configuration $(3(3), 2(2), 1)$. For first step we consider the system

$$\dot{x} = x^3, \ \dot{y} = y^2(c + dx + ey), \ d(|c| + |e|) \neq 0. \tag{24}$$

For (24) the equalities (12) look as:

$$\begin{array}{l} B^2(c + eB) = 0, \ (dB + (2c + 3eB)A)B = 0, \\ cA + (2d + 3eA)B = 0, \ eA^2 + dA - 1 = 0. \end{array} \tag{25}$$

Proposition 1 allows for differential system (24) to have besides the straight lines $l_{1,2,3} = x$, $l_{4,5} = y$ also the invariant straight lines of the form $y = Ax$, $A \neq 0$. Putting in (25) $B = 0$, we obtain that $c = 0$ and $A_{1,2} = (-d \pm \sqrt{d^2 + 4e})/(2e)$. If $d^2 + 4e > 0$ ($d^2 + 4e < 0$), the system (24) has seven (five) real straight lines, and if $d^2 + 4e = 0$, i.e. $e = -d^2/4$, after a transformation $y \to 2y/d$ we come to the system (8) with invariant straight line $l_6 = x - y$. This system has an integrating factor $\mu(x, y) = 1/(xy(x - y)^2)$.

3.6) Configuration $(2, 2, 2)$. Taking into account the propriety **2.3)**, the system (10) with such configuration has at least two singular points through which three invariant straight lines of different directions pass. By a translation one of these points can be brought at the origin. The system (10) realizing this configuration via an affine transformation and time rescaling can be brought to the form

$$\begin{cases} \dot{x} = x(x + 1)(a + bx + y) \equiv P(x, y), \\ \dot{y} = y(y + 1)(c + dx + ey) \equiv Q(x, y), \ GCD(P, Q) = 1. \end{cases} \tag{26}$$

For (26) the equalities (12) look as:

$$\begin{cases} B(B + 1)(c + eB) = 0, \\ (c - a)A + dB + dB^2 + (2c + 2e - 1 + 3eB)AB = 0, \\ d - a - b + (c + e - 1)A + (2d - 1)B + 3eAB = 0, \\ eA^2 + (d - 1)A - b = 0. \end{cases} \tag{27}$$

Besides the invariant straight lines $l_1 = x$, $l_2 = x + 1$, $l_3 = y$, $l_4 = y + 1$, we will seek the conditions on parameters of (27) such that it has exactly two more invariant straight lines of the form $y = Ax$, $y = Ax + B$, $AB \neq 0$. For this, we put $B = 0$ in (27). The second equation of (27) gives $c = a$, and the third one becomes

$$d - a - b + (a + e - 1)A = 0. \tag{28}$$

In assumption that $AB \neq 0$ and $c = a$, the system of equations ((27), (28)) has the following solutions:

1) $b = -a$, $c = a$, $d = 2 - 3a$, $e = 2a - 1$, $A = 1$, $B = -1$.

System (26) with the conditions above has the invariant straight lines $l_5 = y - x$, $l_6 = y - x + 1$. The condition $GCD(P, Q) = 1$ implies the inequality $a(2a - 1)(a - 1) \neq 0$, and the inequality $2 - 3a \neq 0$ excludes the existence of a triplet of invariant straight lines parallel to axis Ox. If $3a-1=0$, then the given system has two more invariant straight lines of the form: $l_7 = y+x+1$ and $l_8 = y-x-1$.

2) $b=(a-1)/2,\ c=a,\ d=(3a+1)/2,\ e=-a,\ A=B=1$
$(l_5 = y-x,\ l_6 = y-x-1,\ a(9a^2-1)(a^2-1)\neq 0)$;

3) $b=1-a,\ c=a,\ d=3a-1,\ e=2a-1,\ A=B=-1$
$(l_5 = y+x,\ l_6 = y+x+1,\ a(a-1)(2a-1)(3a-1)(3a-2)\neq 0)$;

4) $b=2a-1,\ c=a,\ d=3a-1,\ e=1-a,\ A=(1-2a)/(1-a),\ B=a/(a-1)$
$(l_5 = y+(1-2a)x/(a-1),\ l_6 = y+((1-2a)x-a)/(a-1),\ l_7 = y-x)$.

If conditions **4)** hold, then (26) has seven invariant straight lines and, will be not considered. Moreover, it is sufficient to consider only the case **1)**, as the case **2)** (**3)**) can be reduced to the case **1)** via the change

$$a \to \frac{a}{2-3a},\ x \to y,\ y \to x,\ t \to (2-3a)t$$
$$(a \to 1-a,\ x \to x,\ y \to -y-1,\ t \to -t).$$

Inclusion of system (9) in the statement of Theorem in Section I is motivated. This system has the integrating factor

$$\mu(x,y) = \left[y(x+1)(y-x+1)\sqrt{x(y+1)(y-x)}\ \right]^{-1}.$$

3.7) Configuration $(2(2),2,2)$. Let cubic system (10) have distinct invariant straight lines $l_j,\ j=\overline{1,5}$, of which $l_1||l_2$, $l_3||l_4$ and l_5 has the degree of invariance equal to two. According to Proposition 1, the straight line l_5 must go through the points of intersection of straight lines l_1 and l_3, l_2 and l_4 (or l_1 and l_4, l_2 and l_3. This case is reduced to the previous one by changing the enumeration of straight lines). In our assumptions, via affine transformation and time rescaling the system (10) can be written in the form of (26). For (26) the straight lines $l_1 = x$, $l_2 = x+1$, $l_3 = y$ and $l_4 = y+1$ are invariant, and the equalities (13) look as:

$$\begin{array}{l} B(c+e+2eB)=0,\ c+2(c+e)B+3eB^2=0,\\ aA+2(c+e)AB+6eAB^2=0,\\ a+b+(c+e)A+2dB+6eAB=0,\\ d-a-b+(c+e-1)A=0,\\ b+dA+2eA^2=0,\ A(2d-1+3eA)=0. \end{array} \tag{29}$$

The straight line l_5 is given by the formula $x-y=0$. This line is invariant for (26) if $A=1$ together with $B=0$ are the solution of (29). Substituting in (29) these values of A and B, we obtain that $a=c=b+1=d+1=e-1=0$, which implies $GCD(P,Q)=y-x$.

3.8) Configuration $(2(2),2(2),2)$. Proposition 2 does not allow the realization of this configuration.

3.9) Configuration $(2(2),2(2),2(2))$. Taking into account Proposition 2, the invariant straight lines of this configuration should have a common point.

We consider the cubic system (15), where the straight lines $l_1 = x$ and $l_2 = y$ are invariant and have the degree of invariance equal to two. In this case the equalities (13) look as:

$$\begin{gathered} B(c+2eB)=0,\ B(2c+3eB)=0,\ B(2cA+dB+6eAB)=0, \\ a+cA+2dB+6eAB=0,\ a-cA=0, \\ b+dA+2eA^2=0,\ A(2d-1+3eA)=0. \end{gathered} \tag{30}$$

To determine the third invariant straight line $l_3 = Ax - y$, $A \neq 0$, with the same degree of invariance, we put in the equalities (30) $B = 0$ and resolve them for $A \neq 0$. The fourth and fifth equalities of ((30), $B = 0$) give $a = c = 0$. The condition $GCD(P,Q) = 1$ implies $e \neq 0$. From six and seven equalities of (30) we obtain $e = (2-d)(2d-1)/(9b)$ and $A = 3b/(d-2)$. Thus, we come to the system

$$\begin{cases} \dot{x} = x^2(bx+y),\ d(d+1)(2d-1)(d-2) \neq 0, \\ \dot{y} = y^2(dx+(2-d)(2d-1)y/(9b)), \end{cases}$$

which besides the invariant straight lines $x = 0$, $y = 0$, $3bx + (2-d)y = 0$ with the degree of invariance equal to two, also has the invariant straight line $3bx+(1-2d)y = 0$.

4 The phase portraits

We mention that the cubic system with at least four real invariant straight lines has no limit cycles [10]. Hence, the behaviour of trajectories in this system and, in particular, of system with six real invariant straight lines, is imposed by the type of singular points.

We denote by SP singular points; λ_1 and λ_2 the eigenvalues of SP; S – saddle ($\lambda_1\lambda_2 < 0$); N^s – stable node ($\lambda_1,\ \lambda_2 < 0$), N^u – unstable node ($\lambda_1,\ \lambda_2 > 0$); $S - N^{s(u)}$ – saddle-node with stable (unstable) parabolic sector; $P^{s(u)}$ – stable (unstable) parabolic sector; H – hyperbolic sector.

4.1. System (4). The coordinates of singular points of system (4) in the finite and infinite parts of the phase plane Oxy, also the eigenvalues λ_1, λ_2 of the characteristic equation, corresponding to each of these points, are shown in Tab.4.1. In this table the following notations: $\alpha = 1 + a$, $\delta = 1 - d$ were used.

Tab. 4.1

SP	$O_1(0,0)$	$O_2(-1,-1)$	$O_3(a,0)$	$O_4(0,-1)$	$O_5(0,a/\delta)$
$\lambda_1;\lambda_2$	$-a;\ -a$	$\alpha;\ \alpha$	$a\alpha;\ -a\delta$	$-a;\ a+\delta$	$-a;\ a(a+\delta)/\delta$
SP	$O_6(-1,0)$	$O_7(-1,(a+d)/\delta)$	$O_8(a,-1)$	$O_9(a,a)$	$I_1(1,0,0)$
$\lambda_1;\lambda_2$	$\alpha;\ -a-d$	$\alpha;\ \alpha(a+d)/\delta$	$a\alpha;\ \alpha\delta$	$a\alpha;\ a\alpha\delta$	$-1;\ -1$
SP	$I_2(0,1,0)$	$I_3(1,1,0)$	$I_4(1,-1/\delta,0)$		
$\lambda_1;\lambda_2$	$-\delta;\ -\delta$	$-1;\ 2-d$	$-1;\ 1+1/\delta$		

The singular point I_1 is a stable node. Taking into account that $a > 0$, at the point O_1 (O_2) the system (4) has a stable (unstable) node. Whatever are the parameters a, $a > 0$ and d, the types of points O_8 and O_9 coincide. In the case $a + \delta = 0$, i.e. $1 + a - d = 0$, ($a + d = 0$; $d = 2$) the singular points O_4 and O_5 (respectively O_6 and O_7; I_3 and I_4) coincide.

By means of the straight lines $d = 0$, $d = 1$, $d = 2$, $a = 0$, $2+a-d = 0$, $1+a-d = 0$, $a+d-1 = 0$, $a+d = 0$ we divide the half-plane $a > 0$ of parameters space a and d in sectors (Fig. 4.1). In Fig. 4.1 by V we denote the semi-line $1+a-d = 0$, $d > 2$); by VI – the segment of straight line ($1 + a - d = 0$, $1 < d < 2$); by VII – the semi-line ($d = 2$, $a > 1$); by $VIII$ – the segment ($d = 2$, $0 < a < 1$); by IX – the point $(2, 1)$; by XII – the semi-line ($a + d = 0$, $d < 0$); by I – the open domain bounded by straight lines $a = 0$, $d = 2$, $1 + a - d = 0$ without the semi-line ($a - d + 2 = 0$, $2 < d < +\infty$) and so on.

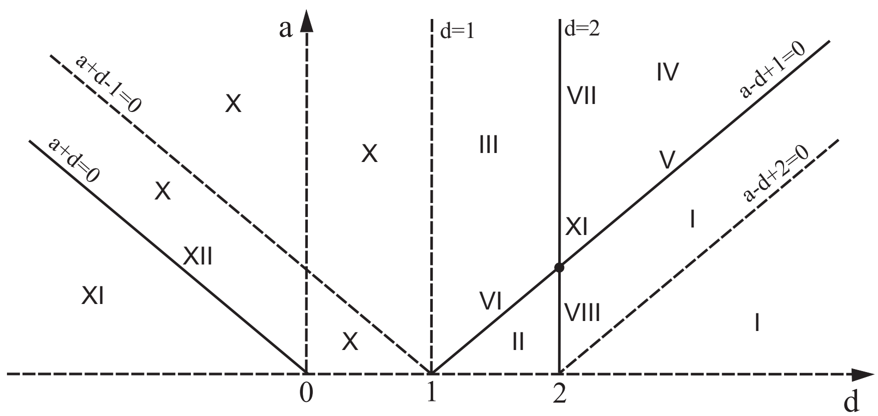


Fig. 4.1

For system (4) the results of qualitative investigation of singular points $O_3 - O_8$, $I_2 - I_4$ in each of the sectors $I - XII$ are given in Tab. 4.2.

Tab. 4.2

SP	I/II	III/IV	V/VI	$VII/VIII$	IX	X/XI	XII
O_3	N^u	N^u	N^u	N^u	N^u	S	S
O_4	N^s	S	$S{-}N^s$	S/N^s	$S{-}N^s$	S	S
O_5	S	N^s	–	N^s/S	–	S	S
O_6	S	S	S	S	S	S/N^u	$S{-}N^u$
O_7	S	S	S	S	S	N^u/S	–
O_8	S	S	S	S	S	N^u	N^u
I_2	N^u	N^u	N^u	N^u	N^u	N^s	N^s
I_3	N^s/S	S/N^s	N^s/S	$S{-}N^s$	$S{-}N^s$	S	S
I_4	S/N^s	N^s/S	S/N^s	–	–	S	S
$Fig.1:$	1)/2)	3)/4)	5)/6)	7)/8)	9)	10)/11)	12)

4.2. System (5). For (5) the singular points and and the eigenvalues of the characteristic equation are shown in Tab. 4.3, where $\alpha = 1 + a$.

Tab. 4.3

SP	$O_1(-1,-1)$	$O_2(a,0)$	$O_3(0,-1)$	$O_4(0,0)$
$\lambda_1;\ \lambda_2$	$\alpha;\ \alpha$	$a\alpha;\ a\alpha$	$-a;\ \alpha-d$	$-a;\ a(\alpha-d)$
SP	$O_5(0,-a/\alpha)$	$O_6(-1,0)$	$O_7(-1,\frac{d-a}{\alpha+d})$	$O_8(a,-1)$
$\lambda_1;$ λ_2	$-a;$ $a(d-\alpha)/\alpha$	$\alpha;$ $\alpha(a-d)$	$\alpha;$ $\alpha(d-a)/(\alpha-d)$	$a\alpha;$ $\alpha(1-d)$
SP	$O_9(a,a/(d-\alpha))$	$I_1(1,0,0)$	$I_2(0,1,0)$	$I_3(1,1/\alpha,0)$
$\lambda_1;$ λ_2	$a\alpha;$ $a\alpha(d-1)/(\alpha-d))$	$-1;$ -1	$\alpha(d-\alpha);$ $\alpha(d-\alpha)$	$-1;$ $2-d/\alpha$
SP	$I_4(1,1/(d-\alpha),0)$			
$\lambda_1;\ \lambda_2$	$-1;\ 1+\alpha/(d-\alpha)$			

For the system (5) the singular points O_1 and O_2 are unstable nodes, but point I_1 is a stable node. At every point of the half-plane $a > 0$ the points O_3 and O_4 are of the same type. If $a - d = 0$ ($d = 1$; $2a - d + 2 = 0$), then the points O_6 and O_7 (respectively: O_8 and O_9; I_3 and I_4) coincide.

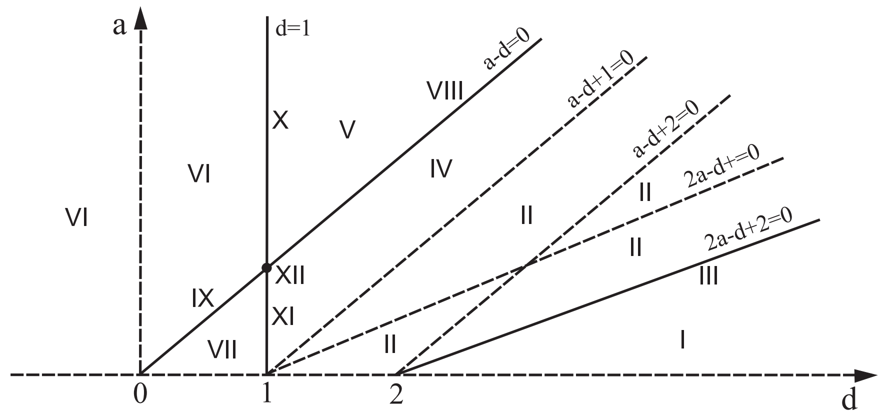


Fig. 4.2

The partition of the half-pane $a > 0$ in sectors and the qualitative study of singular points $O_4 - O_9$, $I_2 - I_4$ are given in Fig. 4.2 and Tab. 4.4 respectively.

Tab. 4.4

SP	I/II	III	IV/V	VI/VII	$VIII/IX$	X/XI	XII
O_4	N^s	N^s	S	S	S	S	S
O_5	S	S	N^s	N^s	N^s	N^s	N^s
O_6	S	S	S/N^u	N^u/S	$S{-}N^u$	N^u/S	$S{-}N^u$
O_7	S	S	N^u/S	S	$-$	S/N^u	$-$
O_8	S	S	S	N^u	N^u/S	$S{-}N^u$	$S{-}N^u$
O_9	S	S	N^u	S/N^u	S/N^u	$-$	$-$
I_2	N^u	N^u	N^s	N^s	N^s	N^s	N^s
I_3	N^s/S	$S{-}N^s$	S	S	S	S	S
I_4	S/N^u	$-$	S	S	S	S	S
$Fig.2:$	1)/2)	3)	4)/5)	6)/5)	7)/8)	8)/7)	9)

4.3. System (6). This system has five singular points in the finite part of the phase plane: $O_1(0,0)$, $O_2(-1,0)$, $O_3(-1,-1)$, $O_4(0,-1)$, $O_5(-1,\ d/(1-d))$; and four singular points at the infinity: $I_1(1,0,0)$, $I_2(0,1,0)$, $I_3(1,1,0)$, $I_4(1,1/(d-1),0)$. Among these singular points only $O_1(0,0)$ has the both eigenvalues of the characteristic equation equal to zero (see Tab. 4.5). To determine the behavior of trajectories in the neighborhood of this point, we write the system (6) in polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$:

$$\begin{cases} \dfrac{d\rho}{d\tau} = \rho[\cos^3\theta(1+\rho\cos\theta)+\sin^2\theta(1+\rho\sin\theta)(d\cos\theta+\delta\sin\theta)], \\ \dfrac{d\theta}{d\tau} = \sin\theta\cos\theta(\sin\theta-\cos\theta)(\rho\cos\theta+\delta(1+\rho\sin\theta)), \end{cases} \tag{31}$$

where $\tau=\rho t$, $\delta=1-d$. The singular points of system (31) with the first coordinate $\rho=0$ and the second $\theta\in[0,2\pi)$, and their eigenvalues are $\{M_1(0,0), M_2(0,\pi) : \lambda_1=1, \lambda_2=d-1\}$; $\{M_3(0,\pi/2),\ M_4(0,3\pi/2) :\ \lambda_1\cdot\lambda_2=-(1-d)^2\}$; $\{M_5(0,\pi/4) : \lambda_1=1/\sqrt{2}, \lambda_2=(1-d)/\sqrt{2}\}$; $\{M_6(0,5\pi/4) :\ \lambda_1=-1/\sqrt{2}, \lambda_2=-(1-d)/\sqrt{2}\}$. The types of these points can differ only if d passes through value 1. If $d<1$, we have Fig. 4.3, and if $d>1$, we have Fig. 4.4.

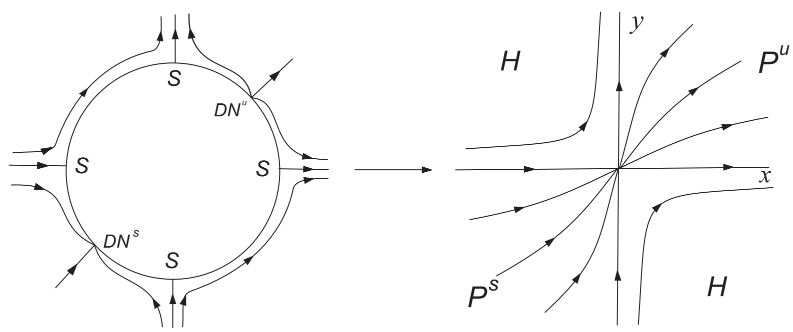


Fig. 4.3 ($d<1$).

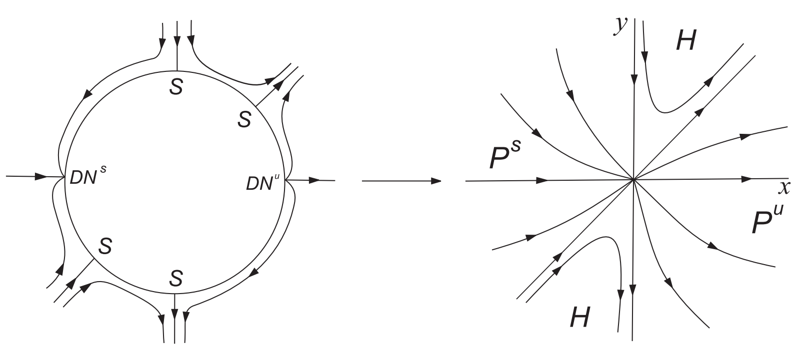


Fig. 4.4 ($d>1$).

In the case of system (6) we have Tab. 4.5.

Tab. 4.5

SP	$\lambda_1;\ \lambda_2$	$d<0$	$0<d<1$	$1<d<2$	$d=2$	$d>2$
O_1	$0;\ 0$	P^uHP^sH	P^uHP^sH	P^uHP^sH	P^uHP^sH	P^uHP^sH
O_2	$1;\ -d$	N^u	S	S	S	S
O_3	$1;\ 1$	N^u	N^u	N^u	N^u	N^u
O_4	$0;\ 1-d$	$S-N^u$	$S-N^u$	$S-N^s$	$S-N^s$	$S-N^s$
O_5	$1;\ \frac{d}{1-d}$	S	N^u	S	S	S
I_1	$-1;\ -1$	N^s	N^s	N^s	N^s	N^s
I_2	$d-1;\ d-1$	N^s	N^s	N^u	N^u	N^u
I_3	$-1;\ 2-d$	S	S	S	$S-N^s$	N^s
I_4	$-1;\ \frac{2-d}{1-d}$	S	S	N^s	$-$	S
$Fig.\,3:$		1)	2)	3)	4)	5)

4.4. System (7). This system has the singular points: $O_1(0,0)$, $O_2(-1,0)$, $O_3(-1,-1)$, $O_4(0,\frac{1}{d-1})$, $I_1(1,0,0)$, $I_2(0,1,0)$, $I_3(1,1,0)$, $I_4(1,\frac{1}{d-1},0)$, whose characterizations are given in Tab. 4.6.

Tab. 4.6

SP	$\lambda_1;\ \lambda_2$	$d<1,\ d\neq 0$	$1<d<2$	$d=2$	$d>2$
O_1	$0;\ 0$	P^uHP^sH	P^uHP^sH	P^uHP^sH	P^uHP^sH
O_2	$0;\ 1$	$S-N^u$	$S-N^u$	$S-N^u$	$S-N^u$
O_3	$1;\ 1-d$	N^u	S	S	S
O_4	$0;\ 1/(1-d)$	$S-N^u$	$S-N^s$	$S-N^s$	$S-N^s$
I_1	$-1;\ -1$	N^s	N^s	N^s	N^s
I_2	$d-1;\ d-1$	N^s	N^u	N^u	N^u
I_3	$-1;\ 2-d$	S	S	$S-N^s$	N^s
I_4	$-1;\ \frac{2-d}{1-d}$	S	N^s	$-$	S
$Fig.\,4:$		1)	2)	3)	4)

As in the case of system (6), the behavior of the trajectories in the neighborhood of singular point $O_1(0,0)$ was established by using the blow-up method for (7) in polar coordinates:

$$\begin{cases} \dfrac{d\rho}{d\tau} = \rho[\cos^3\theta(1+\rho\cos\theta) + \sin^3\theta(1+d\rho\cos\theta+(1-d)\rho\sin\theta)], \\ \dfrac{d\theta}{d\tau} = \sin\theta\cos\theta(\sin\theta-\cos\theta)(1+\rho\cos\theta+(1-d)\rho\sin\theta)), \end{cases} \tag{32}$$

where $\tau=\rho t$. The singular points of (32) with $\rho=0$ and $\theta\in[0,2\pi)$ and their eigenvalues: $\{M_1(0,0), M_2(0,\pi), M_3(0,\pi/2), M_4(0,3\pi/2)\ :\ \lambda_1=-1, \lambda_2=1\}$; $\{M_5(0,\pi/4)\ :\ \lambda_1=\lambda_2=1/\sqrt{2}\}$; $\{M_6(0,5\pi/4)\ :\ \lambda_1=\lambda_2=-1/\sqrt{2}\}$, lead us to Fig. 4.3.

4.5. System (8). This system has in finite parts of the phase plane a singular point $O(0,0)$ with $\lambda_1=\lambda_2=0$ and at infinity singular points $I_1(1,0,0)$; $I_2(0,1,0)$;

$I_3(1,1,0)$ with $\lambda_1 = \lambda_2 = -1$; $\lambda_1 = \lambda_2 = 1$; $\lambda_1 = -1, \lambda_2 = 0$. We have that I_1 is N^s; $I_2 - N^u$; $I_3 - S$–N^s and $O - P^uHHP^uHH$ (see Fig. 5).

4.6. System (9). For (9) the singular points and the eigenvalues of the characteristic equation are shown in Tab. 4.7. In this table we used the notations: $\beta = a - 1$, $\gamma = 2a - 1$.

Tab. 4.7

SP	$O_1(0,0)$	$O_2(-1,0)$	$O_3(-1,-1)$	$O_4(1,0)$
$\lambda_1;\lambda_2$	$a;a$	$-2a;2\gamma$	$-\gamma;-\gamma$	$-2\beta;-2a$
SP	$O_5(-1,-2)$	$O_6(0,-1)$	$O_7(\beta/a,-1)$	$O_8(a/\beta,a/\beta)$
$\lambda_1;\lambda_2$	$-2\beta;2\gamma$	$\beta;\beta$	$-\beta\gamma/a;2\beta\gamma/a$	$-a\gamma/\beta;2a\gamma/\beta$
SP	$O_9(0,a/\gamma)$	$I_1(1,0,0)$	$I_2(0,1,0)$	$I_3(1,1/\beta,0)$
$\lambda_1;\lambda_2$	$-a\beta/\gamma;2a\beta/\gamma$	$a;a$	$\gamma;\gamma$	$\beta;\beta$
SP	$I_4(1,a/\gamma,0)$			
$\lambda_1;\lambda_2$	$-a\beta/\gamma;\ 2a\beta/\gamma$			

We divide the real axis in intervals $J_1 = (-\infty,0)$, $J_2 = (0,1/3)$, $J_3 = (1/3,1/2)$, $J_4 = (1/2,2/3)$, $J_5 = (2/3,1)$, $J_6 = (1,+\infty)$; $J = J_1 \cup J_2 \cup \dots \cup J_6$.

The eigenvalues λ_1 and λ_2 of the characteristic equation corresponding to each singular point, in intervals J_1 and J_6 differ only by sign. Therefore, from the qualitative point of view the phase portraits of system (9) in intervals J_1 and J_6, differ only by directions on trajectories.

Singular points O_7, O_8, O_9 and I_4 are saddles for every $a \in J$. The types of other singular points (i.e. $O_1 - O_6, I_1, I_2, I_3$) are shown in Tab. 4.8.

Tab. 4.8

SP	$J_1\ (J_6)$	$J_2,\ J_3$	$J_4,\ J_5$
O_1	$N^{s(u)}$	N^u	N^u
O_2	S	N^s	S
O_3	$N^{u(s)}$	N^u	N^s
O_4	$N^{u(s)}$	S	S
O_5	S	S	N^u
O_6	$N^{s(u)}$	N^s	N^s
I_1	$N^{s(u)}$	N^u	N^u
I_2	$N^{u(s)}$	N^u	N^s
I_3	$N^{s(u)}$	N^s	N^s
$Fig. 6:$	$1 \rightleftarrows)$	2)	3)

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