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Nash equilibria in the noncooperative informational extended games

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Abstract. In this article* we will analyse informational extended games, i.e. games in which the players choose their actions simultaneously, with assumption that they have some information about the future strategies which will be chosen by other players. All informational extended games of this type will assume that players' payoff functions are common knowledge. Under these assumptions the last section will define the informational extended games and analyse Nash equilibrium and conditions of its existence. The essential result of this article is a theorem of Nash equilibrium existence in informational extended games with n players. Our treatment is based on a standard fixed point theorem which will be stated without proof in the first section.

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1 Preliminary facts

1.1 Fixed points and contraction mappings

Consider the function $f : X \to X$. An element $x \in X$ is called a fixed point of f if $f(x) = x$.

The fixed points of the function f are the intersection points of the graph of f with the product $X \times X$.

Properties of fixed points.

1. If there are two functions f and g from X into Y, then the point $x^* \in X$ for which $f(x^*) = g(x^*)$, is called [2] point of coincidence for the functions f and g.

2. Sometimes it is convenient to use the cyclic points of the function f together with the fixed points, especially in the case when fixed points do not exist. Cyclic points are the points which are images of the iterative function f^n, where n is a natural number. These are cyclic points of the $n-$th order. Often such points do not exist and in these cases we can use boundary cycles. Also we can speak about the invariant sets, i.e. subsets $Y \subset X$, for which $f(Y) = Y$. In such cases the minimal invariant subsets are very important.

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Next the notation $F : X \rightrightarrows 2^Y$ will denote a point-to-set mapping, were 2^Y denotes the set of all subsets of Y. A fixed point of the point-to-set mapping $F : X \rightrightarrows 2^Y$ is a point $x^* \in X$ such that $x^* \in F(x^*)$.

The graph for the application F is the set
$gr(F) = \{(x, y) \in X \times Y \,|x \in X, y \in F(x)\}$. This set can contain some points or can be the empty set.

1.2 The Kakutani fixed point theorem

The existence of the fixed points is considered an important problem. The existence (and other properties) of the fixed point for the function $f : X \to X$ depends on the properties of f and on the properties of the space X. Often it is considered that f is a continuous function.

Definition 1.1. *The function f of the metric space into itself is called [2] contraction mapping if there exists a constant $K < 1$ such that for each two points x and y the inequality $\rho(f(x), f(y)) \leq K\rho(x, y)$ holds, where ρ is the metrics of the space.*

There are some important properties for the fixed points.

Proposition 1.1. *If f is a contraction mapping, then there exists not more than a single fixed point [1, 2].*

Theorem 1.1. *(Principle of the contraction mapping). Consider that f is a contraction mapping of the complete metric space X into itself. Then for each point $x \in X$ the sequence $x, f(x), f^2(x) = f(f(x)), f^3(x), \ldots$ converges to a fixed point. So f has a single fixed point [1, 2].*

The points $x, f(x), f^2(x), \ldots$ are called consequent approximations of the fixed point.

In the case of the contraction mapping we can consider as a start element every element x and the consecutive approximations converge to the fixed point.

The Kakutani fixed point theorem is a fixed-point theorem for point-to-set mapping. It provides sufficient conditions for a point-to-set mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. Kakutani theorem extends this to point-to-set mapping.

The theorem was developed by Shizuo Kakutani in 1941 and was famously used by John Nash in his description of Nash equilibrium. It has subsequently found widespread application in game theory and economics. Many problems in economy appear as problems of maximization and usually the solution of such problems is many-valued.

Before giving this theorem we need to recall some definitions and theorems.

Definition 1.2. *Consider topological spaces X and Y. A point-to-set mapping $F : X \rightrightarrows 2^Y$ is said to be closed if the graph of F is closed as a subset into the product of the spaces $X \times Y$.*

That is if the sequence of points (x_n, y_n) from $gr\,(F)$ converges to a point $(x, y) \in X \times Y$, then the limit point $(x, y) \in gr\,(F)$ [2].

Theorem 1.2 (Kakutani, 1941). *Let X be a Banach space and K a non-empty, compact and convex subset of X. Let $F : K \rightrightarrows 2^K$ be a point-to-set mapping on K with a closed graph and the property that the set $F\,(x)$ is non-empty and convex for all $x \in K$. Then F has a fixed point.*

For proof see [1].

Before giving the applications of the fixed points in the game theory we will recall some other important theorems.

Let $C\,(K)$ be the space of all continuous functions defined on the compactum K.

Theorem 1.3 (Arzelà-Ascoli). *(Compactness criterion). A set of continuous functions $E \subseteq C\,(K)$ is compact if and and only if the set E is uniformly bounded: ($|x\,(t)\,| \leqslant M, \forall t \in K$, for $\forall x \in E$) and the functions from the set E are equicontinuous (i.e. for $\forall \varepsilon, \exists \delta$ so that if $\rho\,(t_1, t_2) < \delta$ then $|x\,(t_1) - x\,(t_2)\,| < \varepsilon$ for $\forall x \in E$).*

Theorem 1.4 (Tikhonov). *A product of a family of compact topological spaces $X = \prod\limits_{\alpha \in A} X_\alpha$ is compact.*

Lemma 1.1. *1) If X and Y are two compacta with the same metric, $f : X \to Y$ is a continuous function, then the set $Arg \max\limits_{x \in X} f\,(x) = \left\{ x \in X \,\middle|\, f\,(x) = \max\limits_{z \in X} f\,(z) \right\}$ is compact too (see [3]).*

2) If X and Y are two compacta with the same metric, and $K\,(x, y)$ is a continuous function on $X \times Y$, then $\varphi\,(y) = \max\limits_{x \in X} K\,(x, y)$ and $\psi\,(x) = \min\limits_{y \in Y} K\,(x, y)$ are continuous functions on Y and X respectively [3].

2 Strategic form games and Nash equilibria

In this part we will analyse games in which the players choose their actions simultaneously (without the knowledge of other player choices). The game will assume that players' payoff functions are common knowledge.

Definition 2.1. *A strategic form of the game consists of: a finite set of players $I = \{1, 2, \ldots, n\}$, action spaces (set of strategies) of players, denoted by $X_i, i \in I$; and payoff functions of players $H_i : X \to R, i \in I$, where $X = X_1 \times \cdots \times X_n$. We refer to such a game as the tuple $< I, (X_i)_{i \in I}\,, (H_i)_{i \in I} >$ denoted by Γ.*

An outcome is an action profile $(x_1, x_2, \ldots, x_n)$, and the outcome space is $X = \times_{i \in I} X_i$. The game is common knowledge among the players.

One of the most common interpretations of Nash equilibrium (introduced by John Nash in 1950) is that it is a steady state in the sense that no rational player has an incentive to unilaterally deviate from it. Let $x_{-i} \equiv (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(x_{-i}, y_i) \equiv (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

Definition 2.2. *A Nash equilibrium of the game* Γ *is an action profile* $x^* \in X$ *such that for every* $i \in I$

$$H_i (x^*) \geqslant H_i \left(x^*_{-i}, x_i\right) \textit{ for all } x_i \in X_i.$$

Another and sometimes more convenient way of defining Nash equilibrium is via the best response correspondences $Br_i : \underset{j \in I \setminus \{i\}}{\times} X_j \rightrightarrows X_i$ such that

$$Br_i (x_{-i}) = \left\{x_i \in X_i : H_i (x) \geqslant H_i \left(x_{-i}, x_i'\right) \text{ for } \forall x_i' \in X_i\right\}. \tag{*}$$

Definition 2.3. *A Nash equilibrium is an action profile* x^**such that* $x_i^* \in Br_i \left(x^*_{-i}\right)$ *for all* $i \in I$.

If the sets X_i are compacts and the functions H_i are continuous, then the best response set (*) for the player i can be represented by:

$$Br_i (x_{-i}) = Arg \max_{x_i \in X_i} H_i (x_{-i}, x_i).$$

Given a strategic form of the game $\Gamma \equiv < I, (X_i)_{i \in I}, (H_i)_{i \in I} >$, the set of Nash equilibria is denoted by $NE(\Gamma)$.

Using the best response sets of the players we consider the point-to-set mapping $Br : \underset{i \in I}{\times} X_i \rightrightarrows 2^X$ such that $Br = (Br_1, Br_2, \ldots, Br_n)$.

Then we can easily prove that $x^* \in NE(\Gamma) \rightleftarrows x^*$ is a fixed point of the set-valued mapping Br, i.e. $x^* \in Br(x^*)$.

3 Nash equilibria in the noncooperative informational extended games with n players

We analyse a static game with n players:

$$\Gamma = \left\langle I, X_i, i = \overline{1,n}, H_i, i = \overline{1,n}\right\rangle \tag{1}$$

where $I = \{1, 2, \ldots, n\}$ is the set of the players, the set of strategies for the i-th player is denoted by X_i, $\left(i = \overline{1,n}\right)$, and the payoff functions are defined by: $H_i : \prod_{i \in I} X_i \to R$, $\left(i = \overline{1,n}\right)$.

Next we will analyse a static informational extended game with n players. In this informational extended game we will consider that each player is informed of the strategies of the other players which will be chosen. In this case the sets of the strategies for each player will be a set of functions defined on the product of the sets of strategies of the rest players from the initial game (1).

The game is realised as follows: the strategies are chosen simultaneously by players (with assumption that each of them knows which strategies will be chosen by all other players), after that each of players determines his payoff and the game is over.

This informational extended game can be described in the normal form by:

$$ {}_n\Gamma = \left\langle I, \overline{X}_i, i=\overline{1,n}, \overline{H}_i, i=\overline{1,n} \right\rangle, $$

where the sets of the strategies for the players are defined by:

$$ \overline{X}_i = \left\{ \varphi_i : \prod_{j\in I, j\neq i} X_j \to X_i \right\}, i=\overline{1,n}, $$

where $\prod\limits_{j\in I,j\neq i} X_j = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$.

The payoff functions are defined on the product of the extended sets of strategies: $\overline{H}_i : \prod\limits_{i\in I} \overline{X}_i \to R,\ \left(i=\overline{1,n}\right)$.

In this case we analyse the informational extended game in which we consider that all players know the chosen strategies of all other players and each player $i\in I$ chooses his strategy from the set $\overline{X}_i$.

If some players do not know which strategies other players will choose, then those players $j\in I$ will choose their strategies from their initial sets X_j. Thus we can define some different informational extended games in which the outcome will consist of strategies $x_j \in X_j, j\in J$ and $\varphi_k \in \overline{X}_k, k\in I\backslash J$, where J is the set of players which do not have some information about chosen strategies of other players.

We denote by $C\left(\prod\limits_{j\in I,j\neq i} X_j, X_i\right), \left(i=\overline{1,n}\right)$ the space of all continuous functions from $\prod\limits_{j\in I,j\neq i} X_j$ into X_i, were $\prod\limits_{j\in I,j\neq i} X_j$ and X_i are compacta.

Next we will apply the fixed point theorem to prove the following theorem of the Nash equilibrium existence for the informational extended game ${}_n\Gamma$ with n players.

Theorem 3.1. *Let us consider that for the game ${}_n\Gamma$ the next conditions hold:*

1) the sets $X_i \neq \emptyset, \left(i=\overline{1,n}\right)$ are compacta of Banach spaces,

2) the sets of functions $\overline{X}_i \subset C\left(\prod\limits_{j\in I,j\neq i} X_j, X_i\right), \left(i=\overline{1,n}\right)$ are uniformly bounded and the functions from the sets $\overline{X}_i$ are equicontinuous;

3) the payoff functions $H_i(\cdot),\ \left(i=\overline{1,n}\right)$ are continuous on the compactum $\prod\limits_{i\in I} X_i$ and the functions $\overline{H}_i(\cdot), \left(i=\overline{1,n}\right)$ are concave on $\overline{X}_i$ for $\forall\varphi_{-i}$, respectively.

Then $NE\left({}_n\Gamma\right) \neq \emptyset$.

Proof. Let $\overline{X} = \prod\limits_{i\in I} \overline{X}_i$ be the outcome space. According to Arzelà-Ascoli theorem the sets $\overline{X}_i, (i \in I)$ are compact, and according to Tikhonov theorem the outcome space $\overline{X}$ is a compactum too.

Let us denote an outcome of the extended game by $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in \overline{X} = \prod\limits_{i\in I} \overline{X}_i$, where $\varphi_i = \varphi_i (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \overline{X}_i$.

Later we will use the next notations: $\varphi_{-i} = (\varphi_1, \varphi_2, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_n)$, $\overline{X}_{-i} = \prod\limits_{\substack{j\in I\\ j\neq i}} \overline{X}_j$.

Since the payoff functions $H_i(\cdot)$, $\left(i = \overline{1,n}\right)$ are continuous on the compact $\prod\limits_{i\in I} X_i$ (from the third condition of the theorem) and because the functions $\varphi_i \in \overline{X}_i$ are continuous on the compact $\prod\limits_{j\in I, j\neq i} X_j$, then the functions $\overline{H}_i, i = \overline{1,n}$ are continuous on the compact $\prod\limits_{i\in I} \overline{X}_i$ as compound functions of continuous functions $\overline{H}_i(\varphi) = H_i(\varphi(x))$.

We define the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$, such that $B(\varphi) = (B_1(\varphi_{-1}), B_2(\varphi_{-2}), \dots, B_n(\varphi_{-n}))$, where $B_i(\varphi_{-i}), (i \in I)$ represents the best response set for the player i for the chosen strategies of all players $j \in I \setminus \{i\}$.

Because the sets $\overline{X}_i, (i \in I)$ are compacts and $\overline{H}_i$, for $i = \overline{1,n}$ are continuous functions, then according to the Weierstrass theorem we can write:

$$B_i(\varphi_{-i}) = Arg \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, \dots, \varphi_n),$$

i. e.:

$$B_i(\varphi_{-i}) = \left\{ \varphi_i \in \overline{X}_i : \overline{H}_i(\varphi_1, \varphi_2, \dots, \varphi_n) = \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, \dots, \varphi_n) \right\}, \left(i = \overline{1,n}\right).$$

In order to use the Kakutani theorem we need to prove that:

1) $\overline{X} = \prod\limits_{i\in I} \overline{X}_i \neq \emptyset$ is a non-empty convex compact set;

2) for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$ the next conditions hold:
   a) for $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$ the set $B(\varphi) \neq \emptyset$ is a convex subset of $\overline{X}$;
   b) the point-to-set mapping B is closed.

Firstly we will prove that $\overline{X}$ is convex and compact.

The set $\overline{X}_i, (i \in I)$ is convex if: for $\forall \varphi_i', \varphi_i'' \in \overline{X}_i$, and $\lambda \in [0,1]$ the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is bounded by the same constant N (see Arzelà-Ascoli theorem) and the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is equicontinuous.

It is easy to prove that the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is bounded by the same constant N:
$|\lambda\varphi_i'(x_{-i}) + (1-\lambda)\varphi_i''(x_{-i})| \leqslant \lambda |\varphi_i'(x_{-i})| + (1-\lambda) |\varphi_i''(x_{-i})| \leqslant \lambda N + (1-\lambda) N = N$ for all $\varphi_i', \varphi_i'' \in \overline{X}_i$, and $\lambda \in [0,1]$.

Evidently the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is equicontinuous. So the set $\overline{X}_i, (i \in I)$ is convex. Then the set $\overline{X}$ is convex and compact too.

Next we need to prove that for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$ the conditions a) and b) hold.

Firstly we will prove the condition a). For $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$ the set $B\left(\varphi\right)$ is non-empty, this follows from the Weierstrass theorem, because $B_i\left(\varphi_{-i}\right), \ \forall i \in I$ are non-empty sets.

Next we need to prove that the set $B\left(\varphi\right)$ is convex for $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$.

So we will prove that the sets $B_i\left(\varphi_{-i}\right), \forall i = \overline{1,n}$ are convex.

The function $\overline{H}_i\left(\varphi_1, \varphi_2, \ldots, \varphi_n\right) = \overline{H}_i\left(\varphi_i, \varphi_{-i}\right)$ is concave on the compact set $\overline{X}_i \subset C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right), \ (i \in I)$, then by definition for $\forall \lambda \in [0,1]$, and $\forall \varphi_i', \varphi_i'' \in \overline{X}_i$ the relation $\overline{H}_i\left(\lambda \varphi_i' + (1-\lambda)\varphi_i'', \varphi_{-i}\right) \geqslant \lambda \overline{H}_i\left(\varphi_i', \varphi_{-i}\right) + (1-\lambda)\overline{H}_i\left(\varphi_i'', \varphi_{-i}\right)$ holds.

For $\forall \varphi_{-i}$ the set $B_i\left(\varphi_{-i}\right)$ will be convex since the function $\overline{H}_i\left(\cdot\right)$ is continuous on $\overline{X}_i$ and $\overline{H}_i\left(\cdot\right)$ is concave by φ_i, for $\forall i = \overline{1,n}$.

From what was proved it follows that for $\forall \varphi_i \in \overline{X}_i, \ \left(i = \overline{1,n}\right)$ we will have a convex subset $B\left(\varphi\right) = \left(B_1\left(\varphi_{-1}\right), B_2\left(\varphi_{-2}\right), \ldots, B_n\left(\varphi_{-n}\right)\right) \neq \emptyset$ from $\overline{X} = \prod\limits_{i \in I} \overline{X}_i$.

Next we will prove the condition b). We need to prove that the point-to-set mapping B is closed.

The point-to-set mapping B is closed if its graph is a closed set [4]. Since $B_i\left(\varphi_{-i}\right)$ is a subset from the compactum $\overline{X}_i$ for all $i = \overline{1,n}$, then $grB_i\left(\varphi_{-i}\right), \left(i = \overline{1,n}\right)$ are compact sets. Here the graph for $B_i\left(\varphi_{-i}\right)$ is defined by:

$$\begin{aligned} grB_i\left(\varphi_{-i}\right) &= \left\{ \left(\varphi_1, \varphi_2, \ldots, \varphi_n\right) \in \overline{X} \,\middle|\, \varphi_i \in Arg \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i\left(\varphi_i', \varphi_{-i}\right), \varphi_{-i} \in \overline{X}_{-i} \right\} = \\ &= \left\{ \left(\varphi_1, \ldots, \varphi_n\right) \in \overline{X} \,\middle|\, \varphi_i \in B_i\left(\varphi_{-i}\right), \varphi_j \in \overline{X}_j, j \in I, j \neq i \right\}. \end{aligned}$$

We will prove that for the chosen strategies φ_{-i} the sets $B_i\left(\varphi_{-i}\right), i = \overline{1,n}$, are closed.

The set $B_i\left(\varphi_{-i}\right)$ can be rewritten as follows:

$$B_i\left(\varphi_{-i}\right) = \left\{ \varphi_i \in \overline{X}_i : \overline{H}_i\left(\varphi_i, \varphi_{-i}\right) - \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i\left(\varphi_i', \varphi_{-i}\right) = 0 \right\}.$$

Because the set $\overline{X}_i$ is compact and the function $\overline{H}_i$ is continuous on X, then the function $\overline{H}_i\left(\varphi_i, \varphi_{-i}\right) - \max\limits_{\varphi_i' \in \overline{X}_i} \overline{H}_i\left(\varphi_i', \varphi_{-i}\right)$ is continuous on $\overline{X}_i$ too. So for $\forall \varphi_{-i}$, the set $B_i\left(\varphi_{-i}\right) \subset \overline{X}_i$ is closed (and compact).

Then according to the Tikhonov theorem, because $grB_i\left(\varphi_{-i}\right)$ is a closed set for all $i = \overline{1,n}$, so it follows that

$$grB = \left\{ \left(\varphi_1, \ldots, \varphi_n\right) \in \overline{X} \,\middle|\, \varphi_i \in B_i\left(\varphi_{-i}\right), \forall i = \overline{1,n} \right\}$$

is a closed set too.

Thus the point-to-set mapping B is closed.

Therefore we can apply the Kakutani theorem.

Let $\varphi^* = (\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) \in \overline{X} = \prod_{i \in I} \overline{X}_i$ be a fixed point for the point-to-set mapping B, i.e. $(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) \in B(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) = \prod_{i \in I} B_i(\varphi_{-i})$, so the relation

$$\overline{H}_i(\varphi_1^*, \ldots, \varphi_i^*, \ldots, \varphi_n^*) = \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, \ldots, \varphi_i, \ldots, \varphi_n^*)$$

holds for all $i = \overline{1, n}$, thus by definition of the Nash equilibrium it follows that $(\varphi_1^*, \ldots, \varphi_i^*, \ldots, \varphi_n^*) \in NE({}_n\Gamma) \neq \emptyset$. $\square$

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