BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(59), 2009, Pages 79–86
ISSN 1024–7696

The distribution of a planar random evolution with random start point

Alexander D. Kolesnik

Abstract. We consider the symmetric Markovian random evolution $\mathbf{X}(t)$ in the Euclidean plane $\mathbb{R}^2$ starting from a random point whose coordinates are the independent standard Gaussian random variables. The integral and series representations of the transition density of $\mathbf{X}(t)$ are obtained.

Mathematics subject classification: 60K35; 60K37; 82B41; 82C70.
Keywords and phrases: Random motion, finite speed, random evolution, random flight, transport process, distribution, Bessel function, Gaussian density, random start point.

The planar random motion at finite speed was dealt with in a series of works [2–4]. In these works the following planar stochastic motion was studied. A particle starts from the origin $\mathbf{0} = (0, 0)$ of the plane $\mathbb{R}^2$ at time $t = 0$ and moves with constant finite speed c. The initial direction is a two-dimensional random vector with uniform distribution on the unit circumference

$$S(\mathbf{0}, 1) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 = 1 \right\} .$$

The particle changes its direction at random instants that form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on $S(\mathbf{0}, 1)$, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t))$ denote the particle's position at an arbitrary instant $t > 0$. At any time $t > 0$ the particle, with probability 1, is located in the planar disc of radius ct

$$\mathbf{B}(\mathbf{0}, ct) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 \le c^2t^2 \right\} .$$

Let $d\mathbf{x}$ be the infinitesimal element of the plane $\mathbb{R}^2$ with the Lebesgue measure $\mu(d\mathbf{x}) = dx_1dx_2$. The distribution $Pr \left\{ \mathbf{X}(t) \in d\mathbf{x} \right\}, \ \mathbf{x} \in \mathbf{B}(\mathbf{0}, ct), \ t \ge 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the circumference

$$S(\mathbf{0}, ct) = \partial\mathbf{B}(\mathbf{0}, ct) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 = c^2t^2 \right\} .$$

In this case, in the moment t, the particle is located on the sphere $S(\mathbf{0}, ct)$ and the probability of this event is

$$Pr \left\{ \mathbf{X}(t) \in S(\mathbf{0}, ct) \right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the disc $\mathbf{B}(\mathbf{0}, ct)$, and the probability of this event is

$$Pr\,\{\mathbf{X}(t) \in \text{int}\,\mathbf{B}(\mathbf{0}, ct)\} = 1 - e^{-\lambda t}.$$

The part of the distribution $Pr\,\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$\text{int}\,\mathbf{B}(\mathbf{0}, ct) = \left\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \; \|\mathbf{x}\|^2 = x_1^2 + x_2^2 < c^2t^2\right\},$$

and forms its absolutely continuous component. Therefore there exists the density of the absolutely continuous component of the distribution $Pr\,\{\mathbf{X}(t) \in d\mathbf{x}\}$.

The principal known result states that the complete density $f(\mathbf{x}, t)$ of the process $\mathbf{X}(t)$ (starting from the origin $\mathbf{0}$), has the form

$$f(\mathbf{x}, t) = \frac{e^{-\lambda t}}{2\pi ct}\,\delta(c^2t^2 - \|\mathbf{x}\|^2) + \frac{\lambda}{2\pi c}\frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x}\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x}\|^2}}\Theta(ct - \|\mathbf{x}\|), \quad (1)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}(\mathbf{0}, ct), \quad \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}, \quad t \geq 0,$$

where $\delta(x)$ is the Dirac delta-function and $\Theta(x)$ is the Heaviside step function. The first term in (1) represents the density of the singular part of the distribution concentrated on the sphere $S(\mathbf{0}, ct)$, while the second term is the density of the absolutely continuous part of the distribution concentrated in int $\mathbf{B}(\mathbf{0}, ct)$.

If the process $\mathbf{X}(t)$ starts from some arbitrary fixed point $\mathbf{x}^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$, then, given the phase space $\mathbb{R}^2$ is isotropic and homogeneous, the density of $\mathbf{X}(t)$ has the form

$$f(\mathbf{x} - \mathbf{x}_0, t) = \frac{e^{-\lambda t}}{2\pi ct}\,\delta(c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2) +$$

$$+ \frac{\lambda}{2\pi c}\frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2}}\Theta(ct - \|\mathbf{x} - \mathbf{x}^0\|), \quad (2)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}(\mathbf{x}^0, ct), \quad \|\mathbf{x} - \mathbf{x}^0\| = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}, \quad t \geq 0.$$

Suppose that the start point $\mathbf{x}^0 = (x_1^0, x_2^0)$ is a two-dimensional random variable (random vector) with given density $p(\mathbf{x})$ on the plane $\mathbb{R}^2$. If the random vectors $\mathbf{X}(t)$ and $\mathbf{x}^0$ are independent for any $t > 0$, then the density of $\mathbf{X}(t)$ is given by the convolution

$$\varphi(\mathbf{x}, t) = f(\mathbf{x}, t) * p(\mathbf{x}) = \int\limits_{\mathbb{R}^2} f(\mathbf{x} - \boldsymbol{\xi}, t)\; p(\boldsymbol{\xi})\; \mu(d\boldsymbol{\xi}). \quad (3)$$

In this paper we obtain a closed-form expression for density (3) when the initial point $\mathbf{x}^0 = (x_1^0, x_2^0)$ is a two-dimensional standard Gaussian vector with independent coordinates. In this case the density $p(\mathbf{x})$ has the form

$$p(\mathbf{x}) = p(x_1, x_2) = \frac{1}{2\pi}\,\exp\left(-\frac{x_1^2 + x_2^2}{2}\right). \quad (4)$$

Due to the fairly simple form of function (4) we are able to obtain the density of the process $\mathbf{X}(t)$ starting from a Gaussian random point of the Euclidean plane $\mathbb{R}^2$.

First, we will prove two auxiliary lemmas.

Lemma 1. *For arbitrary $q > 0$ and any integer $n \geq 0$ the following formula holds*

$$\int_0^1 x^n \, I_0(q\sqrt{1-x^2}) \, dx = 2^{(n-1)/2} \, \Gamma\left(\frac{n+1}{2}\right) \frac{I_{(n+1)/2}(q)}{q^{(n+1)/2}}, \tag{5}$$

where $I_\nu(x)$ is the Bessel function of order ν with imaginary argument given by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{6}$$

Proof. Making the substitution $z = \sqrt{1-x^2}$ in the integral on the left-hand side of (5), we obtain

$$\begin{aligned}
\int_0^1 x^n \, I_0(q\sqrt{1-x^2}) \, dx &= \int_0^1 z \, (1-z^2)^{(n-1)/2} \, I_0(qz) \, dz = \\
&= \frac{1}{2} \int_0^1 (1-\xi)^{(n-1)/2} \, I_0(q\sqrt{\xi}) \, d\xi = \\
&= \frac{1}{2} \int_0^1 (1-\xi)^{(n-1)/2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q\sqrt{\xi}}{2}\right)^{2k} d\xi = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} \int_0^1 \xi^k \, (1-\xi)^{(n-1)/2} \, d\xi = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} B\left(\frac{n+1}{2}, k+1\right) = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} \frac{\Gamma\left(\frac{n+1}{2}\right) \, \Gamma(k+1)}{\Gamma\left(\frac{n+1}{2}+k+1\right)} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma\left(\frac{n+1}{2}+k+1\right)} \left(\frac{q}{2}\right)^{2k} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2}{q}\right)^{(n+1)/2} \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma\left(\frac{n+1}{2}+k+1\right)} \left(\frac{q}{2}\right)^{2k+(n+1)/2} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2}{q}\right)^{(n+1)/2} I_{(n+1)/2}(q) = \\
&= 2^{(n-1)/2} \, \Gamma\left(\frac{n+1}{2}\right) \frac{I_{(n+1)/2}(q)}{q^{(n+1)/2}}.
\end{aligned}$$

The lemma is proved. □

Lemma 2. *For arbitrary $a > 0,\ b > 0$ and $q > 0$ the following formula holds*

$$\int_0^1 e^{ax^2+bx}\ I_0(q\sqrt{1-x^2})\ dx = \\ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{n-k}\ b^k}{k!\ (n-k)!}\ 2^{(2n-k-1)/2}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(q)}{q^{(2n-k+1)/2}}, \tag{7}$$

where $I_\nu(x)$ is the Bessel function of order ν with imaginary argument given by (6).

Proof. By expanding the exponential and applying formula (5) of Lemma 1, we obtain

$$\int_0^1 e^{ax^2+bx}\ I_0(q\sqrt{1-x^2})\ dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 (ax^2+bx)^n\ I_0(q\sqrt{1-x^2})\ dx =$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n} C_n^k\ a^{n-k}\ b^k \int_0^1 x^{2n-k}\ I_0(q\sqrt{1-x^2})\ dx =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{n-k}\ b^k}{k!\ (n-k)!}\ 2^{(2n-k-1)/2}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(q)}{q^{(2n-k+1)/2}},$$

proving (7). The lemma is proved. □

The series on the right-hand side of (7) has a fairly complicated form and seemingly cannot be reduced to a more elegant expression. Nevertheless, it enables us to obtain a series representation of the transition density of $\mathbf{X}(t)$.

Now we are able to establish our main result. It is given by the following theorem.

Theorem 1. *The transition density of the planar random evolution $\mathbf{X}(t)$ started from a random point $\mathbf{x}^0$ with Gaussian density (4) is given by the formula*

$$\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi}\ e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\ I_0(ct\|\mathbf{x}\|) + \\ + \frac{\lambda t\ e^{-\lambda t}}{2\pi}\ e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi}\ I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2})\ d\xi. \tag{8}$$

The density (8) has the following series representation

$$\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi}\ e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\ I_0(ct\|\mathbf{x}\|) + \frac{\lambda t\ e^{-\lambda t}}{2\pi}\ e^{-\|\mathbf{x}\|^2/2}\times \\ \times \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda t)^k\ 2^{(k+1)/2}}{k!\ (n-k)!}\ (c^2t^2)^{n-k}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}}. \tag{9}$$

Proof. According to (3) and taking into account (2) and (4), we have

$$\begin{aligned}
\varphi(\mathbf{x},t) &= \varphi(x_1,x_2,t) = \\
&= \frac{e^{-\lambda t}}{4\pi^2 ct} \iint\limits_{\mathbb{R}^2} \exp\left(-\frac{\xi_1^2+\xi_2^2}{2}\right) \delta(c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2)\, d\xi_1\, d\xi_2 + \\
&+ \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \iint\limits_{\mathbb{R}^2} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}\right)}{\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}} \exp\left(-\frac{\xi_1^2+\xi_2^2}{2}\right) \times \\
&\qquad \times \Theta\left(ct-\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}\right) d\xi_1\, d\xi_2 = \\
&= \frac{e^{-\lambda t}}{4\pi^2 ct} \iint\limits_{\mathbb{R}^2} \exp\left(-\frac{(x_1-\xi_1)^2+(x_2-\xi_2)^2}{2}\right) \delta(c^2t^2-(\xi_1^2+\xi_2^2))\, d\xi_1\, d\xi_2 + \\
&+ \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \iint\limits_{\mathbb{R}^2} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-(\xi_1^2+\xi_2^2)}\right)}{\sqrt{c^2t^2-(\xi_1^2+\xi_2^2)}} \exp\left(-\frac{(x_1-\xi_1)^2+(x_2-\xi_2)^2}{2}\right) \times \\
&\qquad \times \Theta\left(ct-\sqrt{\xi_1^2+\xi_2^2}\right) d\xi_1\, d\xi_2.
\end{aligned}$$

By changing to the polar coordinates $\xi_1 = \rho\cos\alpha,\ \xi_2 = \rho\sin\alpha$, in both integrals, we obtain

$$\begin{aligned}
\varphi(\mathbf{x},t) &= \frac{e^{-\lambda t}}{4\pi^2 ct} \int_0^\infty d\rho \Bigg\{ \rho\, \delta(c^2t^2-\rho^2) \times \\
&\qquad \times \int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2+(x_2-\rho\sin\alpha)^2}{2}\right) d\alpha \Bigg\} + \\
&+ \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \int_0^\infty d\rho \Bigg\{ \frac{\rho\, \exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}}\, \Theta(ct-\rho) \times \\
&\qquad \times \int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2+(x_2-\rho\sin\alpha)^2}{2}\right) d\alpha \Bigg\}.
\end{aligned} \tag{10}$$

Let's evaluate separately the interior integral in (10):

$$\begin{aligned}
&\int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2+(x_2-\rho\sin\alpha)^2}{2}\right) d\alpha = \\
&\quad = \int_0^{2\pi} \exp\left(-\frac{1}{2}\left[x_1^2+x_2^2+\rho^2-2\rho(x_1\cos\alpha+x_2\sin\alpha)\right]\right) d\alpha = \\
&\quad = e^{-(x_1^2+x_2^2+\rho^2)/2} \int_0^{2\pi} e^{\rho(x_1\cos\alpha+x_2\sin\alpha)}\, d\alpha = \\
&\quad = 2\pi\, e^{-(\|\mathbf{x}\|^2+\rho^2)/2}\, I_0(\rho\|\mathbf{x}\|).
\end{aligned}$$

Substituting this into (10) we obtain

$$\begin{aligned}\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi ct} \int_0^{\infty} \rho\, \delta(c^2t^2-\rho^2)\, e^{-(\|\mathbf{x}\|^2+\rho^2)/2}\, I_0(\rho\|\mathbf{x}\|)d\rho + \\ + \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{\infty} \rho \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}} \Theta(ct-\rho) e^{-(\|\mathbf{x}\|^2+\rho^2)/2} I_0(\rho\|\mathbf{x}\|)d\rho = \\ = \frac{e^{-\lambda t}}{2\pi}\, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\, I_0(ct\|\mathbf{x}\|) + \\ + \frac{\lambda e^{-\lambda t}}{2\pi c}\, e^{-\|\mathbf{x}\|^2/2} \int_0^{ct} \rho\, \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}}\, e^{-\rho^2/2}\, I_0(\rho\|\mathbf{x}\|)\, d\rho\end{aligned}$$

Making the substitution $z = \sqrt{c^2t^2-\rho^2}$ in the last integral, we obtain

$$\begin{aligned}\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi}\, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\, I_0(ct\|\mathbf{x}\|) + \\ + \frac{\lambda e^{-\lambda t}}{2\pi c}\, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \int_0^{ct} e^{(\lambda/c)z}\, e^{z^2/2}\, I_0(\|\mathbf{x}\|\sqrt{c^2t^2-z^2})\, dz = \\ = \frac{e^{-\lambda t}}{2\pi}\, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\, I_0(ct\|\mathbf{x}\|) + \\ + \frac{\lambda t\, e^{-\lambda t}}{2\pi}\, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi}\, I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2})\, d\xi,\end{aligned} \tag{11}$$

proving (8).

According to Lemma 2, the last integral in (11) is

$$\begin{aligned}\int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi}\, I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2})\, d\xi = \\ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{2^{(2n-k-1)/2}}{k!(n-k)!} \left(\frac{c^2t^2}{2}\right)^{n-k} (\lambda t)^k\, \Gamma\left(\frac{2n-k+1}{2}\right) \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}} = \\ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda t)^k\, 2^{(k+1)/2}}{k!\,(n-k)!}\, (c^2t^2)^{n-k}\, \Gamma\left(\frac{2n-k+1}{2}\right) \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}}.\end{aligned}$$

Substituting this into (11) we obtain (9).

It remains to check that the (non-negative) function $\varphi(\mathbf{x},t)$ given by (8) is really the density of the process. For this we should show that for any $t>0$

$$\int_{\mathbb{R}^2} \varphi(\mathbf{x},t)\, \mu(d\mathbf{x}) = 1. \tag{12}$$

We have

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) = \iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)/2}\, I_0(ct\sqrt{x_1^2+x_2^2})\, dx_1\, dx_2 =$$

$$= \int_0^\infty dr \int_0^{2\pi} d\theta \{ r\, e^{-r^2/2}\, I_0(ctr) \} = 2\pi \int_0^\infty r\, e^{-r^2/2}\, I_0(ctr)\, dr =$$

$$= \pi \int_0^\infty e^{-z/2}\, I_0(ct\sqrt{z})\, dz = \quad (\text{see}[1], \text{Formula } 6.643(2))$$

$$= \frac{2\pi\sqrt{2}}{ct}\, e^{c^2t^2/4}\, M_{-1/2,0}\left(\frac{c^2t^2}{2}\right),$$

where $M_{\xi,\eta}(z)$ is the Whittaker function. By applying now [1], Formula 9.220(2), we reduce the Whittaker function on the right-hand side of the last equality to the degenerated hypergeometric function and obtain

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) = 2\pi\Phi\left(1; 1; \frac{c^2t^2}{2}\right) = 2\pi\, e^{c^2t^2/2}. \tag{13}$$

From (13) it also follows that

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\sqrt{1-\xi^2}\, \|\mathbf{x}\|)\, \mu(d\mathbf{x}) = 2\pi\, e^{c^2t^2(1-\xi^2)/2}. \tag{14}$$

Therefore, by taking into account (13) and (14), we obtain

$$\int_{\mathbb{R}^2} \varphi(\mathbf{x},t)\, \mu(d\mathbf{x}) = \frac{e^{-\lambda t}}{2\pi}\, e^{-c^2t^2/2} \int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) +$$

$$+ \frac{\lambda t e^{-\lambda t}}{2\pi} e^{-c^2t^2/2} \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi} \left\{ \int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2} I_0(ct\sqrt{1-\xi^2}\|\mathbf{x}\|)\, \mu(d\mathbf{x}) \right\} d\xi =$$

$$= \frac{e^{-\lambda t}}{2\pi} e^{-c^2t^2/2}\, 2\pi e^{c^2t^2/2} + \frac{\lambda t e^{-\lambda t}}{2\pi} e^{-c^2t^2/2} \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi}\, 2\pi e^{c^2t^2(1-\xi^2)/2} d\xi =$$

$$= e^{-\lambda t} + \lambda t\, e^{-\lambda t} \int_0^1 e^{\lambda t\xi}\, d\xi = e^{-\lambda t} + e^{-\lambda t}\left(e^{\lambda t} - 1\right) = 1,$$

proving (12). The theorem is completely proved. □

Remark 1. We have supposed that the start point $\mathbf{x}^0$ was a two-dimensional random vector whose coordinates are the independent standard random variables with Gaussian density (4). However, we can consider in the same manner the case when the coordinates of the start point $\mathbf{x}^0$ are some dependent Gaussian random variables with given characteristics (a_1, σ_1) and (a_2, σ_2), respectively. In this case the density of $\mathbf{x}^0$ has the form

$$p(\mathbf{x}) = p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \times$$
$$\times \exp\left[-\frac{1}{2(1-r^2)} \left\{ \frac{(x_1-a_1)^2}{\sigma_1^2} - 2r\frac{(x_1-a_1)(x_2-a_2)}{\sigma_1\sigma_2} + \frac{(x_2-a_2)^2}{\sigma_2^2} \right\} \right], \tag{15}$$
$$-1 < r < 1.$$

The similar analysis can be done to evaluate the convolution (3) of the transition density (2) with Gaussian density (15), however the computations will be much more difficult and tedious.

References

[1] Gradshteyn I. S., Ryzhik I. M. *Tables of Integrals, Sums, Series and Products.* Academic Press, NY, 1980.

[2] Kolesnik A. D., Orsingher E. *A planar random motion with an infinite number of directions controlled by the damped wave equation.* J. Appl. Probab., 2005, **42**, 1168–1182.

[3] Masoliver J., Porrá J. M., Weiss G. H. *Some two and three-dimensional persistent random walks.* Physica A, 1993, **193**, 469–482.

[4] Stadje W. *The exact probability distribution of a two-dimensional random walk.* J. Stat. Phys., 1987, **46**, 207–216.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academy Street 5, MD-2028 Kishinev
Moldova

E-mail: *kolesnik@math.md*

Received September 10, 2008