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On Topological Groupoids and Multiple Identities

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Abstract. This paper studies some properties of (n, m)-homogeneous isotopies of medial topological groupoids. It also examines the relationship between paramediality and associativity. We extended some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous primitive goupoids with divisions.

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1 Introduction

In this article we study the (n, m)-homogeneous isotopies of topological groupoids with multiple identities and relation between paramediality and associativity. In Section 3 we expand on the notions of multiple identities and homogeneous isotopies introduced in [2]. This concept facilitates the study of topological groupoids with (n, m)-identities and homogeneous quasigroups, which are obtained by using isotopies of topological groups.

The results established in Section 4 are related to the results of M. Choban and L. Kiriyak [2] and to the research papers [5−8, 11]. We prove that if $(G, +)$ is a medial topological groupoid and e is a (k, p)-zero, then every (n, m)-homogeneous isotope $(G, \cdot)$ of $(G, +)$ is medial, with (mk, np)-identity e in $(G, \cdot)$. We present some interesting properties of a class of (n, m)-homogeneous quasigroups.

K. Sigmon, continuing the work of Professor A. D. Wallace, has shown that whenever a medial topological groupoid contains a bijective idempotent, it can be obtained from some commutative topological semigroup [12]. In Section 5, we obtain these and some other results in the case of paramedial topological groupoids. The relationship between mediality, paramediality and associativity was also studied in [9,10]. In Section 6 we extended one well-known statement of the theory of topological groups on the class of topological (n, m)-homogeneous primitive groupoids with divisions.

2 Basic notions

A non-empty set G is said to be a groupoid relative to a binary operation denoted by $\{\cdot\}$ if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called a topological groupoid.

A groupoid G is called a groupoid with division if for every $a, b \in G$ the equations $ax = b$ and $ya = b$ have, not necessarily unique, solutions.

A groupoid G is called reducible or cancellative if for any equality $xy = uv$ the equality $x = u$ is equivalent to the equality $y = v$.

A groupoid G is called a primitive groupoid with divisions if there exist two binary operations $l : G \times G \to G$, $r : G \times G \to G$, such that $l(a, b) \cdot a = b$, $a \cdot r(a, b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

If in a topological groupoid G the primitive divisions l and r are continuous, then we can say that G is a topological primitive groupoid with continuous divisions.

A primitive groupoid G with divisions is called a quasigroup if both equations $ax = b$ and $ya = b$ have unique solutions. In the quasigroup G the divisions l, r are unique.

An element $e \in G$ is called an identity if $ex = xe = x$ for every $x \in X$.

A quasigroup with an identity is called a loop.

If a multiplication operation in a quasigroup $(G, \cdot)$ endowed with a topology is continuous, then G is called a semitopological quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

A groupoid G is called medial if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid G is called paramedial if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

If a medial (paramedial) quasigroup G contains an element e such that $e \cdot x = x (x \cdot e = x)$ for all x in G, then e is called a left (right) identity element of G and G is called a left (right) medial (paramedial) loop.

A groupoid G is said to be hexagonal if it is idempotent, medial and semisymmetric, i.e. the equalities $x \cdot x = x$, $xy \cdot zt = xz \cdot yt$, $x \cdot zx = xz \cdot x = z$ hold for all of its elements.

Let $N = \{1, 2, ...\}$ and $Z = \{..., -2, -1, 0, 1, 2, ...\}$. Furthermore, we shall use the terminology from [1−4].

3 Multiple identities and homogeneous isotopies

Consider a groupoid $(G, +)$. For every two elements a, b from $(G, +)$ we denote:

$$1(a, b, +) = (a, b, +)1 = a + b, \quad \text{and} \quad n(a, b, +) = a + (n - 1)(a, b, +),$$

$$(a, b, +)n = (a, b, +)(n - 1) + b$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n(a, b, +)$ and $(a, b, +)n$.

Definition 1. Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element e of the groupoid $(G, +)$ is called:

– an (n, m)-zero of G if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;

– an $(n, \propto)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;

– an $(\propto, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an (n, m)-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element e is called an (n, m)-identity. The notion of (n, m)-identity was introduced in [6].

Example 1. Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $xe = x$ for every $x \in G$. Then $(G, \cdot)$ is paramedial groupoid with $(2, 1)$-identity e in G. Actually, if $x \in G$, then $e \cdot ex = ee \cdot ex = xe \cdot ee = xe \cdot e = xe = x$.

Example 2. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We define the binary operation $\{\cdot\}$.

$(\cdot)$	1	2	3	4	5	6	7	8	9
1	1	8	6	2	9	4	3	7	5
2	4	2	9	5	3	7	6	1	8
3	7	5	3	8	6	1	9	4	2
4	6	1	8	4	2	9	5	3	7
5	9	4	2	7	5	3	8	6	1
6	3	7	5	1	8	6	2	9	4
7	8	6	1	9	4	2	7	5	3
8	2	9	4	3	7	5	1	8	6
9	5	3	7	6	1	8	4	2	9

Then $(G, \cdot)$ is a non-commutative hexagonal quasigroup and each element from $(G, \cdot)$ is a $(6, 6)$-identity in G.

Definition 2. Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a homogeneous isotope of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$, for all $x, y \in G$.

For every mapping $f : X \to X$ we denote $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for any $n \geq 1$.

Definition 3. Let $n, m \leq \infty$. A groupoid $(G, \cdot)$ is called an (n, m)-homogeneous isotope of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:

1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\psi = \psi\varphi$;
3. If $n < \infty$, then $\varphi^n(x) = x$ for all $x \in G$;
4. If $m < \infty$, then $\psi^m(x) = x$ for all $x \in G$.

Definition 4. A groupoid $(G, \cdot)$ is called an isotope of a topological groupoid $(G, +)$, if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that

$$x \cdot y = \varphi(x) + \psi(y) \text{ for all } x, y \in G.$$

Under the conditions of Definition 4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G, +)$ and write $(G, \cdot) = g(G, +, \varphi, \psi)$.

Example 3. Let $(G, +)$ be a topological commutative additive group with a zero.

1. If $\varphi(x) = x$, $\psi(x) = -x$ and $x \cdot y = x - y$, then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a topological medial quasigroup with a $(2, 1)$-identity 0.

2. If $\varphi(x) = -x$, $\psi(x) = x$ and $x \cdot y = y - x$, then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a topological medial quasigroup with a $(1, 2)$-identity 0.

Example 4. Let $(R, +)$ be a topological Abelian group of real numbers.

1. If $\varphi(x) = x$, $\psi(x) = 2x$ and $x \cdot y = x + 2y$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup. By virtue of Theorem 7 from [2], there exists a right invariant Haar measure on $(R, \cdot)$.

2. If $\varphi(x) = x$, $\psi(x) = x + 7$ and $x \cdot y = x + y + 7$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup and $(R, \cdot)$ does not contain (n, m)-identities. As above, by virtue of Theorem 7 from [2] there exists an invariant Haar measure on $(R, \cdot)$.

Example 5. Denote by $Z_p = Z/pZ = \{0, 1, ..., p - 1\}$ the cyclic Abelian group of order p. Consider the commutative group $(G, +) = (Z_7, +)$, $\varphi(x) = 3x$, $\psi(x) = 4x$ and $x \cdot y = 3x + 4y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup with $(3, 6)$-identity in $(G, \cdot)$, which coincides with the zero element in $(G, +)$.

Example 6. Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x$, $\psi(x) = 3x$ and $x \cdot y = 2x + 3y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and the zero from $(G, \cdot)$ is a $(4, 4)$-identity in G.

Example 7. Consider the Abelian group $(G, +) = (Z_5, +)$, $\varphi(x) = 4x$, $\psi(x) = 2x$ and $x \cdot y = 4x + 2y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is a $(4, 2)$-identity in G.

4 Some properties of (n, m)-homogeneous isotopies

Proposition 1. *If $(G, +)$ is a medial topological groupoid and e is a (k, p)-zero, then every (n, m)-homogeneous isotope $(G, \cdot)$ of the topological groupoid $(G, +)$ is medial with (mk, np)-identity e in $(G, \cdot)$ and $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$ for all $x, y, u, v \in G$ and $n, m, p, k \in N$.*

Proof. The mediality of the (n, m)-homogeneous isotope $(G, \cdot)$ follows from [12]. We will prove that e is an (mk, np)-identity in $(G, \cdot)$ by the method described in [2].

Let $(G, \cdot)$ be an (n, m)-homogeneous isotope of the groupoid $(G, +)$ and e be a (k, p)-zero in $(G, +)$. We mention that $\varphi^q(e) = \psi^q(e) = e$ for every $q \in N$. If $k < +\infty$, then in $(G, +)$ we have $qk(e, x, +) = x$ for each $x \in G$ and for every $q \in N$. Let $m < +\infty$ and $\psi^m(x) = x$ for all $x \in G$. Then $1(e, x, \cdot) = 1(e, \psi(x), +)$ and $q(e, x, \cdot) = q(e, \psi^q(x), +)$ for every $q \geq 1$. Therefore

$$mk(e, x, \cdot) = mk(e, \psi^{mk}(x), +) = mk(e, x, +) = x.$$

Analogously we obtain that

$$(e, x, \cdot)np = (e, \varphi^{np}(x), +)np = (e, x, +)np = x.$$

Hence e is an (mk, np)-identity in $(G, \cdot)$.

Using the algorithm from [12] we will show that $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$. Let $x \cdot y = \varphi(x) + \psi(y)$ and $u \cdot v = \varphi(u) + \psi(v)$. Then

$$(x \cdot y) + (u \cdot v) = [\varphi(x) + \psi(y)] + [\varphi(u) + \psi(v)] =$$

$$= [\varphi(x) + \varphi(u)] + [\psi(y) + \psi(v)] = \varphi(x + u) + \psi(y + v) = (x + u) \cdot (y + v).$$

In this way we have that $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$. The proof is complete. □

Corollary 1. *If $(G, +)$ is a medial topological groupoid, then every homogeneous isotope $(G, \cdot)$ of the topological groupoid $(G, +)$ such that $\varphi\psi = \psi\varphi$ is medial and $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$.*

Definition 5. A topological quasigroup $(G, \cdot)$ is called:

– homogeneous if $(G, \cdot)$ is a homogeneous isotope of the topological group $(G, +)$.

– (n, m)-homogeneous if $(G, \cdot)$ is a (n, m)-homogeneous isotope of the topological group $(G, +)$.

We denote by:

– T the class of all medial quasigroups.

– $Q(n, m)$ the class of all (n, m)-homogeneous quasigroups.

We consider the class: $M(n, m) = T \cap Q(n, m)$.

The class $M(1, 1)$ coincides with the class of topological abelian groups.

Example 8. Let $(G, \cdot)$ be a topological medial quasigroup.

1. If $e \in G$, such that $ex = x$ and $xx = e$ for each $x \in G$, then $(G, \cdot) \in M(1, 2)$ and $(G, \cdot)$ is a topological medial quasigroup with $(1, 2)$-identity e in G.

2. If $e \in G$, such that $xe = x$ and $xx = e$ for each $x \in G$, then $(G, \cdot) \in M(2, 1)$ and $(G, \cdot)$ is a topological medial quasigroup with $(2, 1)$-identity e in G.

Theorem 1. *Let $Q(n,m)$ be a class of (n,m)-homogeneous quasigroups. Then:*

1. For each $G \in Q(n,m)$ there exists an (n,m)-identity $e \in G$ with the following properties:

1.1 $e \cdot e = e$;

1.2 $n(e,x) = x$;

1.3 $(x,e)m = x$;

1.4 $ex \cdot e = e \cdot xe$;

2. If $\varphi(x) = ex$ and $\varphi^n(x) = n(e,x) = x$ then $\varphi^{-1}(x) = (n-1)(e,x)$;

3. If $\varphi^{-1}(x) = (n-1)(e,x)$ and $\varphi^n(x) = n(e,x) = x$ then $(n-1)(e,ex) = x$;

4. If $\psi(x) = xe$ and $\psi^m(x) = (x,e)m = x$ then $\psi^{-1}(x) = (x,e)(m-1)$;

5. If $\psi^{-1}(x) = (x,e)(m-1)$ and $\psi^m(x) = (x,e)m = x$ then $(xe,e)(m-1) = x$.

Proof. **1.** Let $(G,+)$ be a topological group and $\varphi, \psi : G \longrightarrow G$ be topological automorphisms of this group, such that $\varphi^n(x) = \psi^m(x) = x, \varphi \cdot \psi = \psi \cdot \varphi$, for each $x \in G$ and $(G,\cdot) = g(G,+,\varphi,\psi)$. Let e be a zero in $(G,+)$. According to Theorem 3 from [2], e is an (n,m)-identity in $(G,\cdot)$. Hence, $e \cdot e = e$, $n(e,x) = x$ and $(x,e)m = x$. Thus, assertions 1.1, 1.2 and 1.3 are proved.

It is easy to see that $\varphi(x) = ex$ and $\psi(x) = xe$. From the equality $\varphi\psi = \psi\varphi$ we have $\varphi\psi = \varphi(xe) = e \cdot xe$ and $\psi\varphi = \psi(ex) = ex \cdot e$. Therefore $e \cdot xe = ex \cdot e$. Assertion 1 is proved.

2. We will show that if $\varphi(x) = ex$ and $\varphi^n(x) = n(e,x) = x$, then

$$\varphi^{-1}(x) = (n-1)(e,x).$$

We have $\varphi(x) = ex$, hence $\varphi(\varphi^{-1}(x)) = e \cdot \varphi^{-1}(x)$. But $\varphi(\varphi^{-1}(x)) = x$. Therefore, $e \cdot \varphi^{-1}(x) = x$. Since $n(e,x) = x$, we obtain that

$$e \cdot (\varphi^{-1}(x)) = n(e,x). \tag{1}$$

By the definition of multiple identities we have

$$e \cdot (n-1)(e,x) = n(e,x). \tag{2}$$

From (1) and (2) we infer that $\varphi^{-1}(x) = (n-1)(e,x)$, which proves assertion 2.

3. We will prove that if $\varphi^{-1}(x) = (n-1)(e,x)$ and $\varphi^n(x) = n(e,x) = x$ then

$$(n-1)(e,ex) = x.$$

Let be $(n-1)(e,ex) = t$. Then

$$e \cdot (n-1)(e,ex) = et. \tag{3}$$

By the definition of multiple identities

$$e \cdot (n-1)(e,ex) = n(e,ex) = ex. \tag{4}$$

From (3) and (4) it follows $ex = et$ and $t = x$. Hence $(n-1)(e, ex) = x$, as desired.

4. By following the same guidelines as in property 2 we obtain that if $\psi(x) = xe$ and $\psi^m(x) = (x, e)m$, then $\psi^{-1}(x) = (x, e)(m-1)$.

5. Similarly to properties 3 we prove that if $\psi^{-1}(x) = (x, e)(m-1)$ and $\psi^m(x) = (x, e)m = x$, then $(xe, e)(m-1) = x$.
The proof of the theorem is now complete. □

Corollary 2. *A class $Q(n, m)$ of (n, m)-homogeneous quasigroups forms a variety.*

Corollary 3. *A class $M(n, m)$ of topological medial quasigroups with (n, m)-identities forms a variety.*

5 Paramedial topological groupoids

We provide an example of a paramedial groupoid which is not medial.

Example 9. Let $G = \{1, 2, 3, 4\}$. We define the binary operation $\{\cdot\}$.

$(\cdot)$	1	2	3	4
1	1	2	4	3
2	3	4	2	1
3	2	1	3	4
4	4	3	1	2

Then $(G, \cdot)$ is a paramedial quasigroup but it is not medial. For example

$$(2 \cdot 3) \cdot (1 \cdot 4) \neq (2 \cdot 1) \cdot (3 \cdot 4).$$

An element e is called idempotent if $ee = e$. If the maps $x \to xe$ and $x \to ex$ are homeomorphisms then e is also called bijective.

Theorem 2. *Let $(G, \cdot)$ be a paramedial topological groupoid and let e, e_1, e_2 be elements of G for which:*

1. *$ee_1 = e_1$ and $e_2e = e_2$;*
2. *The maps $x \to e_1x$ and $x \to xe_2$ are homeomorphisms of G onto itself;*
3. *The map $x \to xe$ is surjective.*

If there exists a binary operation $\{\circ\}$ on G such that $(e_1x) \circ (ye_2) = yx$, then $(G, \circ)$ is a commutative topological semigroup having e_1e_2 as identity.

Proof. Since $x \to e_1x$ and $x \to xe_2$ are homeomorphisms it is clear that $\{\circ\}$ is continuous.

Using surjectivity and the fact that $(e_1e_2)\circ(ye_2) = ye_2$ and $(e_1x)\circ(e_1e_2) = e_1x$ we see that e_1e_2 is an identity for $(G,\circ)$. Observe that $xe_1 \cdot e_2 = xe_1 \cdot e_2e = ee_1 \cdot e_2x = e_1 \cdot e_2x$.

One can see that

$$xe_1 \cdot zt = (e_1 \cdot zt)\circ(xe_1 \cdot e_2) = (ee_1 \cdot zt)\circ(xe_1 \cdot e_2) =$$
$$= (te_1 \cdot ze)\circ(xe_1 \cdot e_2) = [(e_1 \cdot ze)\circ(te_1 \cdot e_2)]\circ(xe_1 \cdot e_2);$$

$$te_1 \cdot zx = (e_1 \cdot zx)\circ(te_1 \cdot e_2) = (ee_1 \cdot zx)\circ(te_1 \cdot e_2) =$$
$$= (xe_1 \cdot ze)\circ(te_1 \cdot e_2) = [(e_1 \cdot ze)\circ(xe_1 \cdot e_2)]\circ(te_1 \cdot e_2).$$

From paramediality we have

$$[(e_1 \cdot ze)\circ(te_1 \cdot e_2)]\circ(xe_1 \cdot e_2) = [(e_1 \cdot ze)\circ(xe_1 \cdot e_2)]\circ(te_1 \cdot e_2).$$

Setting $z = e_2$ then since $e_2e = e_2$ and e_1e_2 is an identity it follows that:

$$[e_1e_2\circ(te_1 \cdot e_2)]\circ(xe_1 \cdot e_2) = [e_1e_2\circ(xe_1 \cdot e_2)]\circ(te_1 \cdot e_2)$$

and

$$(te_1 \cdot e_2)\circ(xe_1 \cdot e_2) = (xe_1 \cdot e_2)\circ(te_1 \cdot e_2).$$

Hence, $(G,\circ)$ is a commutative topological groupoid and then the associativity is immediate. Indeed

$$[(te_1 \cdot e_2)\circ(e_1 \cdot ze)]\circ(xe_1 \cdot e_2) = (te_1 \cdot e_2)\circ[(e_1 \cdot ze)\circ(xe_1 \cdot e_2)].$$

The proof is complete. □

Theorem 3. *Let $(G,\cdot)$ be a paramedial topological groupoid satisfying the following conditions:*

1. It contains an idempotent e;

2. The maps $x \to xe$ and $x \to ex$ are homeomorphisms of G onto itself;

3. There exists a binary operation $\{\circ\}$ on G such that $(ex)\circ(ye) = yx$.

Then $(G,\circ)$ is a commutative topological semigroup having e as identity. Moreover, the maps $x \to xe$ and $x \to ex$ are antihomomorphisms of $(G,\circ)$ and $xe \cdot e = e \cdot ex$.

Proof. The first part of Theorem 3 follows from Theorem 2 with $e = e_1 = e_2$. Indeed, we have $xe \cdot e = xe \cdot ee = ee \cdot ex = e \cdot ex$. Since

$$(ex \circ ye)\, e = yx \cdot e = yx \cdot ee = ex \cdot ey = (e \cdot ey)\circ(ex \cdot e) = (ye \cdot e)\circ(ex \cdot e),$$

we see that $x \to xe$ is an antihomorphism of $(G,\circ)$. Similarly

$$e\,(ex \circ ye) = e \cdot yx = ee \cdot yx = xe \cdot ye = (e \cdot ye)\circ(xe \cdot e) = (e \cdot ye)\circ(e \cdot ex).$$

Consequently, we obtain that $x \to ex$ is an antihomorphism of $(G,\circ)$. The proof is complete. □

A topological groupoid $(G, \circ)$ is called radical if the map $s : G \to G$, defined by $s(x) = x \circ x$, is a homeomorphism.

If $(G, \circ)$ is paramedial and radical then s, and hence s^{-1}, is an antihomomorphism of $(G, \circ)$.

A topological groupoid $(G, \cdot)$ where $\{\cdot\}$ is defined by

$$x \cdot y = s^{-1}(x) \circ s^{-1}(y) = s^{-1}(y \circ x)$$

is called the radical isotope of $(G, \circ)$.

A radical isotope $(G, \cdot)$ of $(G, \circ)$ is idempotent since

$$x \cdot x = s^{-1}(x \circ x) = s^{-1}(s(x)) = x$$

for each $x \in G$.

Theorem 4. *If $(G, \circ)$ is a topological groupoid with unit e, $(G, \cdot)$ is a commutative, idempotent topological groupoid and*

$$(x \circ y) \cdot (z \circ t) = (ty) \circ (zx),$$

then $(G, \circ)$ is a commutative radical semigroup.

Proof. If we define $t : G \to G$ by $t(x) = ex$ then t is an antihomomorphism of $(G, \circ)$. Indeed, for all $x, y \in G$ we have,

$$t(x \circ y) = e(x \circ y) = (e \circ e)(x \circ y) = (ye) \circ (xe) = (ey) \circ (ex) = t(y) \circ t(x).$$

In particular, we obtain

$$t(s(x)) = t(x \circ x) = t(x) \circ t(x) = s(t(x));$$

where $s : G \to G$ is defined by $s(x) = x \circ x$.

Also, for each $x, y \in G$ and each unit e in $(G, \circ)$

$$xy = (e \circ x) \cdot (e \circ y) = (e \circ x) \cdot (y \circ e) = (ex) \circ (ye) =$$
$$= (ex) \circ (ey) = t(x) \circ t(y) = t(y \circ x).$$

Hence $t(s(x)) = t(x \circ x) = xx = x$.

It follows that t is a continuous inverse for s so that $(G, \circ)$ is radical. Since $(G, \cdot)$ is commutative and $x \circ y = s(yx) = s(xy) = y \circ x$ then $\{\circ\}$ is commutative. Since $xy = t(y \circ x)$ and $t = s^{-1}$ then $(G, \cdot)$ is the radical isotope of $(G, \circ)$. It only remains to show that $\{\circ\}$ is associative.

Since t is bijective and

$$t[(x \circ y) \circ z] = z \cdot (x \circ y) = (e \circ z) \cdot (x \circ y) = (yz) \circ (xe) =$$
$$= (yz) \circ (ex) = t(z \circ y) \circ t(x) = t[x \circ (z \circ y)] = t[x \circ (y \circ z)].$$

we conclude that $(G, \circ)$ is a commutative radical semigroup. The proof is complete. □

6 On topological primitive groupoid with divisions

The following fundamental Theorem was proved in [2].

Theorem. *Let $(G,+)$ be a topological groupoid, $\varphi, \psi : (G,+) \longrightarrow (G,+)$ be homeomorphisms and $(G,\cdot) = g(G,+,\varphi,\psi)$. Then:*

1. $(G,+) = g(G,\cdot,\varphi^{-1},\psi^{-1})$;

2. $(G,\cdot)$ is a topological groupoid;

3. If $(G,+)$ is a reducible groupoid , then $(G,\cdot)$ is a reducible groupoid too;

4. If $(G,+)$ is a groupoid with divisions, then $(G,\cdot)$ is a groupoid with divisions too;

5. If $(G,+)$ is a topological primitive groupoid with divisions, then $(G,\cdot)$ is a topological primitive groupoid with divisions too;

6. If $(G,+)$ is a topological quasigroup, then $(G,\cdot)$ is a topological quasigroup;

7. If $n,m,p,k \in N$ and $(G,\cdot)$ is an (n,m)-homogeneous isotope of the groupoid $(G,+)$ and e is an (k,p)-zero in $(G,+)$, then e is an (mk,np)-identity in $(G,\cdot)$.

We consider a topological groupoid $(G,+)$. If α is a binary relation on G, then $\alpha(x) = \{y \in G : x\alpha y\}$ for every $x \in G$.

An equivalence relation α on G is called a congruence on $(G,+)$ if from $(x\alpha u)$ and $(y\alpha v)$ it follows $(x+y)\alpha\ (u+v)$ for all $x,y,u,v \in G$.

If $(G,+)$ is a primitive groupoid with divisions l and r, then we consider that $l(x,y)\alpha\ l(u,v)$, and $r(x,y)\alpha\ r(u,v)$ provided $(x\alpha u)$ and $(y\alpha v)$.

Let $(G,+,r,l)$ be a topological primitive groupoid with divisions r,l and (k,p)-zero. Let $(G,\cdot) = g(G,+,\varphi,\psi)$ be an (n,m)-homogeneous isotope. Then, by virtue of the aforementioned Theorem, e is an (mk,np)-identity of the topological primitive groupoid with divisions $(G,\cdot)$.

Definition 6. A primitive subgroupoid with divisions H of the primitive groupoid with divisions $(G,+,r,l)$ is called a normal primitive subgroupoid with divisions if $e \in H$ and $H = G(\alpha)$, for some congruence α.

Lemma 1. *Let α be a congruence of the topological primitive groupoid with divisions $(G,+,r,l)$. Then there exists a unique normal primitive subgroupoid with divisions $G(\alpha)$, which is called the primitive subgroupoid with divisions defined by congruence α such that $e \in G$.*

Proof. The set $G(\alpha) = \alpha(e) = \{y \in G : e\alpha y\}$ is the desired primitive subgroupoid with divisions. The proof is complete. □

Definition 7. The primitive subgroupoids with divisions $(H_1,+,r,l)$ and $(H_2,+,r,l)$ of the topological primitive groupoid with divisions $(G,+,r,l)$ are called conjugate if $H_2 = h(H_1)$ for some topological automorphism $h : G \to G$.

Theorem 5. *Let H be a primitive subgroupoid with divisions of the topological primitive groupoid with divisions $(G,+,r,l)$ and let $e \in H$. Then there exists such a primitive subgroupoid with divisions Q of the topological primitive groupoids with divisions $(G,+,r,l)$ and $(G,\cdot,r_1,l_1)$ for which:*

1. $e \in Q \subseteq H$.

2. Q is the intersection of a finite number of the primitive subgroupoids with divisions conjugate to H of the $(G,+,r,l)$.

3. If H is a closed set, then Q is closed too.

4. If H is a G_δ set, then Q is a G_δ set too.

5. If H is an open set, then Q is open too.

6. If H is a normal primitive subgroupoid with divisions, then Q is a normal primitive subgroupoid with divisions of $(G,+,r,l)$ and $(G,\cdot,r_1,l_1)$.

Proof. We put $\{h_p : p \leq n \cdot m\} = \{\varphi^i \circ \psi^j : i \leq n, j \leq m\}, H_p = h_p(H)$ and $Q = \cap\{H_p : p \leq n \cdot m\}$.

We consider that $h_1(x) = x$ for each $x \in H$. Fix $i \leq n$ and $j \leq n$. Let $h_p = \varphi^i \circ \psi^j$. It is clear that h_p is an automorphism of $(G,+,r,l)$. Thus $H_p = h_p(H)$ is a primitive subgroupoid with divisions of $(G,+,r,l)$ conjugate to H in $(G,+,r,l)$. Therefore Q is a primitive subgroupoid with divisions of $(G,+,r,l)$. This establishes assertions 1–5.

Firstly we prove that Q is a primitive sugroupoid with divisions of $(G,\cdot,r_1,l_1)$. Let $x,y,b \in Q$. Then $xy = \varphi(x)+\psi(y)$ and $\varphi(x),\psi(x) \in H_i$ for any i. Thus $xy \in Q$. If $ax = b$, then $a = l_1(x,b) \in H_i$ for every i and $a \in Q$. Similarly, if $xa = b$, then $a = r_1(x,b) \in H_i$ for any i and $a \in Q$. Hence Q is a primitive subgroupoid with divisions of $(G,\cdot,r_1,l_1)$.

Let α be a congruence of $(G,+,r,l)$. Then, by virtue of Lemma 1, there exists a unique normal primitive subgroupoid with divisions $H = G(\alpha)$ and $e \in H$. Because h_p is a topological automorphism of $(G,+,r,l)$, then $H_p = h_p(H)$ is a normal primitive subgroupoid with divisions of $(G,+,r,l)$ conjugate to the normal primitive subgroupoid with divisions H. Therefore Q is a normal primitive subgroupoid with divisions of $(G,+,r,l)$ and Q is a normal primitive subgroupoid of $(G,\cdot,r,l)$. This proves assertion 6 and completes the proof of the theorem. □

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