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On free topological groups

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Abstract. In the present article the existence and unicity of almost $(\mathcal{U}, \mathcal{V})$-free group over given space, where $\mathcal{U}, \mathcal{V}$ are classes of topological groups is studied. If $\mathcal{V}$ is a quasivariety of compact topological groups and $\mathcal{V} \subseteq \mathcal{U}$, then these objects exist for any space. If W is a quasivariety of compact groups, $\mathcal{U} = \mathcal{V}$ is the class of all pseudocompact subgroups of groups from W, then the almost $(\mathcal{U}, \mathcal{V})$-free groups exist only for some special spaces.

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All spaces considered are assumed to be completely regular pointed T_1-spaces. If X is a space, then p_X is the base point of X. If G is a group, then the base point p_G is the identity of G. We consider only mappings $f : X \to Y$ for which $f(p_X) = p_Y$.

For every space X we denote by βX the Stone-Čech compactification and by $|X|$, $w(X)$, $d(X)$ the cardinality, weight and density of the space X, respectively. If $indX = 0$, i.e. X is zero-dimensional, then by $\beta_0 X$ we denote the maximal zero-dimensional compactification of X. If Y is a subspace of a space X, then we consider that $p_Y = p_X$. In particular, $p_{bX} = p_X$ for any compactification bX of X.

Remark 1. If X is not a pointed space, then we put $\bar{X} = X \cup \{p_X\}$, where $p_X \notin X$ and X is an open-and-closed subspace of the space $\bar{X}$. Thus every space may be completed to a pointed space.

1 A free topological group of a space

Let $\mathcal{U}$ and $\mathcal{V}$ be two classes of topological groups.

Definition 1. A pair $(F(X,\mathcal{U},\mathcal{V}), e_X)$ is said to be an almost $(\mathcal{U},\mathcal{V})$-free topological group over a space X if $F(X,\mathcal{U},\mathcal{V}) \in \mathcal{U}$, $e_X : X \to F(X,\mathcal{U},\mathcal{V})$ is a continuous mapping, $e = e_X(p_X)$ is the identity of the group $F(X,\mathcal{U},\mathcal{V})$ and for every continuous mapping $f : X \to G$ with $G \in \mathcal{V}$ there exists a continuous homomorphism $\bar{f} : F(X,\mathcal{U},\mathcal{V}) \to G$ such that $f = \bar{f} \circ e_X$. If $\mathcal{U} = \mathcal{V}$, then we put $F(X,\mathcal{U},\mathcal{V}) = F(X,\mathcal{U})$ and $F(X,\mathcal{U})$ is called the almost $\mathcal{U}$-free group over X.

If for any continuous mapping $f : X \to G \in \mathcal{V}$ the homomorphism $\bar{f} : F(X,\mathcal{U},\mathcal{V}) \to G$ is unique, then $(F(X,\mathcal{U},\mathcal{V}), e_X)$ is called a $(\mathcal{U},\mathcal{V})$-free topological group of X.

Remark 2. Let $\mathcal{U}$ be a multiplicative class of topological groups, $G_0 \in \mathcal{U}$, $|G_0| \geq 2$, X be a space and for X there exists some almost $(\mathcal{U}, \mathcal{V})$-free topological group. Then the almost $(\mathcal{U}, \mathcal{V})$-free topological group over X is not unique. Really, fix some almost $(\mathcal{U}, \mathcal{V})$-free topological group $(F(X,\mathcal{U},\mathcal{V}), e_X)$ over X. Let $\tau \geq 1$ be a cardinal number, e be the identity of $F(X,\mathcal{U},\mathcal{V})$, e_τ be the identity of G_0^τ, $F'(X,\mathcal{U},\mathcal{V}) = F(X,\mathcal{U},\mathcal{V}) \times G_0^\tau$, $e' = (e, e_\tau)$ and $\bar{e}_X(x) = (e_X(x), e_\tau)$ for any $x \in X$. Then $(F'(X,\mathcal{U},\mathcal{V}), \bar{e}_X)$ is an almost $(\mathcal{U}, \mathcal{V})$-free topological group over X.

Remark 3. The concept of an almost $(\mathcal{U}, \mathcal{V})$-free topological group for non-pointed spaces was proposed by W.W.Comfort and J.van Mill (see [1], p.110). We consider that notion for pointed spaces. Moreover, our definition is more general for non-pointed spaces too. In the definition from [1] it is supposed that e_X is an embedding, i.e. $X \subseteq F(X,\mathcal{U},\mathcal{V})$.

We say that the topological group G is complete if it is complete relative to the two-sided uniformity on G (see [6]).

Definition 2. A class $\mathcal{U}$ of topological groups is called a quasivariety if the following properties hold:

– the class $\mathcal{U}$ is multiplicative;

– if a topological group A is topologically isomorphic to a closed subgroup of some group $B \in \mathcal{U}$, then $A \in \mathcal{U}$.

We consider that any quasivariety $\mathcal{U}$ is non-trivial, i.e. there exists $G \in \mathcal{U}$ for which $|G| \geq 2$.

If G is a topological group and X is a subset of G, then X is contained in the minimal subgroup $a(X,G) = \cap\{H : X \subseteq H \text{ and } H \text{ is a subgroup of } G\}$ of G. If $a(X,G) = G$, then we say that X generated G. If $a(X,G)$ is dense in G, then we say that X topologically generated G.

Proposition 1. *Let $\mathcal{U}$ be a quasivariety of topological groups. Then for any pointed space X there exists a unique $\mathcal{U}$-free topological group $(F(X, \mathcal{U}), e_X)$ over X. Moreover, the set $e_X(X)$ topologically generated the group $F(X, \mathcal{U})$.*

Proof. Fix a space X. Let m be an infinite cardinal and $|X| \leq m$. Let τ be an infinite cardinal and $\tau > 2^{2^m}$. We identify the isomorphic topological groups. Then $\mathcal{U}_\tau = \{G \in \mathcal{U} : |G| \leq \tau\}$ is a set. Therefore, the family $\{f_\alpha : X \to G_\alpha : \alpha \in A\}$ of all continuous mappings of X into groups from $\mathcal{U}_\tau$ is a set too. Consider the mapping $e_X : X \to G = \prod\{G_\alpha : \alpha \in A\}$, where $e_X(x) = (f_\alpha(x) : \alpha \in A)$. By $F(X,\mathcal{U})$ we denote the closed subgroup of G topologically generated by the set $e_X(X)$. Then $(F(X,\mathcal{U}), e_X)$ is a $\mathcal{U}$-free topological group over X.

The existence is proved.

Let $F \in \mathcal{U}$, $h : X \to F$ be a continuous mapping and for any continuous mapping $f : X \to G \in \mathcal{U}$ there exists a unique homomorphism $\hat{f} : F \to G$ such that $f = \hat{f} \circ h$. We mention that $h(X)$ topologically generated F. Suppose that F_1 is the closed subgroup of F generated by $h(X)$ and $F_1 \neq F$. Then $F_1 \in \mathcal{U}$ and there exists a continuous homomorphism $h_1 : F \to F_1$ such that $h(x) = h_1(h(x))$ for any $x \in X$.

Then $h_1(y) = y$ for any $y \in F_1$. Now consider the homomorphism $h_2(y) = y$ for any $y \in F$. Then there exist two distinct homomorphisms $h_1, h_2 : F \to F$ such that $h(x) = h_1(h(x)) = h_2(h(x))$ for any $x \in X$, a contradiction. Thus $F = F_1$.

From definition, there exist two continuous homomorphisms $g_1 : F(X, \mathcal{U}) \to F$ and $g_2 : F \to F(X, \mathcal{U})$ such that $g_1(e_X(x)) = h(x)$ and $g_2(h(x)) = e_X(x)$ for any $x \in X$. Therefore $g_2(g_1(y)) = y$ and $g_1(g_2(z)) = z$ for all $y \in e_X(X)$ and $z \in h(X)$. Thus $g_3 = g_1|e_X(X)$ is a homeomorphism of $e_X(X)$ onto $h(X)$, $g_4 = g_2|h(X)$ is a homeomorphism of $h(X)$ onto $e_X(X)$ and $g_4 = {g_3}^{-1}$. Hence g_1, g_2 are isomorphisms and $g_1 = {g_2}^{-1}$. The proof is complete. □

Example 1. Let $\mathcal{U}_c$ be the class of all compact groups. Then $\mathcal{U}_c$ is a quasivariety of topological groups. For every space X the mapping $e_X : X \to F(X,\mathcal{U}_c)$ is an embedding. We can consider that $X = e_X(X) \subseteq F(X,\mathcal{U}_c)$. Then βX is the closure of X in $F(X,\mathcal{U}_c)$.

Example 2. Let $\mathcal{U}_{ac}$ be the class of all commutative compact groups. Then $\mathcal{U}_{ac}$ is a quasivariety of topological groups. For every space X the mapping $e_X : X \to F(X,\mathcal{U}_{ac})$ is an embedding. We can consider that $X = e_X(X) \subseteq F(X,\mathcal{U}_{ac})$. Then βX is the closure of X in $F(X,\mathcal{U}_{ac})$.

Example 3. Let $\mathcal{U}_{0c}$ be the class of all compact zero-dimensional groups. Then $\mathcal{U}_{0c}$ is a quasivariety. The mapping $e_X : X \to F(X,\mathcal{U}_{0c})$ is an embedding if and only if $indX = 0$. If $indX = 0$, then $\beta_0 X$ is the closure of $X = e_X(X)$ in $F(X,\mathcal{U}_{0c})$.

Definition 3. A class $\mathcal{U}$ of topological groups is said to be complete if the following properties hold:
– $(A, \mathcal{T}) \in \mathcal{U}$;
– if $\mathcal{T}'$ is a topology on A and $(A, \mathcal{T}')$ is a topological group, then $(A, \mathcal{T}') \in \mathcal{U}$.

Remark 4. Let $\mathcal{U}$ be a complete quasivariety of topological groups. If A is a subgroup of $B \in \mathcal{U}$, then $A \in \mathcal{U}$. In particular, the set $e_X(X)$ algebraically generated $F(X,\mathcal{U})$ provided $e_X(X)$ topologically generated $F(X,\mathcal{U})$ for any space X. In this case $e_X : X \to F(X,\mathcal{U})$ is an embedding for any space X.

Remark 5. Let $\mathcal{U}$ be a quasivariety, $F(X,\mathcal{U})$ be the almost $\mathcal{U}$-free topological group over a space X and $F_0(X,\mathcal{U})$ be the closed subgroup of $F(X,\mathcal{U})$ generated by the set $e_X(X)$. Then:

1. $F_0(X,\mathcal{U})$ is the $\mathcal{U}$-free topological group over X.

2. There exists a continuous homomorphism $\varphi : F(X,\mathcal{U}) \to F_0(X,\mathcal{U})$ such that $\varphi(y) = y$ for any $y \in F_0(X,\mathcal{U})$. (It is obvious that $\varphi = \bar{e}_X$).

3. If $(F'(X,\mathcal{U}), e'_X)$ is another almost $\mathcal{U}$-free topological group over X and $F'_0(X,\mathcal{U})$ is the closed subgroup of $F'(X,\mathcal{U})$ generated by $e'_X(X)$, then there exists a unique isomorphism $\psi : F_0(X,\mathcal{U}) \to F'_0(X,\mathcal{U})$ such that $\psi(e_X(x)) = e'_X(x)$ for any $x \in X$.

Proposition 2. *Let $\mathcal{U}$ be a quasivariety of topological groups and $\mathcal{V}$ be a class of topological groups. If $\mathcal{V} \subseteq \mathcal{U}$ and $(F(X,\mathcal{U}), e_X)$ is an almost $\mathcal{U}$-free topological group over X, then $(F(X,\mathcal{U}), e_X)$ is an almost $(\mathcal{U}, \mathcal{V})$-free topological group over X.*

Proof. Obvious. □

If $\mathcal{V} \nsubseteq \mathcal{U}$, then Proposition 2 is not true. For example, if $\mathcal{U} = \mathcal{U}_{ac}$ and $\mathcal{V}$ is the class of all commutative groups, then Proposition 2 is not true.

2 On totally bounded groups

A topological group A is totally bounded or precompact if A is a subgroup of some compact group.

Let $\mathcal{U}_b$ be the class of all totally bounded groups. For any space X there exists the $\mathcal{U}_b$-free group $(F(X,\mathcal{U}_b), e_X)$ and $e_X : X \to F(X,\mathcal{U}_b)$ is an embedding. Moreover, $F(X,\mathcal{U}_b)$ is the subgroup of $F(X,\mathcal{U}_c)$ generated by X. The quasivariety $\mathcal{U}_b$ is not complete.

Example 4. Consider the quasivariety $\mathcal{U}_c$. Let X be the space of reals, G be the topological group of reals and $f(x) = x$ for any $x \in X$. Let $F(X,\mathcal{U}_c)$ be the $\mathcal{U}_c$-free topological group over X and X topologically generated $F(X,\mathcal{U}_c)$. Denote by A the subgroup of $F(X,\mathcal{U}_c)$ generated by X. There exists a unique homomorphism $g : A \to G$ for which $f = g|X$. The homomorphism g is not continuous: the group A is totally bounded, the group G is not totally bounded and the continuous homomorphic image of totally bounded group is totally bounded.

3 On pseudocompact groups

A subset L of a space X is bounded in X if for any real-valued continuous function f on X the set $f(L)$ is bounded. If the set X is bounded in the space X, then we say that X is a pseudocompact space.

Any pseudocompact group is totally bounded. For any totally bounded group G there exists a unique compact group bG such that G is a dense subgroup of bG. If the group G is pseudocompact, then $bG = \beta G$ (see [1], p.110).

The following assertion is a generalization of one theorem of M.Ursul (see [7]).

Theorem 1. *Let A be a subgroup of a pseudocompact group B and ω_1 be the first uncountable cardinal. Then in B^{ω_1} there exist two subgroups H and G for which:*

1. *G is a dense pseudocompact subgroup of B^{ω_1}.*
2. *H is a closed subgroup of G.*
3. *The groups A and H are topologically isomorphic.*

Proof. For any ordinal number $\alpha < \omega_1$ let $B_\alpha = B$, $A_\alpha = A$ and e_α be the identity in B_α. It is wellknown that $B^{\omega_1} = \prod\{B_\alpha : \alpha < \omega_1\}$ is a pseudocompact group with the identity $e = (e_\alpha : \alpha < \omega_1)$. Let $G' = \{x = (x_\alpha : \alpha < \omega_1) \in B^{\omega_1}$: *the set* $\{\alpha < \omega_1 : x_\alpha \neq e_\alpha\}$ *is countable* $\}$ and $G = \{x = (x_\alpha : \alpha < \omega_1) \in B^{\omega_1}$: *the set* $\{\alpha < \omega_1 : x_\alpha \notin A_\alpha\}$ *is countable* $\}$. By construction, $G' \subseteq G \subseteq B^{\omega_1}$ and G'

is a pseudocompact subgroup of B^{ω_1}. Since G' is dense in B^{ω_1}, the subspace G is pseudocompact and dense in B^{ω_1}. It is obvious that G' and G are subgroups of B^{ω_1}. For any $x \in B$ we put $\bar{x} = (\bar{x}_\alpha : \alpha < \omega_1)$, where $x_\alpha = x$ for any $\alpha < \omega_1$. Then $\bar{B} = \{\bar{x} : x \in B\}$ is the diagonal of B^{ω_1}. The diagonal $\bar{B}$ is a subgroup topologically isomorphic to B. The subspace $\bar{B}$ is closed in B^{ω_1}. Let $H = \{\bar{x} : x \in A\}$. Then the topological groups A and H are topologically isomorphic. By construction, $H = G \cap \bar{B}$. Thus H is a closed subgroup of the group G. □

Corollary 1. *(see [7]) Every totally bounded subgroup is a closed subgroup of some pseudocompact group.*

Theorem 2. *Let $\mathcal{U}$, $\mathcal{V}$ be two classes of pseudocompact groups with properties:*

1. *$\mathcal{V} \subseteq \mathcal{U}$.*
2. *The classes $\mathcal{U}$ and $\mathcal{V}$ are multiplicative.*
3. *If $A \in \mathcal{U}$, then A is a subgroup of some compact group $B \in \mathcal{V}$.*
4. *If $A \in \mathcal{U}$ and B is a pseudocompact subgroup of A, then $B \in \mathcal{U}$.*
5. *If $A \in \mathcal{V}$ and B is a compact group, then every closed subgroup of B is an element of $\mathcal{V}$.*

Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$.

The following assertions are true:

(A). $\mathcal{U}_0$ is a quasivariety of topological groups.

(B). If X is a space, $F(X,\mathcal{U},\mathcal{V})$ is an almost $(\mathcal{U},\mathcal{V})$-free group over a space X and $X = e_X(X) \subseteq F(X,\mathcal{U},\mathcal{V})$, then:

– the subgroup $F_0(X,\mathcal{U},\mathcal{V})$ generated by the space X in $F(X,\mathcal{U},\mathcal{V})$ is the $\mathcal{U}_0$-free topological group over space X;

– the group $F_0(X,\mathcal{U},\mathcal{V})$ is finite or $F_0(X,\mathcal{U},\mathcal{V})$ is not pseudocompact.

Proof. If $F_0(X,\mathcal{U},\mathcal{V})$ is a compact group, then $F_0(X,\mathcal{U},\mathcal{V}) \in \mathcal{V}$ and $F_0(X,\mathcal{U},\mathcal{V})$ is a $(\mathcal{U},\mathcal{V})$-free group over X.

Suppose that the space $F_0(X,\mathcal{U},\mathcal{V})$ is not compact. Consider the $\mathcal{U}_0$-free group $F(X,\mathcal{U}_0)$ over X. It is obvious that $X \subseteq F(X,\mathcal{U}_0)$ and X generated $F(X,\mathcal{U}_0)$. By assumptions, $F(X,\mathcal{U}_0)$ is a dense subgroup of some compact group $G \in \mathcal{V}$. Thus there exists a continuous homomorphism $h : F(X,\mathcal{U},\mathcal{V}) \to G$ such that $h(x) = x$ for any $x \in X$. Therefore $h|F_0(X,\mathcal{U},\mathcal{V})$ is a topological isomorphism of $F_0(X,\mathcal{U},\mathcal{V})$ onto $F(X,\mathcal{U}_0)$. We can consider that $F_0(X,\mathcal{U},\mathcal{V}) = F(X,\mathcal{U}_0)$.

Let $Y = cl_G X$, $Y^{-1} = \{y^{-1} : y \in Y \subseteq G\}$ and $Y_n = \{y_1 \cdot y_2 \cdot \ldots \cdot y_n : y_1, y_2, \ldots, y_n \in Y \cup Y^{-1}\}$ for any $n \geq 1$. Then $H = \cup\{Y_n : n \in \mathbb{N}\}$ is a subgroup of G and every set Y_n is compact. The group $F(X,\mathcal{U}_0)$ is a dense subgroup of H and H is a dense subgroup of G. Let $X_n = Y_n \cap F(X,\mathcal{U}_0)$. By construction, X_n is a closed subset of $F(X,\mathcal{U}_0)$, $X_1 = X \cup X^{-1}$ and $X_n = \{x_1 \cdot x_2 \cdot \ldots \cdot x_n : x_1, x_2, \ldots, x_n \in X_1\}$ for $n \geq 2$. If $F(X,\mathcal{U}_0)$ is infinite, then $F(X,\mathcal{U}_0)$ is not a discrete space.

It is obvious that $F(X,\mathcal{U}_0) = \cup\{X_n : n \in \mathbb{N}\}$.

Case 1. X is a discrete space.

In this case every X_n is a discrete closed subspace of $F(X,\mathcal{U}_0)$. Since $F(X,\mathcal{U}_0)$ is not discrete, X_n is nowhere dense subset of $F(X,\mathcal{U}_0)$. Suppose that $F(X,\mathcal{U}_0)$

is pseudocompact. Then $W_n = F(X,\mathcal{U}_0) \setminus X_n$ is a dense open subset of $F(X,\mathcal{U}_0)$. From the Baire category theorem, the set $W = \cap\{W_n : n \in \mathbb{N}\}$ is dense in $F(X,\mathcal{U}_0)$. By construction, we have $W = \cap\{W_n : n \in \mathbb{N}\} = F(X,\mathcal{U}_0) \setminus \cup\{X_n : n \in \mathbb{N}\} = \emptyset$, a contradiction.

Thus $F(X,\mathcal{U}_0)$ is not pseudocompact.

Case 2. X is not a discrete space.

Let a be a non-isolated point of the space X. We put $a_{2i-1} = a$ and $a_{2i} = a^{-1}$ for any $i \geq 1$. Fix $n \in \mathbb{N}$, $b \in X_n$ and a neighbourhood W of b in G. Fix $m > n$. Then $b = b \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_{2m-1} \cdot a_{2m} \in W$. There exist open subsets $W_0, W_1, \ldots, W_{2m}$ of G such that $b \in W_0, a_1 \in W_1, \ldots, a_{2m} \in W_{2m}$ and $W_0 \cdot W_1 \cdot \ldots \cdot W_{2m} \subseteq W$. There exist distinct elements $x_1, x_2, \ldots, x_{2m} \in X$ such that $y_{2i-1} = x_{2i-1} \in W_{2i-1}$ and $y_{2i} = x_{2i}^{-1} \in W_{2i}$. Then $b' = b \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m \in W \setminus X_n$. Thus X_n is nowhere dense in $F(X,\mathcal{U}_0)$. Suppose that $F(X,\mathcal{U}_0)$ is pseudocompact. Then $W_n = F(X,\mathcal{U}_0) \setminus X_n$ is a dense open subset of $F(X,\mathcal{U}_0)$. From the Baire category theorem, the set $W = \cap\{W_n : n \in \mathbb{N}\}$ is dense in $F(X,\mathcal{U}_0)$. By construction, we have $W = \cap\{W_n : n \in \mathbb{N}\} = F(X,\mathcal{U}_0) \setminus \cup\{X_n : n \in \mathbb{N}\} = \emptyset$, a contradiction.

Moreover, we prove that $F_0(X,\mathcal{U},\mathcal{V})$ is not pseudocompact provided it is infinite. Thus $F_0(X,\mathcal{U},\mathcal{V})$ is finite if and only if $F_0(X,\mathcal{U},\mathcal{V})$ is compact. □

Theorem 3. *Let $\mathcal{U}$, $\mathcal{V}$ be two classes of topological groups with the properties:*

1. $\mathcal{V} \subseteq \mathcal{U}$.
2. *The classes $\mathcal{U}$ and $\mathcal{V}$ are multiplicative.*
3. *Every group $A \in \mathcal{V}$ is compact.*
4. *If $A \in \mathcal{V}$ and B is a closed subgroup of A, then $B \in \mathcal{V}$.*
5. *If $A \in \mathcal{U}$ and B is a pseudocompact subgroup of A, then $B \in \mathcal{U}$.*
6. *If $A \in \mathcal{U}$, then A is a subgroup of some compact group $B \in \mathcal{V}$.*

Then for every space X there exists some almost $(\mathcal{U}, \mathcal{V})$-free group $(F(X, \mathcal{U}, \mathcal{V}), e_X)$ over X such that $F(X, \mathcal{U}, \mathcal{V})$ is pseudocompact and $e_X(X)$ is a closed subspace of $F(X,\mathcal{U},\mathcal{V})$.

Proof. Denote by $\mathcal{U}_0$ the class of subgroups of groups from $\mathcal{U}$. Then $\mathcal{U}_0$ is a quasi-variety of topological groups.

Fix a space X. Then there exists the $\mathcal{U}_0$-free group $(F(X,\mathcal{U}_0), e_X)$ over X. The subspace $e_X(X)$ is closed in $F(X,\mathcal{U}_0)$. The group $F(X,\mathcal{U}_0)$ is a dense subgroup of some compact group $G \in \mathcal{V}$.

By construction, for every continuous mapping $f : X \to A \in \mathcal{V}$ there exists a unique continuous homomorphism $\bar{f} : G \to A$ such that $f = \bar{f} \circ e_X$. Thus (G, e_X) is a $(\mathcal{U},\mathcal{V})$-free group over X.

Consider the projection $\pi : G^{\omega_1} \to G$ where $\pi((x_\alpha : \alpha < \omega_1)) = x_1$ for any $(x_\alpha : \alpha < \omega_1) \in G^{\omega_1}$.

Now we apply the construction from the proof of Theorem 1.

In G^{ω_1} there exists a dense pseudocompact subgroup H and a closed subgroup B of H such that $\pi(H) \supseteq \pi(B) = F(X,\mathcal{U}_0)$ and $\pi|H : H \to F(X,\mathcal{U}_0)$ is a topological isomorphism. We consider that $F(X,\mathcal{U}_0) = B \subseteq H$. Let $e'_X(x) = e_X(x) \in B$ for any $x \in X$. Then $e_X(x) = \pi(e'_X(x)) \in e_X(X) \subseteq F(X,\mathcal{U}_0)$.

Fix a continuous mapping $f : X \to A \in V$. There exists a continuous homomorphism $f_1 : G \to A$ such that $f = f_1 \circ e_X$. Now we put $\bar{f}(y) = f_1(\pi(y))$ for any $y \in H$. Then $f = \bar{f} \circ e'_X$. By construction, the mapping $\bar{f}$ is a unique continuous homomorphism of H into A for which $f = \bar{f} \circ e'_X$. Therefore, (H, e'_X) is a $(\mathcal{U}, \mathcal{V})$-free group over X. □

Remark 6. For $\mathcal{V} = \mathcal{U}_{ac}$ Theorem 3 was proved by Comfort and van Mill ([1], Theorem 4.1.9b).

Theorem 4. *Let $\mathcal{U} = \mathcal{V}$ be a class of pseudocompact groups with the properties:*

1. *The class $\mathcal{U}$ is multiplicative.*
2. *If $A \in \mathcal{U}$ and B is a pseudocompact subgroup of A, then $B \in \mathcal{U}$.*
3. *If $A \in \mathcal{U}$, then A is a subgroup of some compact group $B \in \mathcal{U}$.*

Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$.

If X is a space and there exists an almost $(\mathcal{U}, \mathcal{V})$-free group over X $(F(X,\mathcal{U},\mathcal{V}), e_X)$, then $F(X,\mathcal{U}_0)$ is a finite group.

Proof. Let X be a space for which the $(\mathcal{U}, \mathcal{V})$-free group $(F(X,\mathcal{U},\mathcal{V}), e_X)$ be given. We put $Y = e_X(X)$ and $G = F(X,\mathcal{U},\mathcal{V})$. Then $Y \subseteq G$, p_Y is the identity in G and for any continuous mapping $f : Y \to A \in \mathcal{V}$ there exists a continuous homomorphism $\bar{f} : G \to A$ for which $f = \bar{f}|Y$. In particular, G is the $(\mathcal{U}, \mathcal{V})$-free group over Y. Moreover, for any continuous mapping $g : X \to A \in \mathcal{V}$ there exists a unique continuous mapping $f : Y \to A$ such that $g = f \circ e_X$. Let G be a dense subgroup of the compact group $\bar{G} \in \mathcal{U}$. Then there exists a compact subgroup H of $\bar{G}$ such that $Y \subseteq H$ and Y topologically generated H.

We can consider that $Y \subseteq F(Y,\mathcal{U}_0) \subseteq H$, i.e. the group $F(Y,\mathcal{U}_0)$ generated by Y in H is the $\mathcal{U}_0$-free group over Y. If $F(Y,\mathcal{U}_0)$ is finite, then $F(Y,\mathcal{U}_0)$ is the $(\mathcal{U}, \mathcal{V})$-free group over X and over Y. Suppose that $F(Y,\mathcal{U}_0)$ is infinite. Then $F(Y,\mathcal{U}_0)$ is not pseudocompact.

Let $\tau = |\bar{G}|$.

There exist an uncountable cardinal $\lambda > \tau$, a set Λ and a family $\{A_\alpha : \alpha \in \Lambda\}$ of pseudocompact groups such that:

1) any A_α is a subgroup of H and $Y \subseteq A_\alpha$;

2) if A is a pseudocompact subgroup of H and $Y \subseteq A$, then $|\{\alpha \in \Lambda : A_\alpha$ *is isomorphic to* $A\}| = \lambda$;

3) $|\Lambda| = \lambda$.

Denote by H' the subgroup of H generated by the set Y. Then H' is a subgroup of A_α for any $\alpha \in \Lambda$.

Let $H_\alpha = H$ and $H'_\alpha = H'$ for any $\alpha \in \Lambda$. We put $B = \prod\{H_\alpha : \alpha \in \Lambda\}$ and $B' = \prod\{H'_\alpha : \alpha \in \Lambda\}$. If $x \in H$, then $\bar{x} = (x_\alpha : \alpha \in \Lambda)$, where $x_\alpha = x$ for any $\alpha \in \Lambda$. For any $L \subseteq H$ we put $\bar{L} = \{\bar{x} : x \in L\} \subseteq H$. Then $\bar{H}$ is the diagonal of $B = H^\lambda$. If L is a subgroup of H, then $\bar{L}$ is a subgroup of B.

Consider $A = \prod\{A_\alpha : \alpha \in \Lambda\}$. Let $a \in H'$. Then $E_a = \{x = (x_\alpha : \alpha \in \Lambda) \in A :$ *the set* $\{\alpha \in \Lambda : x_\alpha \neq a\}$ *is countable* $\}$. Then E_a is a dense pseudocompact subspace of the pseudocompact group A. If a is the identity element of the group H, then E_a is a subgroup of the group A.

Let $E = \cup\{E_a : a \in H'\}$. Then E is a dense pseudocompact subgroup of A. Moreover, A is a dense subgroup of the compact group B. By construction, $\bar{Y} \subseteq \bar{H}' \subseteq E$. The space Y is homeomorphic to $\bar{Y}$. We consider that $y = \bar{y}$ for any $y \in Y$. Then $Y = \bar{Y} \subseteq E$. Since $E \in \mathcal{U} = \mathcal{V}$, there exists a continuous homomorphism $g : G \to E$ such that $g(y) = y = \bar{y}$ for any $y \in Y$.

Let $\beta \in \Lambda$ and $\pi_\beta : B \to H_\beta$ be the projection $\pi_\beta(x_\alpha : \alpha \in \Lambda) = x_\beta$ for any $(x_\alpha : \alpha \in \Lambda) \in B$. Then $\pi_\beta(g(G))$ is a pseudocompact subgroup of the pseudocompact group A_β for any $\beta \in \Lambda$. Since H' is not a pseudocompact group, $\pi_\beta(g(G)) \setminus H'_\beta \neq \emptyset$ for any $\beta \in \Lambda$. Thus, for any $\beta \in \Lambda$ there exists $z_\beta \in G$ for which $\pi_\beta(\pi_\beta(z_\beta)) \notin H'_\beta = H'$. By assertion, $|G| \leq |\bar{G}| = \tau < \lambda$. Thus, there exists $b \in G$ such that $|\{\beta \in \Lambda : z_\beta = b\}| = \lambda$. Let $\Lambda_b = \{\beta \in \Lambda : z_\beta = b\}$ and $c = g(b) \in g(G) \subseteq E$. There exists $a \in H'$ such that $c = (c_\alpha : \alpha \in \Lambda) \in E_a$. Thus, the set $\Lambda'_c = \{\alpha : c_\alpha \neq a\}$ is countable. Then $|\Lambda_b \setminus \Lambda'_c| = \lambda$ and $c_\alpha = a$ for any $\alpha \in \Lambda_b \setminus \Lambda'_c$, a contradiction. The theorem is proved. □

Corollary 2. *Let $\mathcal{U} = \mathcal{V}$ be a class of pseudocompact groups with the properties:*

1. *The class $\mathcal{U}$ is multiplicative.*
2. *If $A \in \mathcal{U}$ and B is a pseudocompact subgroup of A, then $B \in \mathcal{U}$.*
3. *If $A \in \mathcal{U}$, then A is a subgroup of some compact group $B \in \mathcal{U}$.*
4. *There exist $A \in \mathcal{U}$ and $a \in A$ such that $a^m \neq a^n$ for any distinct $m, n \in \mathbb{N}$.*

If $F(X,\mathcal{U},\mathcal{V})$ is an almost $(\mathcal{U},\mathcal{V})$-free topological group over pointed space X, then either $|X| = 1$ or X is connected and every connected subset of any $H \in \mathcal{U}$ is a singleton set.

Example 5. Let A be a finite non-trivial group. Denote by $\mathcal{U} = \mathcal{V}$ the family of all pseudocompact subgroups of the groups A^τ, where τ is an arbitrary cardinal number. The class $\mathcal{U}$ is multiplicative and every group $A \in \mathcal{U}$ is a subgroup of some compact group $B \in \mathcal{U}$. Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$. Then $\mathcal{U}_0$ is a quasivariety of topological groups. If the space X is finite, then $F(X,\mathcal{U}_0)$ is a finite group. If X_1 is a finite space and X_2 is a connected space, then $F(X_1 \times X_2,\mathcal{U}_0) \equiv F(X_1,\mathcal{U}_0)$ is a finite group.

Remark 7. Corollary 2 improved the results obtain by M. Tkachenco and R. Fokkink (see [1], p. 110).

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