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Applications of the integral operator to the class of meromorphic functions

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Abstract. By using the Sălăgean integral operator $I^n f(z),\ z \in U$, we introduce a class of holomorphic functions denoted by $\Sigma_k(\alpha, n)$ and we obtain an inclusion relation related to this class and some differential subordinations.

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1 Introduction and preliminaries

We denote the complex plane by $\mathbb{C}$ and the open unit disc by U

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

with $\dot{U} = U - \{0\}$.

Let $\mathcal{H}(U)$ denote the class of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we have

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \ldots,\ z \in U\},$$
$$A_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\right\}$$

with $A_1 = A$.

For integer $k \geq 0$, denote by Σ_k the class of meromorphic functions, defined in $\dot{U}$, which are of the form

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n.$$

A function $f \in \mathcal{H}(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain. The function f is convex if and only if $f'(0) \neq 0$ and $\operatorname{Re}\left[\dfrac{zf''(z)}{f'(z)} + 1\right] > 0$, for $z \in U$ (see [2]).

We denote

$$K = \left\{f \in A,\ \operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0,\ z \in U\right\}$$

the set of convex functions.

Let f and g be two analytic functions in U. The function f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$, $z \in U$.

If g is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1 ([2])**.** Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \ z \in U \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec p$ for all dominants q of (1) is said to be the best dominant of (1).

Note that the best dominant is unique up to a rotation of U.

If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then p will be called an (a, n) solution, q an (a, n) dominant and $\widetilde{q}$ the best (a, n) dominant.

We will need the following lemma, which is due to D. J. Hallenbeck and St. Ruscheweyh.

Lemma 1 ([1])**.** *Let h be a convex in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \ z \in U$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and it is the best (a, n) dominant.

The following lemma is due to S. S. Miller and P. T. Mocanu.

Lemma 2 ([3])**.** *Let q be a convex function in U and let*

$$h(z) = q(z) + n\beta zq'(z)$$

where $\beta > 0$ and n is a positive integer. If $p \in \mathcal{H}[q(0), n]$ and

$$p(z) + \beta zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

Lemma 3 ([2]). *Let $f \in A$, $\gamma > 1$ and F is given by*

$$F(z) = \frac{1+\gamma}{z^{\frac{1}{\gamma}}} \int_0^z f(t) t^{\frac{1}{\gamma}-1} dt.$$

If

$$\mathrm{Re} \frac{z f''(z)}{f'(z)} + 1 > -\frac{1}{2}, \; z \in U$$

then $F \in K$.

Definition 2 ([5]). For $f \in \mathcal{H}(U)$, $f(0) = 0$ and $n \in \mathbb{N}$ we define the operator $I^n f$ by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt,$$

$$I^n f(z) = I[I^{n-1} f(z)], \; z \in U.$$

Remark 1. For $n = 1$, $I^n f$ is the Alexander operator.

Remark 2. If we denote $l(z) = -\log(1-z)$, then

$$I^n f(z) = [\underbrace{(l * l * \cdots * l)}_{n-times} * f](z), \; f \in \mathcal{H}(U), f(0) = 0.$$

By " $*$ " we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$, then $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$).

Remark 3. $I^n f(z) = \int_0^z \int_0^{t_n} \ldots \int_0^{t_2} \frac{f(t_1)}{t_1 t_2 \ldots t_n} dt_1 dt_2 \ldots dt_n$.

Remark 4. $D^n I^n f(z) = I^n D^n f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, where $D^n f$ is the Sălăgean differential operator.

2 Main results

Definition 3. If $0 \le \alpha < 1$, k positive integer and $n \in \mathbb{N}$, let $\Sigma_k(\alpha, n)$ denote the class of functions $f \in \Sigma_k$ which satisfy the inequality

$$\mathrm{Re} \left[I^n (z^2 f(z)) \right]' > \alpha, \; z \in \dot{U}. \tag{2}$$

Theorem 1. *If* $0 \le \alpha < 1$, k *positive integer and* $n \in \mathbb{N}$, *then*

$$\Sigma_k(\alpha, n) \subset \Sigma_k(\delta, n+1), \tag{3}$$

where

$$\delta = \delta(\alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right)$$

and

$$\beta(x) = \int_0^z \frac{t^{x-1}}{1+t}dt.$$

Proof. Assume that $f \in \Sigma_k(\alpha, n)$. By using the properties of the operator $I^n f$ we have

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]', \; z \in \dot{U}. \tag{4}$$

Differentiating this equality, we obtain

$$\left[I^n(z^2 f(z))\right]' = \left[I^{n+1}(z^2 f(z))\right]' + z\left[I^{n+1}(z^2 f(z))\right]''. \tag{5}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

with $p(z) \in \mathcal{H}[1, k+1]$, $z \in \dot{U}$, then (5) becomes

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z) + zp'(z), \; z \in \dot{U}.$$

Since $f \in \Sigma_k(\alpha, n)$, from Definition 3 we have

$$\mathrm{Re}[p(z) + zp'(z)] > \alpha, \; z \in \dot{U}$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z), \; z \in \dot{U}.$$

Therefore, from Lemma 1 for $\gamma = 1$, it results that

$$p(z) \prec q(z) \prec h(z), \; z \in \dot{U},$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}}\int_0^z \frac{1+(2\alpha-1)t}{1+t}t^{\frac{1}{k+1}-1}dt$$

$$= (2\alpha - 1) + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right)\frac{1}{z^{\frac{1}{k+1}}}.$$

Moreover, the function q is convex and is the best dominant.

From $p(z) \prec q(z),\ z \in \dot{U}$ it results that

$$\mathrm{Re}p(z) > \mathrm{Re}q(1) = \delta = (2\alpha - 1) + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right).$$

But

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

and

$$\mathrm{Re}\left[I^{n+1}(z^2 f(z))\right]' > \delta,$$

from Definition 3 we have $f \in \Sigma_k(\delta, n+1)$. □

Theorem 2. *Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + z(k+1)q'(z),\ z \in U.$$

If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{6}$$

then

$$\left[I^{n+1}(z^2 f(z))\right]' \prec q(z),\ z \in \dot{U}$$

and this result is sharp.

Proof. By using the properties of the operator $I^n f$ we have

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]',\ z \in \dot{U}. \tag{7}$$

By differentiating (7), we obtain

$$\left[I^n(z^2 f(z))\right]' = \left[I^{n+1}(z^2 f(z))\right]' + z\left[I^{n+1}(z^2 f(z))\right]''. \tag{8}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z),$$

with $p(z) \in \mathcal{H}[1, k+1]$ then we obtain

$$p(z) + zp'(z) \prec h(z) = q(z) + z(k+1)q'(z),\ z \in \dot{U}.$$

By using Lemma 2 for $\beta = 1$, we have

$$p(z) \prec q(z),\ z \in \dot{U},$$

or

$$\left[I^{n+1}(z^2 f(z))\right]' \prec q(z),\ z \in \dot{U}$$

and this result is sharp. □

Theorem 3. *Let q be a convex function with $q(0) = 1$ and*

$$h(z) = q(z) + z(k+1)q'(z),\ z \in U.$$

If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{9}$$

then

$$\frac{I^n(z^2 f(z))}{z} \prec q(z),\ z \in \dot{U}$$

and this result is sharp.

Proof. We let

$$p(z) = \frac{I^n(z^2 f(z))}{z},\ z \in \dot{U}. \tag{10}$$

By differentiating this relation, we obtain

$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z),\ z \in \dot{U}.$$

Then (9) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + z(k+1)q'(z),\ z \in \dot{U}.$$

By using Lemma 2 we have

$$p(z) \prec q(z),\ z \in \dot{U}$$

i.e.

$$\frac{I^n(z^2 f(z))}{z} \prec q(z),\ z \in \dot{U}$$

and this result is sharp. □

Theorem 4. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, and $h'(0) \neq 0$ which satisfies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2},\ z \in U.$$

If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{11}$$

then

$$\left[I^{n+1}(z^2 f(z)\right]' \prec g(z),\ z \in \dot{U}$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t)t^{\frac{1}{k+1}-1}dt,\ z \in U. \tag{12}$$

The function q is convex and it is the best $(1, k+1)$ dominant.

Proof. By applying Lemma 3 for the function given by (12) and function h, for $\gamma = k+1$, we obtain that the function q is convex.

By using the properties of the operator $I^n f$ we let

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]', \; z \in \dot{U}. \tag{13}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

with

$$p(z) \in \mathcal{H}[1, k+1]$$

and differentiating (13) we obtain

$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z), \; z \in \dot{U}$$

and (11) becomes

$$p(z) + zp'(z) \prec h(z), \; z \in \dot{U}.$$

By using Lemma 1 for $\gamma = 1$ and $n = k+1$ we have

$$p(z) \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t)t^{\frac{1}{k+1}-1}dt, \; z \in U,$$

i.e.

$$\left[I^n(z^2 f(z))\right]' \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t)t^{\frac{1}{k+1}-1}dt, \; z \in U.$$

Moreover the function q is the best $(1, k+1)$ dominant. □

Theorem 5. *Let $h \in H(U)$ with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \; z \in U.$$

If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination

$$\left[I^n(z^2 f(z))\right]' \prec h(z), \; z \in \dot{U} \tag{14}$$

then

$$\frac{I^n(z^2 f(z))}{z} \prec q(z), \; z \in \dot{U}$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t)t^{\frac{1}{k+1}-1}dt, \; z \in U.$$

The function q is convex and is the best $(1, k+1)$ dominant.

Proof. We let

$$p(z) = \frac{I^n(z^2 f(z))}{z}, \ z \in \dot{U} \tag{15}$$

with $p(z) \in \mathcal{H}[1, k+1]$.

By differentiating (15), we obtain

$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z), \ z \in \dot{U}, \tag{16}$$

then (14) becomes

$$p(z) + zp'(z) \prec h(z), \ z \in \dot{U}.$$

By using Lemma 1, we have

$$p(z) \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \ z \in U,$$

i.e.

$$\left[I^n(z^2 f(z))\right]' \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \ z \in U.$$

Moreover the function q is the best $(1, k+1)$ dominant. □

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