BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(59), 2009, Pages 31–36
ISSN 1024–7696

The graded Jacobson radical of associative rings

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Abstract. We introduce a consistent definition for the graded Jacobson radical for group graded rings without unity. We compare the graded Jacobson radical for rings with unity and those without. We find that for group graded rings, the descriptions are equivalent.

Mathematics subject classification: 34C05, 58F14.
Keywords and phrases: Graded ring, Jacobson radical.

In the book of Năstăsescu and Van Oystaeyen [1] on group graded rings, two equivalent descriptions of the graded Jacobson radical for rings with unity are given. Several investigations of the graded Jacobson radical have appeared over the last two decades (see [2–6] or [7] for example) all for rings with unity. In [9] a comprehensive account of special radicals of graded rings without unity was presented. Unfortunately the descriptions given in the section for the Jacobson radical came (in the most part) from [1] on group graded rings with unity. After an extensive literature search, it seems that no actual definition of the graded Jacobson radical for rings without unity has appeared. The definition given here is the most natural one – the intersection of the annihilators of all simple graded modules – and it is meaningful more generally for semigroup graded rings, though for semigroups in general it may not be a graded ideal.

As an example of a consequence of this investigation, we show that a 1984 result of Năstăsescu [3] that $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ (for a finite group G of order $n \in \mathbb{Z}^+$ where R is a G-graded ring with unity and $\mathcal{J}_{gr}$ is the G-graded Jacobson radical) can be extended to group graded rings without unity.

1 Unital Extensions of graded rings

By a unital extension of a ring, we mean an embedding of a ring R without unity into a ring R^u with unity. We do this in the standard way (see [10] for example). We reserve the use of R^u to always mean the unital extension of R. Thus R^u is made up of the additive group $R \bigoplus \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. Elements in $R \bigoplus \mathbb{Z}$ are denoted by ordered pairs $\{(r, n) : r \in R, n \in \mathbb{Z}\}$ with componentwise addition and multiplication defined by

$$(r, n)(s, m) = (rs + mr + ns, nm)$$

where $r, s \in R$ with $n, m \in \mathbb{Z}$.

Lemma 1 ([8], p.136)**.** *Let R be a ring without identity, and let R^u be the standard unital extension of R. Then $\mathcal{J}(R) = \mathcal{J}(R^u)$, where $\mathcal{J}$ is the Jacobson radical.*

For this investigation we require specifically that R be group graded. This allows us to place the identity element carefully into our graded ring without causing major offence to the structure of our ring. So we begin with a ring R graded by a group G with group identity e. Any element $r \in R$ can be written uniquely as $r = \sum_{g \in G} r_g$, where $r_g \in R_g$ for each $g \in G$. We embed our G-graded ring R into R^u as above. We identify R with its copy in R^u and since $(0,1)R_g \subseteq R_g$ and $R_g(0,1) \subseteq R_g$ for all $g \in G$, we can grade R^u by putting the identity element $(0,1)$ in the e component, whence R^u becomes G-graded with

$$R^u = R^u_e \oplus \bigoplus_{g \in G \setminus \{e\}} R_g.$$

For any $r \in R$ we have

$$(r,n) = (r_e,n) + \sum_{\substack{g \in G \\ g \neq e}} (r_g,0).$$

(Recalling that R_e is a subring of R, we can see that the e component in R^u is just given by the standard unital extension of R_e in R.)

2 Graded ideals and modules

Let G be a group or semigroup and suppose I is an ideal (left, right or two-sided) of a G-graded ring R. Then I is said to be a *G-graded ideal* if

$$I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$$

(so that I as a ring is G-graded). A G-graded left (right, two-sided) ideal M of R is a *G-graded-maximal* left (right, two-sided) ideal if $M \neq R$ and M is not contained in any other proper G-graded left (right, two-sided) ideals of R.

Let G be a group or semigroup. A left module T over a G-graded ring R is a *G-graded left module* if there exist additive subgroups T_g of T with

$$T = \bigoplus_{g \in G} T_g$$

and $R_x T_y \subseteq T_{xy}$ for all $x, y \in G$. We suppress the adjective "left" throughout, but note that a development based on right modules is also possible.

Let G be a group or semigroup. A G-graded module T over a G-graded ring R is a *graded-simple module* if $T \neq 0$ and 0 and T are its only graded submodules.

The annihilator of any G-graded module T is

$$\mathcal{A}(T) = \{a \in R : at = 0 \ \text{ for all } t \in T\}.$$

Annihilators of modules are ideals.

3 $\mathcal{J}_{gr}(R)$ for rings with unity

We describe the graded Jacobson radical here for group graded rings *with unity.* In Section 4 we give an equivalent description of the graded Jacobson radical for rings without unity.

Let G be a group with identity element e and R a G-graded ring with unity. In this case the *graded Jacobson radical* $\mathcal{J}_{gr}(R)$ of R is defined to be the intersection of all S-graded-maximal left ideals of R.

In [1] the equivalence of other definitions of the graded Jacobson radical of a group graded ring is shown. One of these defines the graded Jacobson radical as the intersection of all left annihilators of all G-graded-simple R-modules.

Recently, Abrams and Menini [7] considered the graded Jacobson radical of graded rings with unity, extending the definition to include semigroup-graded rings. In this case the graded Jacobson radical is defined to be the intersection of all left annihilators of all G-graded-simple R-modules.

4 $\mathcal{J}_{gr}(R)$ for rings without unity

The Jacobson radical of a ring without unity has a handful of equivalent descriptions (see [11] for example) including one as the intersection of modular maximal left ideals. For rings with unity, all ideals are modular and so the wording of the definition is altered slightly. In both cases, the equivalent definition of the Jacobson radical as the intersection of all the left annihilators of simple left modules is the same. It seems then that the natural choice for defining the graded Jacobson radical for group graded rings without unity is as the intersection of annihilators of simple modules, coincident with the definition of the graded Jacobson radical in the case the ring has unity.

For a ring R graded by a semigroup G, we define the *graded Jacobson radical* of R as the intersection of left annihilators of all G-graded-simple R-modules. We use the $_{gr}$ here to indicate a graded structure. It turns out that as long as G is a group, $\mathcal{J}_{gr}(R)$ is always a graded ideal. For more general semigroups this need not be so.

Example 1 ([9], Example 6)**.** Consider the set $S = \{(1,1),(1,2),(2,1),(2,2)\}$ under

$$(r,s)\cdot(t,u) = (r,u) \qquad (r,s,t,u \in \{1,2\})\,.$$

Then $(S,\cdot)$ forms a rectangular band. The semigroup ring $A = kS$ with coefficients in a field k is S-graded in the usual way, that is

$$A = kS = k(1,1)\oplus k(1,2)\oplus k(2,1)\oplus k(2,2).$$

Then the element $(1,1)-(1,2)$ annihilates M, for any simple A-module M. (Note that $S \subseteq A$ as k is unital.) This puts $(1,1)-(1,2)$ in $\mathcal{J}_{gr}(A)$. Now let $N = A(1,1)$. Then N is a graded-simple A-module but $(1,1)$ doesn't annihilate N. The consequence is that $(1,1) \notin \mathcal{J}_{gr}(A)$. This means that $\mathcal{J}_{gr}(A)$ is actually an *ungraded* ideal of A.

Theorem 1. *Let G be a group and let R be a G-graded ring without unity. Then $\mathcal{J}_{gr}(R) = \mathcal{J}_{gr}(R^u)$ where $\mathcal{J}_{gr}$ is the G-graded Jacobson radical and R^u is the unital extension of R.*

Proof. Any R-module M becomes a unital R^u-module if we define

$$(r, n)m = rm + nm$$

for $(r, n) \in R^u$ and $m \in M$.

Let Y be a G-graded R-module with $Y = \oplus_g Y_g$ and $R_h Y_g \subseteq RY_{hg}$. For $(r_e, n) \in R^u_e$ and any $y_g \in Y_g$ we have $(r_e, n)y_g = r_e y_g + n y_g \in Y_g$ and so Y is a G-graded unital R^u-module. Any G-graded unital R^u-module is a G-graded R-module

Similarly, if K is an G-graded R-submodule of a G-graded R-module M, then K is a unital G-graded R^u-submodule of the G-graded unital R^u-module M, and *vice versa*.

So any G-graded-simple R-module is also a unital G-graded-simple R^u-module, and *vice versa*.

Suppose M is any G-graded-simple R-module with left annihilator $\mathcal{A}(M)$. Take any $a \in \mathcal{A}(M)$. Then $a = \sum_{g \in G} a_g$. For any $h \in G$, pick an $m \in M_h$ (since M is graded). Then

$$0 = \left(\sum_{g \in G} a_g \right) m = \sum_{g \in G} a_g m$$

where $a_g m \in M_{gh}$ for each $g \in G$. Since the sum runs over distinct gh (here G is a group), we have $a_g m = 0$ for all $g \in G$, and so all the homogeneous components $a_g \in \mathcal{A}(M)$. Thus the annihilator is a G-graded ideal:

$$\mathcal{A}(M) = \bigoplus_{g \in G} R_g \cap \mathcal{A}(M) = \bigoplus_{g \in G} \mathcal{A}(M)_g.$$

In the same way the set

$$\mathcal{A}(M)^u = \{ r \in \mathbb{R}^u : rm = 0 \ \ \forall\, m \in M \}$$

is a graded ideal of R^u.

For any $a \in \mathcal{A}(M)$, the element $(a, 0) \in R^u$ is in $\mathcal{A}(M)^u$ since $(a, 0)m = am = 0$, so, $\mathcal{A}(M) \subseteq \mathcal{A}(M)^u$. It is clear to see that the elements $(a, 0) \in R^u$ behave exactly as the elements $a \in R$. So to compare $\mathcal{A}(M)$ with its unital extension, we need only consider the e-component.

Suppose there is an $(r_e, n) \in \mathcal{J}(R^u) \backslash \mathcal{J}(R)$. Then $(r_e, n) \in \mathcal{A}(M)^u_e = \mathcal{A}(M)^u \cap R^u_e$ with $n \neq 0$. Thus for all graded simple R-modules M we have

$$(r_e, n)m = r_e m + nm = 0, \qquad \text{for all} \quad m \in M.$$

This means that multiplication of an element in any simple module by r_e has the same effect as multiplying by $-n \in \mathbb{Z}$. For every prime p, $\mathbb{Z}_p$ is a simple R-module with trivial multiplication. It is graded-simple if we let $\mathbb{Z}_p$ be the e component, with all other components being equal to zero. For all $x \in \mathbb{Z}_p$ we now have $0 = (r_e, n)x = r_e x + nx = nx$, so n is divisible by p. This being so for every p, we conclude that $n = 0$.

This means that no extra killers are admitted by unital extension. Hence, if R is a ring without unity, then $\mathcal{J}_{gr}(R) = \mathcal{J}_{gr}(R^u)$. □

As an example of the potential application of Theorem 1 we extend a theorem of Năstăsescu for finite group graded rings with unity to include rings without unity.

Theorem 2 ([12], [3], Theorem 5.4)**.** *Let G be a finite group of order $n \in \mathbb{Z}^+$ and let R be a G-graded ring with unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}(R)$ is the Jacobson radical of R and $\mathcal{J}_{gr}$ is the G-graded Jacobson radical.*

Corollary 1. *Let G be a finite group of order $n \in \mathbb{Z}^+$ and let R be a G-graded ring with or without unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}_{gr}$ is the G-graded Jacobson radical.*

Proof. If R is a ring graded by a finite group with unity, then this is just Theorem 2. Otherwise, applying Lemma 1 and Theorem 1 yields $n\mathcal{J}(R) = n\mathcal{J}(R^u) \subseteq \mathcal{J}_{gr}(R^u) = \mathcal{J}_{gr}(R)$ which completes the proof. □

References

[1] Năstăsescu C., Van Oystaeyen F. *Graded Ring Theory.* North-Holland, Amsterdam-New York, 1982.

[2] Cohen M., Rowen L. H. *Group graded rings.* Commun. Algebra, 1983, **11**, 1253–1270.

[3] Năstăsescu C. *Group rings of graded rings.* J. Pure Appl. Algebra, 1984, **33**, 313–335.

[4] Cohen M., Montgomery S. *Group-graded rings, smash products, and group actions.* Trans. Amer. Math. Soc., 1984, **282**, 237–258.

[5] Beattie M. A. *A generalization of the smash product of a graded ring.* J. Pure Appl. Algebra, 1988, **51**, 219–226.

[6] Karpilovsky G. *The Jacobson Radical of Classical Rings.* Wiley and Sons, New York, 1991.

[7] Abrams G., Menini C. *Embedding modules in graded modules over a semigroup-graded ring.* Comm. Algebra, 2001, **29**, 2611-2625.

[8] Kertész A. *Vorlesungen über artinsche Ringe.* Akadémiai Kiadó, Budapest, 1968.

[9] Balaba I. N. *Special radicals of graded rings.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2004, No. 1(44), 2004, 26–33.

[10] Hungerford T. W. *Algebra*. Springer, New York, 1974.

[11] Gardner B. J., Wiegandt R. *Radical Theory of Rings*. Marcel Dekker, New York, 2004.

[12] Bergman G. M. *On the Jacobson radical of graded rings*. Unpublished typescript.

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Received February 2, 2008

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