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Continuity of the norm of a composition operator between weighted Banach spaces of holomorphic functions

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Abstract. We consider composition operators C_ϕ between given weighted Banach spaces of analytic functions defined on the open unit disk and explore the continuity of the map, which given an analytic self-map of the disk, takes as its value the associated composition operator resp. the norm of this operator.

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1 Introduction

Let v and w be strictly positive bounded continuous functions (*weights*) on the open unit disk D in the complex plane. Define the weighted space

$$H_v^\infty \quad := \quad \{f \in H(D);\ \|f\|_v = \sup_{z\in D} v(z)|f(z)| < \infty\},$$

where $H(D)$ denotes the space of all analytic functions which is usually endowed with the compact-open topology *co*. Endowed with the weighted sup-norm $\|.\|_v$, H_v^∞ becomes a Banach space (for further information on these spaces see e.g.[1]). Spaces of this type appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e. g.[1–3, 10, 14, 16, 17, 19, 22–24].

Let ϕ be an analytic self-map of D. The equation $C_\phi f = f\circ\phi$ defines a composition operator on $H(D)$. We are interested in composition operators acting between weighted Banach spaces of holomorphic functions as defined above.

Composition operators and differences of composition operators have recently been of much interest and have been studied on various spaces of analytic functions, see e. g.[4–8, 13, 18, 21].

Next, let $\mathcal{C}$ be the set of all composition operators endowed with the usual norm. Moreover, let $ASM(D)$ be the set of all analytic self-maps of D and φ_z be the Möbius transformation of D which interchanges z and the origin, namely,

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w},\ w \in D.$$

Then the pseudohyperbolic distance $\rho(z,w)$ for $z,w \in D$ is defined as

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|.$$

Next, on $ASM(D)$ we want to introduce the following metric

$$d(\phi,\psi) := \sup_{z\in D} \rho(\phi(z),\psi(z)) \text{ for every } \phi,\psi \in ASM(D).$$

Recall $\|\phi\|_\infty = \sup_{z\in D} |\phi(z)|$ for every $\phi \in ASM(D)$. Considering the maps

$$\begin{aligned} N_\infty &: (ASM(D), \|.\|_\infty) \to \mathbb{R}, \phi \to \|C_\phi\| \\ N_d &: (ASM(D), d) \to \mathbb{R}, \phi \to \|C_\phi\| \end{aligned}$$

as well as

$$\begin{aligned} K_\infty &: (ASM(D), \|.\|_\infty \to \mathcal{C}, \phi \to C_\phi \\ K_d &: (ASM(D), d) \to \mathcal{C}, \phi \to C_\phi, \end{aligned}$$

we pose the following question:

For which functions $\phi \in ASM(D)$ is N_∞ or N_d or K_∞ or K_d continuous?

2 Notations and definitions

For notation on composition operators and spaces of analytic functions we refer the reader to [8, 11, 12, 21]. An important tool in the theory of weighted spaces is the so-called *associated weight* (see [3]) defined by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\}},\ z \in D.$$

By [3] the associated weight has the following properties:

(i) $\tilde{v}$ is continuous,

(ii) $\tilde{v} \geq v > 0$,

(iii) for each $z \in D$ there is $f_z \in H_v^\infty$, $||f_z||_v \leq 1$, such that $|f_z(z)| = \dfrac{1}{\tilde{v}(z)}$.

In general it is not so easy to compute the associated weight. Thus, an important class of weights is the class of *essential* weights, that is, the weights such that there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in D.$$

For examples of essential weights and conditions when weights are essential see [3, 5] and [4]. Especially interesting are *radial weights* v, i.e. weights which satisfy

$v(z) = v(|z|)$ for every $z \in D$. Every radial weight which is non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1} v(z) = 0$ is called a *typical weight*. In the sequel every radial weight is assumed to be non-increasing.

We say that a radial weight v has condition $(L1)$ (which is due to Lusky, see [16]) if for every $z \in D$ the following holds:

$$(L1) \quad \inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0.$$

Note, that each radial weight satisfying $(L1)$ is essential (see [9] Proposition 2(b)). Lusky showed (see [16]) that each of the following weights satisfies condition $(L1)$

$$\begin{aligned} v(z) &= (1-|z|)^p,\ 0 < p < \infty, \\ v(z) &= (1-\log(1-|z|))^{-\beta},\ \beta > 0, \\ v(z) &= (1-|z|)^p(1-\log(1-|z|))^{-\beta},\ 0 < p < \infty \text{ and } \beta > 0. \end{aligned}$$

If v is a radial weight on D which is continuously differentiable with respect to $|z|$, then by [9] we know that condition $(L1)$ is equivalent to each of the following two conditions:

(A) there are $0 < r < 1$ and $1 < C < \infty$ with $\dfrac{v(z)}{v(p)} \le C$ for every $z, p \in D$ with $\rho(z,p) \le r$,

(U) there exists $\alpha > 0$ such that $\dfrac{v(z)}{(1-|z|)^\alpha}$ is increasing near the boundary of D.

Next, let us list up some auxiliary results which are essential for the proofs of this article's results.

Theorem 1. ([5] Proposition 2.1) *Let v and w be weights. Then $C_\phi : H_v^\infty \to H_w^\infty$ is continuous (or, equivalently, bounded) if and only if* $\sup_{z\in D} \dfrac{w(z)}{\tilde{v}(\phi(z))} < \infty$.

Thus, the norm $\|C_\phi\|$ may be identified with $\sup_{z\in D} \dfrac{w(z)}{\tilde{v}(\phi(z))}$.

Theorem 2. ([7] Proposition 2) *Let v and w be weights such that v is radial and satisfies $(L1)$. Then $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is continuous (or, equivalently, bounded) if and only if*

$$\sup_{z\in D} w(z)\rho(\phi(z),\psi(z)) \max\left\{\frac{1}{\tilde{v}(\phi(z))}, \frac{1}{\tilde{v}(\psi(z))}\right\} < \infty.$$

Thus, the norm $\|C_\phi - C_\psi\|$ can be identified with the expression

$$\sup_{z\in D} w(z)\rho(\phi(z),\psi(z)) \max\left\{\frac{1}{\tilde{v}(\phi(z))}, \frac{1}{\tilde{v}(\psi(z))}\right\}.$$

Lemma 1. ([15] Lemma 1) *Let v be a radial weight on D satisfying the Lusky condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. There is a constant $M < \infty$ such that if $f \in H_v^\infty$, then*

$$|v(p)f(p) - v(q)f(q)| \leq M\|f\|_v \rho(p,q)$$

for all $p, q \in D$.

The following lemma is taken from [15] and can be found there as Lemma 2. For the sake of completeness as well as for a better understanding of this note we want to repeat the proof given in [15].

Lemma 2. ([15] Lemma 2) *Let v be a radial weight on D satisfying the Lusky condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. Then $\dfrac{v(p)}{v(z)} \to 1$ when $\rho(z,p) \to 0$.*

Proof. Since condition (A) holds, Lemma 14 in [9] gives that there exist $1 < C < \infty$ and $0 < s < 1$ such that

$$|v(p)f(z) - v(p)f(p)| \leq \frac{4C}{s}\|f\|_v \rho(p,z)$$

for all $f \in H_v^\infty$ and all $p, z \in D$ with $\rho(p,z) < \dfrac{s}{2}$. Combining this inequality with Lemma 1, we obtain that

$$\begin{aligned} |v(p)f(z) - v(z)f(z)| &\leq |v(p)f(z) - v(p)f(p)| + |v(p)f(p) - v(z)f(z)| \\ &\leq \left(\frac{4C}{s} + M\right)\|f\|_v \rho(p,z) \end{aligned}$$

for all $f \in H_v^\infty$ and all $p, z \in D$ with $\rho(p,z) < \dfrac{s}{2}$. The Lusky condition $(L1)$ implies condition (U), so by Proposition 3.4 in [3], there is a constant $C_1 > 0$ such that $v(z) \leq \tilde{v}(z) \leq C_1 v(z)$ for all $z \in D$. Moreover, for each $z \in D$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$ such that $f_z(z) = \dfrac{1}{\tilde{v}(z)}$. Hence

$$\left|\frac{v(p)}{v(z)} - 1\right| \leq C_1 \left|\frac{v(p) - v(z)}{\tilde{v}(z)}\right| = C_1|v(p)f_z(z) - v(z)f_z(z)| \leq C_2\rho(p,z) \to 0$$

when $\rho(p,z) \to 0$. □

3 Results

Theorem 3. *Let v and w be weights such that v is radial and satisfies condition $(L1)$. If K_d is continuous at ϕ, i.e.*

$$d(\phi, \phi_n) \to 0 \Longrightarrow \|C_{\phi_n} - C_\phi\| \to 0, \tag{1}$$

then C_ϕ is continuous. If we assume additionally that v is continuously differentiable with respect to $|z|$, then the converse is also true.

Proof. First, we suppose that K_d is continuous at ϕ. By Theorem 2, (1) is equivalent with

$$d(\phi,\phi_n) \to 0 \Longrightarrow \sup_{z\in D} w(z)\rho(\phi(z),\phi_n(z)) \max\left\{\frac{1}{\tilde{v}(\phi(z))},\frac{1}{\tilde{v}(\phi_n(z))}\right\} \to 0.$$

We prove the assertion indirectly, i.e. we assume that C_ϕ is not continuous. By Theorem 1, there is a sequence $(z_n)_n \subset D$, $|\phi(z_n)| \to 1$, such that $\dfrac{w(z_n)}{\tilde{v}(\phi(z_n))} \geq n$ for every $n \in \mathbb{N}$. Next, we choose $\phi_n(z) := \dfrac{\phi(z)-\frac{1}{n}}{1-\frac{1}{n}\overline{\phi(z)}}$ for every $n \in \mathbb{N}$. Each ϕ_n is an element of $ASM(D)$ since $|\phi_n(z)| \leq \dfrac{|\phi(z)|-\frac{1}{n}}{1-\frac{1}{n}|\overline{\phi(z)}|} < \dfrac{1-\frac{1}{n}}{1-\frac{1}{n}} = 1$ for every $z \in D$. We have

$$\rho(\phi(z),\phi_n(z)) = \left|\frac{\phi(z)-\dfrac{\phi(z)-\frac{1}{n}}{1-\frac{1}{n}\overline{\phi(z)}}}{1-\overline{\phi(z)}\dfrac{\phi(z)-\frac{1}{n}}{1-\frac{1}{n}\overline{\phi(z)}}}\right| = \left|\frac{\phi(z)-\frac{1}{n}|\phi(z)|^2-\phi(z)+\frac{1}{n}}{1-\frac{1}{n}\overline{\phi(z)}-|\phi(z)|^2+\frac{1}{n}\overline{\phi(z)}}\right| = \frac{1}{n}$$

for every $z \in D$. Hence $\sup_{z\in D}\rho(\phi(z),\phi_n(z)) = \frac{1}{n} \to 0$ if $n \to \infty$. Then we obtain

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))}\rho(\phi(z_n),\phi_n(z_n)) \geq \frac{n}{n} = 1.$$

This is a contradiction.

In order to show the converse we assume additionally that v is continuously differentiable with respect to $|z|$. Let us assume that $d(\phi,\phi_n) = \sup_{z\in D}\rho(\phi(z),\phi_n(z)) \to 0$ if $n \to \infty$. Hence, for every $z \in D$, $\rho(\phi(z),\phi_n(z)) \to 0$ if $n \to \infty$. By Lemma 2, $\dfrac{v(\phi(z))}{v(\phi_n(z))} \to 1$ for every $z \in D$ if $n \to \infty$. Thus, there is $n_0 \in \mathbb{N}$ such that C_{ϕ_n} is continuous for every $n \geq n_0$, i.e. we can find $M > 0$ with $\sup_{z\in D} w(z)\max\left\{\dfrac{1}{\tilde{v}(\phi(z))},\dfrac{1}{\tilde{v}(\phi_n(z))}\right\} \leq M$ for every $n \geq n_0$. Now, since $\sup_{z\in D}\rho(\phi(z),\phi_n(z)) \to 0$ if $n \to \infty$, an application of Theorem 2 yields

$$\|C_\phi - C_{\phi_n}\| = \sup_{z\in D} w(z)\max\left\{\frac{1}{\tilde{v}(\phi(z))},\frac{1}{\tilde{v}(\phi_n(z))}\right\}\rho(\phi(z),\phi_n(z)) \to 0,$$

and the claim follows. □

The following proposition follows directly from Theorem 3 but we want to give a direct proof at this point.

Proposition 1. *Let v and w be weights such that v is radial and satisfies condition $(L1)$. If K_∞ is continuous at ϕ, i.e.*

$$\|\phi_n - \phi\|_\infty \to 0 \Longrightarrow \|C_{\phi_n} - C_\phi\| \to 0, \tag{2}$$

then C_ϕ is continuous.

Proof. By Theorem 2, (2) is equivalent with

$$\|\phi_n - \phi\|_\infty \to 0 \Longrightarrow \sup_{z\in D} w(z)\rho(\phi(z), \phi_n(z)) \max\left\{\frac{1}{\tilde{v}(\phi_n(z))}, \frac{1}{\tilde{v}(\phi(z))}\right\} \to 0.$$

We prove the proposition indirectly, i.e. we assume that C_ϕ is not continuous. By Theorem 1, we can find a sequence $(z_n)_n \subset D$, $|\phi(z_n)| \to 1$, such that $\dfrac{w(z_n)}{\tilde{v}(\phi(z_n))} \geq n$ for every $n \in \mathbb{N}$. W.l.o.g. we may assume that $|\phi(z_n)|^2 \geq \left(1 - \dfrac{1}{n}\right)$. Next, we choose $\phi_n(z) := \left(1 - \dfrac{1}{n}\right)\phi(z)$. Obviously we have $\|\phi_n - \phi\|_\infty \to 0$. Then we obtain

$$\begin{aligned}
\frac{w(z_n)}{\tilde{v}(\phi(z_n))}\rho(\phi(z_n), \phi_n(z_n)) &= \frac{w(z_n)}{\tilde{v}(\phi(z_n))}\frac{1}{n}\frac{|\phi(z_n)|}{1 - \left(1 - \frac{1}{n}\right)|\phi(z_n)|^2} \\
&\geq \frac{w(z_n)}{\tilde{v}(\phi(z_n))}\frac{1}{n}\frac{|\phi(z_n)|^2}{1 - \left(1 - \frac{1}{n}\right)|\phi(z_n)|^2} \\
&\geq n\frac{1}{n}\frac{1 - \frac{1}{n}}{1 - \left(1 - \frac{2}{n} + \frac{1}{n^2}\right)} \geq \frac{n-1}{n}\frac{n^2}{2n-1} \\
&\geq (n-1)\frac{n}{2n-1}.
\end{aligned}$$

This is a contradiction. □

The other assertion of Theorem 3 does not remain true if we consider K_∞ instead of K_d as the following example shows.

Example 1. Choose $w(z) = 1, v(z) = 1 = \tilde{v}(z)$, $z \in D$, as well as $\phi(z) = \dfrac{z+1}{2}$, $\phi_n(z) = \left(1 - \dfrac{1}{n}\right)\dfrac{z+1}{2}$, $z \in D$, $n \in \mathbb{N}$. We obviously have

$$\|C_\phi\| = \sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi(z))} = 1 = \sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi_n(z))} = \|C_{\phi_n}\|$$

for every $n \in \mathbb{N}$ and

$$\|\phi - \phi_n\|_\infty = \frac{1}{n}\sup_{z\in D}\left|\frac{z+1}{2}\right| \leq \frac{1}{n} \to 0$$

if $n \to \infty$, but

$$\sup_{z\in D}\rho(\phi(z),\phi_n(z)) = \sup_{z\in D}\left|\frac{\frac{1}{n}\frac{z+1}{2}}{1-\left(1-\frac{1}{n}\right)\left|\frac{z+1}{2}\right|^2}\right| = \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \not\to 0$$

if $n \to \infty$. Hence by Theorem 1 C_ϕ and each C_{ϕ_n} are continuous, but

$$\begin{aligned}\|C_\phi - C_{\phi_n}\| &= \sup_{z\in D} w(z)\rho(\phi(z),\phi_n(z))\max\left\{\frac{1}{\tilde{v}(\phi(z))},\frac{1}{\tilde{v}(\phi_n(z))}\right\}\\ &= \sup_{z\in D}\rho(\phi(z),\phi_n(z)) = 1 \not\to 0.\end{aligned}$$

Proposition 2. *Let w be a weight and v be a radial weight satisfying the condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. If K_∞ resp. K_d is continuous at ϕ, then N_∞ resp. N_d is continuous at ϕ.*

Proof. By the proof of Lemma 2 we can find a constant $C > 0$ such that

$$\begin{aligned}|\|C_{\phi_n}\| - \|C_\phi\|| &= \left|\sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi_n(z))} - \sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi(z))}\right|\\ &\le \sup_{z\in D} w(z)\left|\frac{1}{\tilde{v}(\phi_n(z))} - \frac{1}{\tilde{v}(\phi(z))}\right|\\ &\le \sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi(z))}\left|\frac{\tilde{v}(\phi(z))}{\tilde{v}(\phi_n(z))} - 1\right|\\ &\le C\sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\phi_n(z))\\ &\le \|C_\phi - C_{\phi_n}\|,\end{aligned}$$

and the claim follows. □

Remark 1. (i) The continuity of N_∞ at ϕ does not imply the continuity of K_∞ at ϕ as Example 1 shows.

(ii) Let w be a weight and v be a radial weight satisfying the condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. If N_∞ is continuous at ϕ, then C_ϕ is continuous.

The converse of (ii) in the remark above is not true as the following example shows.

Example 2. Choose $v(z) = 1 - |z| = \tilde{v}(z)$, $w(z) = |1-z|$, $z \in D$, as well as $\phi(z) = \frac{z+1}{2}$ and $\phi_n(z) = \frac{z+\xi_n}{2}$, $n \in \mathbb{N}$, $z \in D$, where $(\xi_n)_{n\in\mathbb{N}} \subset \partial D$ is a sequence going to 1. Then, for each $n \in \mathbb{N}$ we obtain

$$\|C_{\phi_n}\| = \sup_{z\in D}\frac{w(z)}{\tilde{v}(\phi_n(z))} = \sup_{z\in D}\frac{|1-z|}{1-\left|\frac{z+\xi_n}{2}\right|} = \infty,$$

but

$$\|C_\phi\| = \sup_{z\in D} \frac{|1-z|}{1-\left|\dfrac{z+1}{2}\right|} < \infty.$$

Hence $\|\phi - \phi_n\|_\infty \to 0$, but $\|C_{\phi_n}\| \not\to \|C_\phi\|$.

Remark 2. Let v be a radial weight on D satisfying the Lusky condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. Then Lemma 2 yields that N_d is always continuous.

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