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Exact solutions for a rotational flow of generalized second grade fluids through a circular cylinder

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Abstract. In this note the velocity field and the associated tangential stress corresponding to the rotational flow of a generalized second grade fluid within an infinite circular cylinder are determined by means of the Laplace and Hankel transforms. At time $t = 0$ the fluid is at rest and the motion is produced by the rotation of the cylinder, around its axis, with the angular velocity Ωt. The velocity field and the adequate shear stress are presented under integral and series forms in terms of the generalized G-functions. Furthermore, they are presented as a sum between the Newtonian solutions and the adequate non-Newtonian contributions. The corresponding solutions for the ordinary second grade fluid and Newtonian fluid are obtained as particular cases of our solutions for $\beta = 1$, respectively $\alpha = 0$ and $\beta = 1$.

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1 Introduction

The motion of a fluid in a rotating or sliding cylinder is of interest to both theoretical and practical points of view. It is very important to study the mechanism of viscoelastic fluids flow in many industry fields, such as oil exploitation, chemical and food industry and bio-engineering [1]. Fetecau *et al.* [2] have considered the general case of helical flow of an Oldroyd-B fluid and have determined the velocity fields and the associated tangential stresses in forms of series in terms of Bessel functions. Recently fractional calculus has encountered much success in the description of complex dynamics, such as relaxation, oscillation, wave and viscoelastic behaviour. Bagley [3], He [4], Tan [5] used fractional calculus to handle various problems regarding to flow of the second grade fluid.

In this note we will study the rotational flow of a generalized second grade fluid within an infinite circular cylinder of radius R. The motion is due to the cylinder that at time $t = 0^+$, begins to rotate around its axis with the angular velocity Ωt. Exact analytic solutions of this problem are obtained by using Hankel and Laplace transforms and generalized G-functions. Some classical results can be obtained as special cases of our solutions.

2 Governing equations

The constitutive equation of an incompressible generalized second grade fluid is given by [4–6]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A_1}$ and $\mathbf{A_2}$ are the kinematic tensors defined through

$$\mathbf{A_1} = grad\,\mathbf{v} + (grad\,\mathbf{v})^T, \tag{2}$$

$$\mathbf{A_2} = D_t^\beta\mathbf{A_1} + \mathbf{A_1}(grad\,\mathbf{v}) + (grad\,\mathbf{v})^T\mathbf{A_1}. \tag{3}$$

In the above relations $\mathbf{v}$ is the velocity, the superscript T denotes the transpose operator, and D_t^β is the Riemann-Liouville fractional derivative operator defined by [7]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}d\tau; \quad 0 < \beta \le 1, \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function. For $\beta = 1$ the generalized model reduces to classical model of second grade fluid because $D_t^1 f = df/dt$.
Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$div\,\mathbf{v} = tr\,\mathbf{A_1} = 0. \tag{5}$$

If this model is required to be compatible with thermodynamics, then the material moduli must meet the following restrictions [8]

$$\mu \ge 0\ , \quad \alpha_1 \ge 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{6}$$

In cylindrical coordinates (r, θ, z), the rotational flow velocity is given by [2, 6]

$$\mathbf{v} = \mathbf{v}(r,t) = \omega(r,t)\mathbf{e}_\theta, \tag{7}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction. For such flows the constraint of incompressibility is automatically satisfied.
Introducing (7) into constitutive equation, we find that

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)(\frac{\partial}{\partial r} - \frac{1}{r})\omega(r,t), \tag{8}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress which is different of zero. The last equation together with the equations of motion lead to the governing equation

$$\frac{\partial\omega(r,t)}{\partial t} = (\nu + \alpha D_t^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\omega(r,t)\,, \quad r \in (0,R)\,,\ \ t > 0, \tag{9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, ρ is the constant density of the fluid and $\alpha = \alpha_1/\rho$.

3 On the rotational flow through an infinite circular cylinder

Let us consider an incompressible generalized second grade fluid at rest in an infinite circular cylinder of radius R. At time zero, the cylinder suddenly begins to rotate about its axis with the angular velocity Ωt. Owing to the shear, the fluid is gradually moved, its velocity being of the form (7) and governing equation is (9). The appropriate initial and boundary conditions are

$$\omega(r,0)=0\,; \quad r\in[0,R)\,, \qquad \omega(R,t)=R\Omega t\,; \quad t\geq 0. \tag{10}$$

To solve this problem we shall use as in [6, 9] the Laplace and Hankel transforms.

3.1 Calculation of the velocity field

Applying the Laplace transform to Eqs. (9) and (10) and using the Laplace transform formula for sequential fractional derivatives [7], we obtain

$$(\nu+\alpha q^{\beta})\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{\omega}(r,q)-q\overline{\omega}(r,q)=0, \tag{11}$$

where the image function $\overline{\omega}(r,q)=\int_0^\infty \omega(r,t)e^{-qt}dt$ of $\omega(r,t)$ has to satisfy the condition

$$\overline{\omega}(R,q)=\frac{R\Omega}{q^2}, \tag{12}$$

q being the transform parameter. In the following we denote by

$$\overline{\omega}_H(r_{1n},q)=\int_0^R r\overline{\omega}(r,q)J_1(rr_{1n})dr, \tag{13}$$

the Hankel transform of $\overline{\omega}(r,q)$, where $J_1(\cdot)$ is the Bessel function of first kind of order one and $r_{1n}\,,n=1,2,3,...$ are the positive roots of the transcendental equations $J_1(Rr)=0$.

Multiplying now both sides of Eq. (11) by $rJ_1(rr_{1n})$, integrating with respect to r from 0 to R and taking into account the condition (12) and the equality

$$\int_0^R r\left[\frac{\partial^2\overline{\omega}(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{\overline{\omega}(r,q)}{r^2}\right]J_1(rr_{1n})dr= \\ =Rr_{1n}J_2(Rr_{1n})\overline{\omega}(R,q)-r_{1n}^2\overline{\omega}_H(r_{1n},q), \tag{14}$$

we find that

$$\overline{\omega}_H(r_{1n},q)=\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{\nu+\alpha q^{\beta}}{q^2\left[q+\alpha r_{1n}^2 q^{\beta}+\nu r_{1n}^2\right]}. \tag{15}$$

Now, for a more suitable presentation of the final results, we rewrite Eq. (15) in the following equivalent form

$$\overline{\omega}_H(r_{1n},q)=\overline{\omega}_{1H}(r_{1n},q)+\overline{\omega}_{2H}(r_{1n},q)+\overline{\omega}_{3H}(r_{1n},q), \tag{16}$$

where

$$\overline{\omega}_{1H}(r_{1n},q)=\frac{\Omega R^2}{q^2 r_{1n}}J_2(Rr_{1n}),\tag{17}$$

$$\overline{\omega}_{2H}(r_{1n},q)=-\frac{\Omega R^2 J_2(Rr_{1n})}{\nu r_{1n}^3}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)\tag{18}$$

and

$$\overline{\omega}_{3H}(r_{1n},q)=\alpha\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{1}{q+\nu r_{1n}^2}\frac{q^{\beta-1}}{\left[q+\alpha r_{1n}^2 q^\beta+\nu r_{1n}^2\right]}.\tag{19}$$

Using the formula

$$\int_0^R r^2 J_1(rr_{1n})dr=\frac{R^2}{r_{1n}}J_2(Rr_{1n}),\tag{20}$$

we get that inverse Hankel transform of the function $\overline{\omega}_{1H}(r_{1n},q)$ is

$$\overline{\omega}_1(r,q)=\frac{\Omega r}{q^2}.\tag{21}$$

The inverse Hankel transforms of the functions $\overline{\omega}_{kH}(r_{1n},q)$, $k=2,3$, are the functions

$$\overline{\omega}_{kH}(r,q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{J_2^2(Rr_{1n})}\overline{\omega}_{kH}(r_{1n},q).\tag{22}$$

Introducing Eqs. (21) and (22) into Eq. (16) we find that the Laplace transform $\overline{\omega}(r,q)$ has the form

$$\overline{\omega}(r,q)=\frac{\Omega r}{q^2}-\frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)+$$
$$+2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\frac{1}{q+\nu r_{1n}^2}\frac{q^{\beta-1}}{\left[q+\alpha r_{1n}^2 q^\beta+\nu r_{1n}^2\right]}.\tag{23}$$

To obtain the velocity field $\omega(r,t)=L^{-1}\{\overline{\omega}(r,q)\}$ we will apply the discrete inverse Laplace transform method [6, 7, 9]. For this we use the expansion

$$F(q)=\frac{q^{\beta-1}}{q+\alpha r_{1n}^2 q^\beta+\nu r_{1n}^2}=\frac{q^{-1}}{(q^{1-\beta}+\alpha r_{1n}^2)+\nu r_{1n}^2 q^{-\beta}}=$$
$$=\sum_{k=0}^{\infty}(-\nu r_{1n}^2)^k\frac{q^{-\beta k-1}}{\left(q^{1-\beta}+\alpha r_{1n}^2\right)^{k+1}}.\tag{24}$$

Introducing (24) into (23), applying the discrete inverse Laplace transform and using the following properties

$$L^{-1}\{F_1(q)F_2(q)\} = (f_1 * f_2)(t) = \int_0^t f_1(t-s)f_2(s)ds, \tag{25}$$

where

$$f_k(t) = L^{-1}\{F_k(q)\}\,, \qquad k = 1, 2\,,$$

$$L^{-1}\Big\{\frac{q^b}{(q^a-d)^c}\Big\} = G_{a,b,c}(d,t), \quad Re\,(ac-b) > 0, \tag{26}$$

and [10]

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{27}$$

are the generalized G-functions, we find for $\omega(r,t)$ the expression

$$\omega(r,t) = \omega_N(r,t) + 2\alpha\,\Omega \sum_{n=1}^{\infty} \frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})} \sum_{k=0}^{\infty}\Big(-\nu\, r_{1n}^2\Big)^k \times$$

$$\times \int_0^t exp[-\nu r_{1n}^2(t-s)]G_{1-\beta,\,-\beta k-1,\,k+1}\Big(-\alpha r_{1n}^2, s\Big)ds, \tag{28}$$

where [2, Eq. (4.5)]

$$\omega_N(r,t) = r\Omega t - \frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})}\big[1 - exp(-\nu r_{1n}^2 t)\big], \tag{29}$$

is the similar solution for Newtonian fluids, performing the same motion.

3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (8) we find that

$$\overline{\tau}(r,q) = (\mu + \alpha_1 q^{\beta})(\frac{\partial}{\partial r} - \frac{1}{r})\overline{\omega}(r,q). \tag{30}$$

The image function $\overline{\omega}(r,q)$ can be obtained using Eqs. (27)-(29) and the formula

$$L\Big\{\frac{t^a}{\Gamma(a+1)}\Big\} = \frac{1}{q^{a+1}}\,, \qquad a > -1. \tag{31}$$

Consequently, applying the Laplace transform to Eq. (28), differentiating the result with respect to r and using the identity

$$rJ_1'(rr_{1n}) - J_1(rr_{1n}) = -rr_{1n}J_2(rr_{1n}), \tag{32}$$

we find that

$$\frac{\partial\overline{\omega}}{\partial r} - \frac{\overline{\omega}}{r} = \frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2J_2(Rr_{1n})}\left(\frac{1}{q} - \frac{1}{q+\nu r_{1n}^2}\right) -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha\, r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\frac{1}{q+\nu r_{1n}^2}\frac{1}{q^{k+(1-\beta)(j+1)+1}}. \tag{33}$$

Introducing (33) into (30) we get

$$\overline{\tau}(r,q) = 2\rho\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2J_2(Rr_{1n})}\left(\frac{1}{q} - \frac{1}{q+\nu r_{1n}^2}\right) + 2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}\frac{q^{\beta-1}}{q+\nu r_{1n}^2} -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\left\{\frac{1}{q+\nu r_{1n}^2}\left[\frac{\mu}{q^{k+(1-\beta)(j+1)+1}} - \frac{\nu\alpha_1 r_{1n}^2}{q^{k+3+(1-\beta)j-2\beta}}\right] + \frac{\alpha_1}{q^{k+3+(1-\beta)j-2\beta}}\right\}. \tag{34}$$

Applying the inverse Laplace transform to Eq. (34), we find that the shear stress $\tau(r,t)$ has the form

$$\tau(r,t) = \tau_N(r,t) + 2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}G_{1,\beta-1,1}(-\nu r_{1n}^2,t) -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha\, r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)]\left\{\frac{\mu s^{k+(1-\beta)(j+1)}}{\Gamma[k+(1-\beta)(j+1)+1]} -\right.$$

$$-\frac{\nu\alpha_1 r_{1n}^2 s^{k+2+(1-\beta)j-2\beta}}{\Gamma[k+3+(1-\beta)j-2\beta]}\Big\}ds\ -2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\times$$

$$\times\sum_{k,j=0}^{\infty}\frac{\Big(-\nu\, r_{1n}^2\Big)^k\Big(-\alpha\, r_{1n}^2\Big)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{\alpha_1 t^{k+2+(1-\beta)j-2\beta}}{\Gamma[k+3+(1-\beta)j-2\beta]}, \tag{35}$$

where [2, Eq. (5.3) for $\alpha=0$]

$$\tau_N(r,t)=2\rho\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})}\big[1-exp(-\nu r_{1n}^2 t)\big], \tag{36}$$

is the shear stress corresponding to a Newtonian fluid performing the same motion.

4 Special cases

Making $\beta=1$ into Eq. (28), we obtain the velocity field

$$\omega(r,t)=\omega_N(r,t)+2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\sum_{k=0}^{\infty}\Big(-\nu\, r_{1n}^2\Big)^k\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)]G_{0,-k-1,k+1}\Big(-\alpha r_{1n}^2,s\Big)ds, \tag{37}$$

corresponding to an ordinary second grade fluid, performing the same motion. Similarly, from (35), we obtain the shear stress

$$\tau(r,t)=\tau_N(r,t)+2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}G_{1,0,1}(-\nu r_{1n}^2,t)-$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{(-\nu\, r_{1n}^2)^k(-\alpha r_{1n}^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)](\mu+\nu\alpha_1 r_{1n}^2)\frac{s^k}{\Gamma(k+1)}ds-$$

$$-2\alpha\alpha_1\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{(-\nu\, r_{1n}^2)^k(-\alpha r_{1n}^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{t^k}{\Gamma(k+1)}, \tag{38}$$

corresponding to an ordinary second grade fluid, performing the same motion. The above relations can be simplified if we use the following relations:

$$G_{0,-k-1,k+1}(-\alpha r_{1n}^2, s) = \frac{s^k}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{(-\alpha r_{1n}^2)^j \Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)} = \\ = \frac{s^k}{\Gamma(k+1)}(1+\alpha r_{1n}^2)^{-(k+1)}, \tag{39}$$

$$\sum_{k=0}^{\infty}(-\nu r_{1n}^2)^k G_{0,-k-1,k+1}(-\alpha r_{1n}^2, s) = \frac{1}{1+\alpha r_{1n}^2} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{\nu r_{1n}^2 s}{1+\alpha r_{1n}^2}\right)^k = \\ = \frac{1}{1+\alpha r_{1n}^2} exp\left(-\frac{\nu r_{1n}^2 s}{1+\alpha r_{1n}^2}\right), \tag{40}$$

and

$$G_{1,0,1}(-\nu r_{1n}^2, t) = exp\left(-\nu r_{1n}^2 t\right). \tag{41}$$

As a result, we find the velocity field and the adequate shear stress under simplified forms

$$\omega(r,t) = r\Omega t - \frac{2\Omega}{\nu} \sum_{n=1}^{\infty} \frac{J_1(r r_{1n})}{r_{1n}^3 J_2(R r_{1n})}\left[1 - exp\left(-\frac{\nu r_{1n}^2}{1+\alpha r_{1n}^2} t\right)\right] \tag{42}$$

and

$$\tau(r,t) = 2\rho\Omega \sum_{n=1}^{\infty} \frac{J_2(r r_{1n})}{r_{1n}^2 J_2(R r_{1n})}\left[1 - \frac{1}{1+\alpha r_{1n}^2} exp\left(-\frac{\nu r_{1n}^2}{1+\alpha r_{1n}^2} t\right)\right], \tag{43}$$

which are identical to Eqs. (5.1) and (5.3) from [2].

If in Eqs. (42) and (43), we make $\alpha = 0$, then the corresponding solutions of the Newtonian fluids are recovered.

5 Conclusions

In this note, the velocity field and the adequate shear stress corresponding to the rotational flow induced by an infinite circular cylinder in an incompressible generalized second grade fluid, have been determined using Hankel and Laplace transforms. The motion is produced by the circular cylinder that at the initial moment begins to rotate around its axis with angular velocity Ωt. The solutions that have been obtained, written under integral and series forms in terms of generalized G-function, satisfy all imposed initial and boundary conditions. Furthermore, they are presented as a sum between the Newtonian solutions and the adequate non-Newtonian contributions. In the special case when $\beta = 1$, or $\beta = 1$ and $\alpha = 0$, the corresponding solutions for ordinary second grade fluid and Newtonian fluid, respectively, performing the same motion, are obtained.

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