About Division of d-Convex Simple Graphs in M-Prime Graphs

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Abstract. In this article we research the structure of d-convex simple graphs in order to extend the already known classes of graphs of this type. We do this using some new operations and new graphs. We introduce the notion of M-prime graphs and split all d-convex simple graphs into M-prime graphs using the M operation. After that we describe all M-prime graphs we know.

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1 Preliminary Considerations

After we obtained an iterative method of the characterisation of all d-convex simple graphs [3], we observe that this method allow us to construct d-convex simple graphs, which are complicated enough, but it says few things about the structure of these graphs, what they look like and how diverse they are. Therefore, to learn this structure, which would allow us to resolve some application problems, like in [4], we start to study these graphs via special classes, to be precise we want to extend the already known classes of d-convex simple graphs from [5, 6]. So we introduce a new operation $M$ [1, 2], that is algebraic on all known classes of these graphs and allows us to do this extensions by using only one new graph. In this article, using the $M$-operation and some new operations on this set of graphs, we will define new classes of d-convex simple graphs, which extend the already known classes of d-convex simple graphs and have visible structure, our goal being to characterise as many d-convex simple graphs as it is possible.

Definition 1. [5, 6] An undirected graph $G = (X, U)$ is called d-convex simple if any subset of vertexes $A \subset X$, $2 < |A| < |X|$ is not d-convex.

Let us denote by $\Gamma(x)$ the neighbourhood of vertex $x$, i. e. $\Gamma(x) = \{y \in X | x \sim y\}$. We will say that vertex $x$ dominates the vertex $y$ if $\Gamma(x) \supset \Gamma(y)$ and vertex $y$ is called a copy for vertex $x (x \neq y)$, in graph $G = (X; U)$ if $\Gamma(x) = \Gamma(y)$ [5].

Definition 2. [5] The subset of vertexes $D$ is called dominating in graph $G = (X, U)$ if for $\forall x \in X$, $\exists y \in D$ such that $y$ dominates $x$, where it is possible that $y = x$. 
In [5] it is shown that all dominating sets that are minimal by inclusions, of a graph \( G \) are isomorphic and each of them generates the same subgraph \( G_0 \), called the **atom** of graph \( G \). To construct \( G_0 \) we have first to find the sets:

\[
S = \{ x \in X : \forall y \in X \Rightarrow \Gamma(x) \not\subseteq \Gamma(y) \}; \\
R = \{ x \in X \setminus S : \forall y \in X \Rightarrow \Gamma(x) \not\subset \Gamma(y) \}.
\]

Then for \( \forall x \in R \) form the set \( R(x) = \{ x \} \cup \{ y \in R : \Gamma(x) = \Gamma(y) \} \). By this way \( R \) is divided into classes of equivalence. \( G_0 \) is formed from \( S \) and one vertex from each class of equivalence. For each vertex \( x_0 \in G_0 \) we can find a vertex \( x \in G \) such that \( x \) corresponds to \( x_0 \), because \( G_0 \) is a copy of a subgraph of \( G \). We denote by \( L(G, G_0) \) a new graph that is obtained from \( G \), \( G_0 \) and the following edges: for each vertex \( x_0 \in G_0 \) we will add all edges between \( x_0 \) and all vertexes from \( \Gamma(x) \). It is easy to see that in the graph \( L(G, G_0) \) the pair \( x, x_0 \) will be adjacent to the same vertexes, so the pairs of such kind will be pairs of copies. The next theorem is true:

**Theorem 1.** [5, 6] If \( G \) is a connected graph, without cycles of length three (called triangles), then the graph \( L(G, G_0) \) is \( d \)-convex simple, where \( G_0 \) is the atom of the graph \( G \).

Let \( G_1 \) and \( G_2 \) be two \( d \)-convex simple graphs, which have one pair of copies \( x_1, x_2 \in G_1 \) and \( y_1, y_2 \in G_2 \). Let us denote by \( M_{x_1=y_1}^{x_2=y_2}(G_1, G_2) \) the graph obtained from \( G_1 \) and \( G_2 \) by pasting together \( x_1 \) with \( y_1 \) and \( x_2 \) with \( y_2 \). The new graph \( G = M_{x_1=y_1}^{x_2=y_2}(G_1, G_2) \) contains two vertexes less than the union of graphs \( G_1 \) and \( G_2 \) and as many edges as have \( G_1 \) and \( G_2 \) together (Fig. 1). We will write \( G = M(G_1, G_2) \) if we know the pairs of vertexes that participate in forming the new graph \( G \).

![Figure 1](image-url)

**Theorem 2.** [1, 2] If \( G_1 \) and \( G_2 \) are two \( d \)-convex simple graphs, where one pair of copies \( x_1, x_2 \in G_1 \) and \( y_1, y_2 \in G_2 \) exists, then the graph \( G = M_{x_1=y_1}^{x_2=y_2}(G_1, G_2) \) is also \( d \)-convex simple.

From this theorem it results that the operation \( M \) introduced above is an algebraic operation on the set of \( d \)-convex simple graphs. In [1] this operation \( M \) is studied on some known classes of \( d \)-convex simple graphs [5] namely:
1. $\mathcal{A}$ is the set of all d-convex simple graphs without cycles of length 3, where each vertex is dominated by other;

2. $\mathcal{F}$ are graphs without cycles of length 3 and without generated subgraphs $F$ (Fig. 2a);

3. $\mathcal{H}_1$ are hereditary-modular graphs, i.e. bipartite graphs where each isometric cycle is of length 4;

4. $\mathcal{H}_2$ are chordal graphs, i.e. bipartite graphs where each generated cycle is of length 4;

5. $\mathcal{H}_3$ are hereditary by distance graphs, i.e. graphs without cycles of length 3 and where each generated connected subgraph is isometric;

6. $\mathcal{P}$ is the set of d-convex simple and planar graphs.

Let $S\mathcal{F}$, $S\mathcal{H}_1$, $S\mathcal{H}_2$, $S\mathcal{H}_3$ be all d-convex simple graphs from classes of graphs $\mathcal{F}$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ respectively. The next lemmas are true:

**Lemma 1.** [5] If $G$ is a graph from the class $\mathcal{A}$, then $G$ is d-convex simple.

**Lemma 2.** [5] If $G$ is a d-convex simple graph without generated subgraphs $F$ (Fig. 2a), then $G$ is from $\mathcal{A}$.

Also, in [5] is proved next relation:

$$\mathcal{P} \subset S\mathcal{H}_3 \subset S\mathcal{H}_2 \subset S\mathcal{H}_1 \subset S\mathcal{F} \subset \mathcal{A}.$$ 

We have to say that reverse affirmations of Lemmas 1, 2 are false, because a d-convex simple graph that is not in $\mathcal{A}$ is the graph H illustrated in Fig. 2b and a d-convex simple graph that contains $F$ as a generated subgraph and belongs to the class $\mathcal{A}$ is the graph $L(F, F_0)$.

**Theorem 3.** [5] Let $G$ be a locally finite graph then:

1. $G \in \mathcal{A}$ if and only if $G = L(\Gamma, \Gamma_0)$, where $\Gamma$ is a connected graph without cycles of length 3, $\Gamma_0$ is the atom of $\Gamma$;
2. $G \in SF$ if and only if $G = L(\Gamma, \Gamma_0)$, where $\Gamma \in F$;

3. $G \in SH_i$ if and only if $G = L(\Gamma, \Gamma_0)$, where $\Gamma \in H_i$, $i = 1, 2, 3$;

4. $G \in P$ if and only if $G = L(\Gamma, \Gamma_0)$, where $\Gamma$ is a tree with at least 3 vertexes.

From this theorem and L operation it results that the graphs of classes $P$, $SH_1$, $SH_2$, $SF$, $A$ have at least one pair of vertexes copies and then we can apply the M operation to them.

**Theorem 4.** [1] For any two finite graphs $G_1$ and $G_2$ are true the next affirmations:

1. If $G_1, G_2 \in A$, then $G = M(G_1, G_2) \in A$;

2. If $G_1, G_2 \in SF$, then $G = M(G_1, G_2) \in SF$;

3. If $G_1, G_2 \in SH_i$, then $G = M(G_1, G_2) \in SH_i$, $i = 1, 2, 3$;

4. If $G_1, G_2 \in P$, then $G = M(G_1, G_2) \in P$;

5. If $G_1, G_2$ are two d-convex simple and bipartite graphs which have at least one pair of copies, then $G = M(G_1, G_2)$ is also d-convex simple and bipartite.

In other words, theorem 6 asserts that the introduced operation M is algebraic on all mentioned classes, where the class $A$ is the vastest. Lemma 4 asserts that all d-convex simple graphs without generated subgraph $F$ are from $A$. It results that a new class of d-convex simple graphs should have only graphs with $F$ as a subgraph and some vertexes that are not dominated.

### 2 Extensions of Classes of d-Convex Simple Graphs

Let us denote by $C$ a class of d-convex simple graphs, for example one of classes we have mentioned above.

Let $G$ be a d-convex simple graph, not from $C$, which has at least one pair of copies vertexes, because we need that $G$ could participate in the operation M with other graphs. For example if we consider that $C = A$ then as $G$ the graph $H$ (Fig. 2b) can be toked.

**Definition 3.** The set of all graphs that could be obtained from the graph $G$ and graphs of set $C$, by using the M operation a finite number of times, is called the extension of class $C$ by graph $G$ and denoted $C[G]$.

The next properties are true:

1. $C[G]$ is a class of d-convex simple graphs;

2. $C \subset C[G]$;

3. If $C_1 \subset C_2$, then $C_1[G] \subset C_2[G]$;
4. If $C_1 \subseteq C_2$ and $G \notin C_2$, then $C_1[G] \not\subseteq C_2$.

Now we can form the extensions of known classes of d-convex simple graphs by graph $H$ (Fig. 2b). We obtain $P[H], S\mathcal{H}_i[H], i = 1, 2, 3, S\mathcal{F}[H], A[H]$. The next relation is true:

$$P[H] \subset S\mathcal{H}_3[H] \subset S\mathcal{H}_2[H] \subset S\mathcal{H}_1[H] \subset S\mathcal{F}[H] \subset A[H].$$

Moreover, it is also true that $A \subset A[H]$, so the class $A[H]$ is larger than all classes of d-convex simple graphs known by now.

But the graph $H$ is not the unique graph that can make extensions, and any other graph, that has the same properties generate with $A$ new extensions. We can also make extension of extension of some classes of graphs. Let $\sigma$ be a set of d-convex simple graphs and $C$ be a class of d-convex simple graphs such that the graphs of the set $\sigma$ are not from class $C$, then:

**Definition 4.** The set of all graphs that could be obtained from the graphs of the set $\sigma$ and graphs of set $C$, by using the M operation a finite number of times, is called the extension of class $C$ by the set $\sigma$ and denoted $C[\sigma]$.

The next properties are true:

1. $C \subset C[\sigma] \subset G$, where $G$ is the set of all d-convex simple graphs;
2. $C[\sigma_1 \cup \sigma_2] = C[\sigma_1][\sigma_2] = C[\sigma_2][\sigma_1]$.

**Definition 5.** We will say that the d-convex simple graph $G$ is divisible with respect to the M operation if there exist two d-convex simple graphs $G_1$ and $G_2$ such that $G = M(G_1, G_2)$. In this case the graphs $G_1$ and $G_2$ will be called divisors of the graph $G$.

**Definition 6.** The d-convex simple graph $G$ is called M-prime if it is not divisible with respect to the M operation.

It is easy to see that the graph $H$ (Fig. 2b) is a M-prime graph.

**Theorem 5.** The d-convex simple graph $G$ is divisible with respect to the M operation if and only if there exists a pair of copies vertexes $z_1$ and $z_2$ in $G$ such that as result of the elimination of the vertexes $z_1$, $z_2$ from $G$, we obtain an unconnected graph.

**Proof.** Necessity: Let $G$ be a d-convex simple graph that is divisible with respect to the M operation, then by the definition of divisibility there are d-convex simple graphs $G_1, G_2$, with pairs of copies vertexes $x_1, x_2$ and $y_1, y_2$ respectively, such that $G = M_{x_1=y_1}(G_1, G_2)$. As result of the elimination from the graph $G$ of the vertexes $z_1 = x_1 = y_1, z_2 = x_2 = y_2$, the obtained graph is obviously unconnected.

Sufficiency: Let $G$ be a d-convex simple graph, where there exists a pair of vertexes copies $z_1, z_2$, such that as result of the elimination of the vertexes $z_1, z_2$...
from $G$, we obtain a graph with two components, not necessarily connected. Let us denote by $G_1$ and $G_2$ each of components and make the next changes: in the first component we add two vertexes $x_1, x_2$, and all vertexes from $G_1$ which were adjacent with $z_1, z_2$ in $G$ now will be adjacent with $x_1, x_2$; in the second component we also add two vertexes $y_1, y_2$ and make the same thing, i.e. all vertexes from $G_2$ which were adjacent with $z_1, z_2$ in $G$ now will be adjacent with $y_1, y_2$. Of course $G = M_{x_1=y_1}(G_1, G_2)$. It remains to prove that the graphs $G_1$ and $G_2$ are $d$-convex simple. Let us show that $G_1$ is $d$-convex simple, the fact that $G_2$ is $d$-convex simple can be proved by analogy. First we want to show that $d - \text{conv}_{G_1}(\{x_1, x_2\}) = X_{G_1}$. Indeed, if we construct $d$-convex hull in $G$ of any two vertexes $v_1$ and $v_2$ that are not from $G_1$, then there must be obligatory the vertexes $z_1$ and $z_2$, which would attract in this hull all vertexes from $G_1$. From this result

$$d - \text{conv}_{G_1}(\{x_1, x_2\}) = X_{G_1}.$$  

Let now $x, y$ be two vertexes from $G_1$ at distance two. As above, because $G$ is $d$-convex simple we have

$$d - \text{conv}_{G}(\{x, y\}) = \bigcup_{i=0}^{\infty} B_i = X_G.$$  

Results that $\exists k \geq 0$ such that $z_1, z_2 \in B_k$ and $z_1, z_2 \notin B_{k-1}$ and all vertexes from $B_k$, except $z_1, z_2$, are from $G_1$, because after the elimination of $z_1, z_2$ from $G$ an unconnected graph remained. We construct by the same way the convex hull of vertexes $x, y$ in $G_1$ until we come to the set $B_k$, where instead of vertexes $z_1, z_2$, we have $x_1, x_2$. So we have that $\{x_1, x_2\} \subset d - \text{conv}(\{x, y\})$, from (*) it results that

$$G_1 = d - \text{conv}_{G_1}(\{x_1, x_2\}) \subseteq d - \text{conv}_{G_1}(\{x, y\}).$$  

The reverse inclusions is obvious so we have that $G_1$ is a $d$-convex simple graph. 

□

From this theorem follows the next corollary:

**Corollary.** *Decomposition of an arbitrary $d$-convex simple graph into M-prime graphs is unique.*

### 3 The Sets of M-Prime Graphs

Let us denote by $\mathcal{B}$ the set of all M-prime graphs from $\mathcal{G} \backslash \mathcal{A}$, by $\mathcal{B}_1$ those graphs from $\mathcal{B}$ which have at least one pair of vertexes copies, and denote by $\mathcal{B}_2$ the rest graphs from $\mathcal{B}$, i.e. those graphs where don’t exist pairs of copies vertexes. We have $\mathcal{B}_1 \neq \emptyset$ because $H \in \mathcal{B}_1$ (Fig. 2b). Let us see that $\mathcal{B}_2 \neq \emptyset$, too. For that we construct the graphs $J_k = (X_k, U_k), \forall k \in \mathbb{N}$, where $X_k = \{z_1, z_2, z_3, z_4, z_5, z_6, x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$, $U_k = \{(z_1, z_4); (z_1, z_5); (z_2, z_3); (z_2, z_5); (z_2, z_6); (z_3, z_4); (z_3, z_6)\} \cup \{(x_i, z_4); (x_i, z_5); (x_i, z_6) | \forall i \in \{1, 2, \ldots, k\}\} \cup \{(y_i, z_1); (y_i, z_2); (y_i, z_3) | \forall i \in \{1, 2, \ldots, k\}\},$ Fig. 3.
By direct verification we can see that graphs $J_k, \forall k \in \mathbb{N}$ are d-convex simple and that no one vertex is dominated and respectively does not exist any pair of copies vertexes, therefore it results $\{J_k, \forall k \in \mathbb{N}\} \subset B_2$. So we have that $B_2 \neq \emptyset$.

Figure 3.

Let now $G = (X, U)$ be an arbitrary, undirected, d-convex simple graph that contains one pair of copies vertex $x_1$ and $x_2$. From this graph we form a new graph, where we add one more vertex copy $x_3$ of vertexes $x_1, x_2$. Let us denote this graph by $G^{++}$ (this notation is borrowed from the language C++, where i++ increases the value of i by one entity).

**Lemma 3.** For any finite graph $G$ are true the next assertions:

1. If $G \in \mathcal{G}$, then $G^{++} \in \mathcal{G}$;
2. If $G \in \mathcal{A}$, then $G^{++} \in \mathcal{A}$;
3. If $G \in \mathcal{B}$, then $G^{++} \in \mathcal{B}$;

**Proof.** 1. Indeed, let $G \in \mathcal{G}$ be any d-convex simple graph that contains one pair of copies $x_1, x_2$, and $x_3$ is their new copy in $G^{++}$. Then the d-convex hull of any two nonadjacent vertexes from $G$ will contain together with $x_1, x_2$ the vertex $x_3$, in $G^{++}$. The d-convex hull of vertexes $x_3, y, \forall y \in G, y \not\sim x_3$, will contain the same vertexes as the convex hull of vertexes $x_1, y$ or $x_2, y$, because they are copies, but the d-convex hull of each of the last pair contains all vertexes of $G$, because $G$ is d-convex simple. So we have showed that the d-convex hull of any pair of nonadjacent vertexes of $G^{++}$ contains all vertexes of this graph, from which it results that $G^{++}$ is d-convex simple: $G^{++} \in \mathcal{G}$.
2. Let $G \in A$, so $G$ is a d-convex simple graph, where any vertex is dominated by other. From the first part of this proof we have that $G^{++}$ is d-convex simple, remains to show that $G^{++}$ does not contain vertexes, that are not dominated. The new added vertex $x_3$ is dominated by their copies $x_1$ and $x_2$, all other vertexes being dominated by condition, it results $G^{++} \in A$.

3. Let $G \in B$, so $G$ is a d-convex simple graph which has at least one vertex $v$ that is not dominated and let $G$ have a pair of copies vertexes $x_1, x_2$. Then from the first part of this proof we have that $G^{++}$ is d-convex simple, where the vertex $v$ is not dominated, because $x_3$ cannot dominate $v$ as a copy of $x_1$, otherwise $x_1$ dominates $v$ in $G$, too. Other new vertexes that should dominate the vertex $v$ also do not exists, it results $G^{++} \in B$.

So we have introduced an operation that allows us the multiplication of copies vertexes, i.e. if we have a d-convex simple graph $G$ with a pair of copies vertexes, then we can form a new d-convex simple graph, analogical with $G$, but where instead of two copies vertexes we have $n$ copies vertexes, $\forall n \in \mathbb{N}$, $n$ is fixed. For example, by using operation of multiplication, we obtain from the graph $H$ (Fig. 2b) a countable set of M-prime graphs and with vertexes that are not dominated: $\{H_k \mid k \in \mathbb{N}\}$, Fig. 4.

![Figure 4](image_url)

If a graph $G$ contains more than one pair of copies vertexes, then we can use the operation of multiplication over all of them, or only on some of them arbitrary, not necessarily equal numbers of times.

The reverse operation it is also true, i.e. if we have a d-convex simple graph $G$ and $x_1, x_2, x_3$ are three copies vertexes of $G$, then the graph $G^{-}$, where $x_3$ is missing, will be also d-convex simple.

Let us now have a d-convex simple graph $G$ and $v$ an arbitrary vertex of it. Let us form the graph $G^{++}$, where we have added a copy vertex for $v$, which we denote $v'$. The next lemma is true:

**Lemma 4.** If $G$ is a d-convex simple graph then $G^{++}$ is also d-convex simple.
Proof. Let $G \in \mathcal{G}$ be any $d$-convex simple graph and $v$ an arbitrary vertex of it. Let us form the graph $G^{++}$, where $v$ has a copy $v'$. The $d$-convex hull of any two nonadjacent vertexes from $G$, different from $v$ and $v'$, will contain together with $v$ the vertex $v'$ in $G^{++}$. $G$ is $d$-convex simple, we have that $v$ is not a suspended vertex, i.e. $v$ is adjacent to at least two nonadjacent vertexes, because $G$ does not contain triangles, so we have that $d$-segment $\langle v, v' \rangle$ contains at least two nonadjacent vertexes $x, x'$, and $d$-convex hull of these will contains all vertexes of $G$. By this way we obtain $\langle v, v' \rangle = X_{G^{++}}$. Let now $y$ be a vertex from $G$ such that $d(v, y) = 2$. $d$-Segment $\langle v, y \rangle$ will contain in $G$ at least two nonadjacent vertexes $w_1, w_2$, which in $G^{++}$ will be also adjacent with $v'$, so we have $\langle v, y \rangle = G^{++}$. But $\langle v', y \rangle = \langle v, y \rangle = X_{G^{++}}$. We have already proved that $d$-convex hull of any two vertexes at distance two in $G^{++}$ contains all vertexes of this graph, result the graph $G^{++}$ is $d$-convex simple.

The last lemma allows to use the operation of multiplication in one $d$-convex simple graph on any vertex we want and new graph will be also $d$-convex simple. So we can construct from graphs of set $\mathcal{B}_2$ the graphs, that will belong to $\mathcal{B}_1$, or even to $\mathcal{A}$, if we will duplicate all vertexes that are not dominated. We observe that the graphs $H, H_k$ (Fig. 4) can be obtained by using the operation of multiplication on vertex $y_1$ in graph $J_1$ (Fig. 3a). In Fig. 5 we have other graphs that are derived from the graph $J_1$.

Thus we are as close as it was possible to the description of $d$-convex simple $M$-prime graphs, with vertexes that are not dominated. Now let us prove the next theorem:

**Theorem 6.** The next relations are true:

$$\mathcal{A}[\mathcal{B}] = \mathcal{A}[\mathcal{B}_1] \cup \mathcal{B}_2 = \mathcal{G};$$

**Proof.** The equality $\mathcal{A}[\mathcal{B}] = \mathcal{A}[\mathcal{B}_1] \cup \mathcal{B}_2$ is true because the graphs of $\mathcal{B}_2$ have not pairs of copies vertexes, so they cannot participate in the $M$-operation and respectively cannot generate new graphs in this way.

The inclusion $\mathcal{A}[\mathcal{B}] = \mathcal{A}[\mathcal{B}_1] \cup \mathcal{B}_2 \subseteq \mathcal{G}$ is true, because we have already proved that the $M$-operation is algebraic on $\mathcal{G}$. Let us prove the reverse inclusion.

Let $G \in \mathcal{G}$ be any $d$-convex simple graph. If $G$ has not any pair of copies vertexes then $G \in \mathcal{B}_2$. Otherwise $G \in \mathcal{A}[\mathcal{B}_1]$. So we have $\mathcal{G} \subseteq \mathcal{A}[\mathcal{B}_1] \cup \mathcal{B}_2$. □

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Figure 5.
The last theorem is very close to our goal, the goal which would be achieved if we could describe in some way all graphs from $\mathcal{B}$.

References


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