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The sets of the classes $\widetilde{\mathrm{M}}_{p,k}$ and their subsets

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Abstract. In this paper the sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property in the generalized metric spaces (E,l) are considered. Certain properties for these sets and their subsets in the generalized metric spaces (E,l) and in the Cartesian space have been given here.

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1 Introduction

Let (E,l) be a generalized metric space. E denotes here an arbitrary non-empty set, and l is a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let k be any, but fixed positive real number, and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0. \tag{1}$$

We say that a pair (A,B) of sets of the family E_0 is (a,b)-clustered at the point p of the space (E,l) if 0 is the cluster point of the set of all numbers $r > 0$ such that the sets $A \cap S_l(p,r)_{a(r)}$ and $B \cap S_l(p,r)_{b(r)}$ are non-empty.

The sets $S_l(p,r)_{a(r)}$, $S_l(p,r)_{b(r)}$ (see [12]) denote here, respectively, so-called $a(r)$-, $b(r)$-neighbourhoods of the sphere $S_l(p,r)$ with the centre at the point $p \in E$ and the radius $r > 0$ in the space (E,l).

The tangency relation $T_l(a,b,k,p)$ of sets of the family E_0 in the generalized metric space (E,l) is defined as follows (see [12]):

$$T_l(a,b,k,p) = \Big\{(A,B):\ A,B \in E_0, \text{ the pair } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0\Big\}. \tag{2}$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is (a,b)-tangent (or shortly: is tangent) of order k to the set $B \in E_0$ at the point p of the space (E,l).

Let ρ be an arbitrary metric of the set E. By $d_\rho A$ we shall denote the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A, B \in E_0$ in the metric space (E,ρ), i.e.

$$d_\rho A = \sup\{\rho(x,y) : x, y \in A\} \text{ and } \rho(A,B) = \inf\{\rho(x,y) : x \in A, \ y \in B\} \quad (3)$$

for $A, B \in E_0$.

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0 such that $f(0) = 0$.

By $\mathfrak{F}_{f,\rho}$ we will denote the class of all functions l fulfilling the conditions:

1^0 $l : E_0 \times E_0 \longrightarrow \langle 0, \infty)$,

2^0 $f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A \cup B))$ for $A, B \in E_0$.

It is easy to show that every function $l \in \mathfrak{F}_{f,\rho}$ generates in the set E the metric l_0 defined by the formula:

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad x, y \in E. \quad (4)$$

We say (see [5]) that the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), and we shall write this as: $A \in D_p(E,l)$ if there exists a number $\tau > 0$ such that $A \cap S_l(p,r) \neq \emptyset$ for $r \in (0,\tau)$.

It is easy to notice that, if the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), then every set $E_0 \ni B \supset A$ has also this property at the point p of this space, i.e.

$$(A \in D_p(E,l) \ \wedge \ A \subset B \in E_0) \ \Rightarrow \ B \in D_p(E,l). \quad (5)$$

In this paper we shall consider some problems concerning the sets of the classes $\widetilde{M}_{p,k}$ on the Darboux property at the point p of the generalized metric spaces (E,l) for the function $l \in \mathfrak{F}_{f,\rho}$. Some theorems for the sets of the classes $\widetilde{M}_{p,k}$ will be given.

2 On the sets of the classes $\widetilde{M}_{p,k}$

Let ρ be a metric of the set E, and let A be any set of the family E_0 of all non-empty subsets of the set E. By A' we shall denote the set of all cluster points of the set A of the family E_0.

Let us put

$$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\} \quad \text{for} \quad x \in E. \quad (6)$$

The classes of sets $\widetilde{M}_{p,k}$, mentioned in the Introduction of this paper, are defined as follows (see [5]):

$$\begin{aligned} \widetilde{M}_{p,k} = \Big\{ & A \in E_0 : \ p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ & \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \text{for every pair of points } (x,y) \in [A,p;\mu,k] \\ & \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon \Big\}, \end{aligned} \quad (7)$$

where

$$[A, p; \mu, k] = \{(x, y) : \ x \in E, \ y \in A \ \text{and} \ \mu\rho(x, A) < \rho^k(p, x) = \rho^k(p, y)\}. \tag{8}$$

Let A, B be arbitrary sets of the family E_0.

Lemma 1. *If* $A \subset B$, *then* $[A, p; \mu, k] \subset [B, p; \mu, k]$.

Proof. Let us assume that $(x, y) \in [A, p; \mu, k]$ for $x \in E$ and $y \in A$. Hence and from the definition of the set $[A, p; \mu, k]$ it results

$$\mu\rho(x, A) < \rho^k(p, x) = \rho^k(p, y). \tag{9}$$

Because $A \subset B$, then $\rho(x, B) \leq \rho(x, A)$ for $x \in E$. From here and from (9) it follows that

$$\mu\rho(x, B) < \rho^k(p, x) = \rho^k(p, y) \quad \text{for} \quad x \in E \ \text{and} \ y \in B,$$

which yields $(x, y) \in [B, p; \mu, k]$. Therefore the inclusion $[A, p; \mu, k] \subset [B, p; \mu, k]$ is satisfied. □

Using this lemma we prove the following theorem:

Theorem 1. *If* $A \in E_0$ *is an arbitrary subset of the set* $B \in \widetilde{M}_{p,k}$ *and* $p \in A'$, *then* $A \in \widetilde{M}_{p,k}$.

Proof. Let us assume that $B \in \widetilde{M}_{p,k}$. From here and from the definition of the classes of sets $\widetilde{M}_{p,k}$ it follows that for an arbitrary $\varepsilon > 0$ there exists a number $\delta_1 > 0$ such that for every pair of points $(x, y) \in [B, p; \mu, k]$

$$\frac{\rho(x, y)}{\rho^k(p, x)} < \varepsilon \tag{10}$$

if only

$$\rho(p, x) < \delta_1 \quad \text{and} \quad \frac{\rho(x, B)}{\rho^k(p, x)} < \delta_1. \tag{11}$$

We shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_2 > 0$ such that for every pair of points $(x, y) \in [A, p; \mu, k]$ the inequality (10) is fulfilled if

$$\rho(p, x) < \delta_2 \quad \text{and} \quad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta_2. \tag{12}$$

If the inequalities (11) are fulfilled, then from here and from Lemma 1 of this paper it follows that the inequality (10) is satisfied for every pair of points $(x, y) \in [A, p; \mu, k]$.

Let us put $\delta_2 = \min(\frac{1}{\mu}, \delta_1)$. Hence, from the assumption that $p \in A'$ and from the condition (12) we obtain the inequality (10), what means that $A \in \widetilde{M}_{p,k}$. This ends the proof. □

In the paper [10] was proved the following (see Theorem 4.3):

Theorem 2. *If* $l \in \mathfrak{F}_{f,\rho}$, *the sets* $A, B \in E_0$ *on the Darboux property at the point* p *of the space* (E, l) *are subsets of a certain set* $C \in \widetilde{M}_{p,k}$ *and the functions* a, b *fulfil the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r \to 0^+]{} 0, \tag{13}$$

then $(A, B) \in T_l(a, b, k, p)$.

From above theorem, Theorem 1 of this paper, and from symmetry and transitivity of the tangency relation $T_l(a, b, k, p)$ result the following corollaries:

Corollary 1. *If* $l \in \mathfrak{F}_{f,\rho}$, *the functions* a, b *fulfil the condition* (13), *then*

$$(A, B) \in T_l(a, b, k, p) \quad \wedge \quad (B, A) \in T_l(a, b, k, p) \tag{14}$$

for arbitrary sets $A, B \in E_0$ *such that* $A \subset B$, $A \in D_p(E, l)$ *and* $B \in \widetilde{M}_{p,k}$.

Corollary 2. *If* $l \in \mathfrak{F}_{f,\rho}$, *the functions* a, b *fulfil the condition* (13), *then for arbitrary sets* $A, B, C \in E_0$ *such that* $A \subset B$, $A \in D_p(E, l)$, $B \in \widetilde{M}_{p,k}$ *and* $C \in \widetilde{M}_{p,k} \cap D_p(E, l)$

$$(A, C) \in T_l(a, b, k, p) \quad \Leftrightarrow \quad (B, C) \in T_l(a, b, k, p). \tag{15}$$

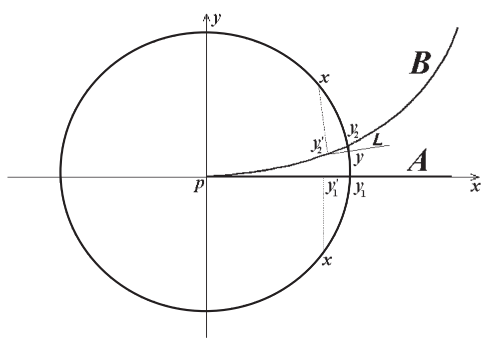

Figure 1

Below we shall give the examples of sets of the class $\widetilde{M}_{p,k}$ which are tangent in the two-dimensional Cartesian space $E = \mathbf{R}^2$.

Example 1. Let $E = \mathbf{R}^2$ be the two-dimensional Cartesian space. Let φ be an increasing differentiable function defined in a certain right-hand side neighbourhood of 0 such that $\varphi(0) = 0$.

Let $A \subset E$, $B \subset E$ be the sets of the form (see Figure 1)

$$A = \{(t, 0) : \ t \geq 0\}, \quad B = \{(t, \varphi^{k+1}(t)) : \ t \geq 0, \ k \in \mathbf{N}\}. \tag{16}$$

The sets A, B defined by the formula (16) are the sets of the class $\widetilde{M}_{p,k}$, where $p = (0, 0)$.

In the paper [11] I proved (see Example 2.1) that B is the set of the class $\widetilde{M}_{p,k}$.

Now we shall prove that the set A defined by (16) also belongs to the class $\widetilde{M}_{p,k}$. In the first place we shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every pair of points $(x, y_1) \in [A, p; \mu, k]$

$$\frac{\rho(x, y_1)}{\rho^k(p, x)} < \varepsilon, \tag{17}$$

when

$$r = \rho(p, x) < \delta_1 \quad \text{and} \quad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta_1. \tag{18}$$

Let y_1' be the projection of the point $x \in E$ the set A, i.e. such point of the set A that $\rho(x, y_1') = \rho(x, A)$. Because $x = (t, \pm\sqrt{r^2 - t^2})$ for $0 \le t < r$, then

$$\rho(y_1', y_1) = r - t = \sqrt{(r-t)^2} \le \sqrt{(r+t)(r-t)} = \sqrt{r^2 - t^2} = \rho(x, y_1'),$$

that is to say,

$$\rho(y_1', y_1) \le \rho(x, A). \tag{19}$$

Let $\mu = 2$, $\delta_1 = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$. Hence, from (18), (19) and from the triangle inequality we have

$$\frac{\rho(x, y_1)}{\rho^k(p, x)} \le \frac{\rho(x, y_1') + \rho(y_1', y_1)}{\rho^k(p, x)} \le \frac{2\rho(x, A)}{\rho^k(p, x)} < \varepsilon,$$

which yields the inequality (17). From here it follows that the set $A \in \widetilde{M}_{p,k}$.

Let now φ be a increasing function of the class C_1 (homogenous function together with 1[st] derivative) defined in a certain right-hand side neighbourhood of 0 such that $\varphi(0) = 0$. Using the de L′Hospital′s rule and mathematical induction we can easily prove that

$$\frac{\varphi^{k+1}(t)}{t^k} \xrightarrow[t \to 0^+]{} 0 \quad \text{for} \quad k \in \mathbf{N}. \tag{20}$$

From this it follows immediately

$$\frac{\varphi^{2k+2}(t)}{t^{2k}} \xrightarrow[t \to 0^+]{} 0 \quad \text{for} \quad k \in \mathbf{N}. \tag{21}$$

Example 2. Similarly as in Example 1, let $E = \mathbf{R}^2$ be the two-dimensional Cartesian space, and let A, B be sets defined by (16). Let $f = \mathrm{id}$, where id is the identity function defined in a right-hand side neighbourhood of 0. Let moreover a, b be functions defined in a right-hand side neighbourhood of 0 and filgilling the condition (13).

We shall prove that $(A, B) \in T_l(a, b, k, p)$ for $k \in \mathbf{N}$ and $p = (0, 0)$. Let y_2 be an arbitrary point of the set B. Then $y_2 = (t, \varphi^{k+1}(t))$ and

$$r = \rho(p, y_2) = \sqrt{t^2 + \varphi^{2k+2}(t)}. \tag{22}$$

Hence it follows that $y_1 = (\sqrt{t^2 + \varphi^{2k+2}(t)}, 0) \in A \cap S_\rho(p, r)$, where $S_\rho(p, r)$ denotes the sphere with the centre at the point p and the radius r in the metric space (E, ρ). From the assumptions on the funcion φ and from (22) it follows that $r \to 0^+$ if and only if $t \to 0^+$. Hence and from (20), (21), (22) for $r > 0$ we have

$$\frac{1}{r^{2k}}\rho^2(y_1, y_2) = \frac{(\sqrt{t^2 + \varphi^{2k+2}(t)} - t)^2 + \varphi^{2k+2}(t)}{(t^2 + \varphi^{2k+2}(t))^k}$$

$$= 2\frac{t^2 + \varphi^{2k+2}(t) - t\sqrt{t^2 + \varphi^{2k+2}(t)}}{(t^2 + \varphi^{2k+2}(t))^k}$$

$$= 2\frac{\varphi^{2k+2}(t) + t^2 - t\sqrt{t^2 + \varphi^{2k+2}(t)}}{t^{2k}} \; \frac{1}{(1 + \varphi^{2k+2}(t)/t^2)^k} \xrightarrow[t \to 0^+]{}$$

$$2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}} + \frac{t - \sqrt{t^2 + \varphi^{2k+2}(t)}}{t^{2k-1}}\right) = 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}} - \frac{\varphi^{2k+2}(t)}{t^{2k-1}(\sqrt{t^2 + \varphi^{2k+2}(t)} + t)}\right)$$

$$= 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}} - \frac{\varphi^{2k+2}(t)}{t^{2k}(\sqrt{1 + \varphi^{2k+2}(t)/t^2} + 1)}\right)$$

$$= 2\frac{\varphi^{2k+2}(t)}{t^{2k}}\left(1 - \frac{1}{1 + \sqrt{1 + \varphi^{2k+2}(t)/t^2}}\right) \xrightarrow[t \to 0^+]{} \left(\frac{\varphi^{k+1}(t)}{t^k}\right)^2 \xrightarrow[t \to 0^+]{} 0,$$

that is to say,

$$\frac{1}{r^k}\rho(y_1, y_2) \xrightarrow[r \to 0^+]{} 0. \tag{23}$$

From here, from the inequality

$$d_\rho(A \cup B) \le d_\rho A + d_\rho B + \rho(A, B) \quad \text{for} \quad A, B \in E_0, \tag{24}$$

from the fact that $f = \mathrm{id}$ and from Theorem 2.1 of the paper [8] we obtain

$$\begin{aligned}
\frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) &\le \frac{1}{r^k} d_\rho((A \cap S_l(p, r)_{a(r)}) \cup (B \cap S_l(p, r)_{b(r)})) \\
&\le \frac{1}{r^k} d_\rho(A \cap S_l(p, r)_{a(r)}) + \frac{1}{r^k} d_\rho(B \cap S_l(p, r)_{b(r)}) \\
&\quad + \frac{1}{r^k}\rho(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \\
&\le \frac{1}{r^k} d_\rho(A \cap S_l(p, r)_{a(r)}) + \frac{1}{r^k} d_\rho(B \cap S_l(p, r)_{b(r)}) + \frac{1}{r^k}\rho(y_1, y_2) \xrightarrow[r \to 0^+]{} 0,
\end{aligned}$$

i.e.,

$$\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{25}$$

Because the pair of sets (A, B) is (a, b)-clustered at the point p of the metric space (E, ρ), then from here and from (25) it follows that $(A, B) \in T_l(a, b, k, p)$ for $k \in \mathbf{N}$.

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