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Hopf bifurcations analysis of a three-dimensional nonlinear system

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Abstract. Bifurcations analysis of a $3D$ nonlinear chaotic system, called the T system, is treated in this paper, extending the work presented in [5] and [6]. The system T belongs to a class of cvasi-metriplectic systems having the same Poisson tensor and the same Casimir.

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1 Introduction

The nonlinear differential systems are studied both from theoretical point of view and from the point of view of their potential applications in various areas. The nonlinear systems present many times the property of sensitivity with respect to the initial conditions (some authors consider this property sufficient for a system to be chaotic). Applications of such systems have been found lately in secure communications [1, 4]. Of the pioneering papers which proposed to use the chaotic systems in communications are the papers of Pecora and Carrol [8, 9]. Consequently, an appropriate chaotic system can be chosen from a catalogue of chaotic systems to optimize some desirable factors, idea suggested in [4].

These facts led us to study a new 3D polynomial differential system given by:

$$\dot{x} = a(y - x),\ \dot{y} = (c - a)x - axz,\ \dot{z} = -bz + xy, \tag{1}$$

with a, b, c real parameters and $a \neq 0$. Call it the T system. Some results regarding the T system are already presented in [5] and [6], where we pointed out two particular cases. The system T is related to the Lorenz, Chen and Lü system [3] being a small generalization of the latter one.

The paper is organized as follows. In the first Section we recall some results regarding the stability of equilibria. In Section 2 we present the pitchfork and Hopf bifurcations occurring at the equilibrium points in the general case. Also, we show that the T system belongs to a class of cvasi-metriplectic systems.

2 Equilibrium points of the system

Because the dynamics of the system is characterized by the existence and the number of the equilibrium points as well as of their type of stability, we recall in the

Figure 1. a) The orbit of the Lorenz system for $a = 10,\ b = 8/3,\ c = 28$ and the initial condition $(-0.04,\ -0.3,\ 0.52)$ (left); b) The orbit of the Chen system for $a = 35,\ b = 3,\ c = 28$ and the initial condition $(-0.1,\ 0.1,\ 0.4)$ (right)

Figure 2. a) The orbit of the Lü system for $a = 36, b = 3, c = 19$ and the initial condition $(-1, 0.1, 4)$ (left) ; b) The orbit of the T system for $(a, b, c) = (2.1, 0.6, 30)$ and the initial condition $(0.1, -0.3, 0.2)$ (right)

following the equilibrium points of the system T and their stability.

Proposition 1. *If* $\frac{b}{a}(c-a) > 0$, *then the system* T *possesses three equilibrium isolated points:*

$$O(0,0,0), E_1(x_0, x_0, c/a-1), E_2(-x_0, -x_0, c/a-1)$$

where $x_0 = \sqrt{\frac{b}{a}(c-a)}$, *and for* $b \neq 0, \frac{b}{a}(c-a) \leq 0$, *the system* T *has only one isolated equilibrium point, namely* $O(0,0,0)$,

Theorem 1. *For* $b \neq 0$ *the following statements are true:*

a) If $(a>0, b>0, c \leq a)$, *then* $O(0,0,0)$ *is asymptotically stable;*

b) If $(b<0)$ *or* $(a<0)$ *or* $(a>0, c>a)$, *then* $O\,(0,0,0)$ *is unstable.*

For the other two equilibria, $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$, for $b/a(c-a) > 0$, using the Routh-Hurwitz conditions we have:

Theorem 2. *([5]) If the conditions* $a+b>0$, $ab(c-a)>0$, $b(2a^2+bc-ac)>0$ *hold, the equilibrium points* $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$, *are asymptotically stable.*

3 Pitchfork and Hopf bifurcations of the system T

Consider the parameter a as bifurcation parameter.

a) *Bifurcations at O(0,0,0)*

Proposition 2. *([7]) If* $\beta = a - c = 0$ *the equilibrium O(0,0,0) of the system* T *undergoes a pitchfork bifurcation that generates the asymptotic stable equilibrium point O(0,0,0) if* $a > c$, *and for* $a < c$ *three equilibria: O(0,0,0) (unstable),* $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$ *(locally stable).*

Notice that the equilibrium O(0,0,0) can not undergo a Hopf bifurcation because the roots of the characteristic polynomial of the Jacobian matrix of the system T at $O(0,0,0)$ are $\lambda_1 = -b$, $\lambda_2 = \frac{1}{2}\left(-a-\sqrt{4ac-3a^2}\right)$, $\lambda_3 = \frac{1}{2}\left(-a+\sqrt{4ac-3a^2}\right)$ and the last two roots can not be purely imaginary because $a \neq 0$.

b) Bifurcations of the equilibria E_1 *and* E_2.

We observe that the characteristic polynomial in this case is:

$$f(\lambda) := \lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) \tag{2}$$

Because $ab(c-a) > 0$, the system T does not undergo pitchfork bifurcations at $E_{1,2}$, so we study the Hopf bifurcations at these points. The following proposition characterizes the imaginary roots of (2).

Proposition 3. *Consider* $\frac{b}{a}(c-a) > 0$. *The polynomial (2) has one real negative root and two purely imaginary roots if and only if* $(a,b,c) \in \Omega$, *where*

$$\Omega = \{(a,b,c) \in \mathbb{R}^3 \mid a > b > 0, 2a^2 + bc = ac\}.$$

In this case the roots are: $\lambda_1 = -a-b$, $\lambda_{2,3} = \pm i\omega, \omega := \sqrt{bc}$.

In the following we show that the system T undergoes a Hopf bifurcation at E_1 (for E_2 is similar). Remember that a is the bifurcation parameter.

From $2a^2+bc = ac$ one gets $a = a_s := \dfrac{c \pm \sqrt{c^2-8bc}}{4}$. Denote $\lambda := \alpha\,(a) \pm i\omega\,(a)$ the complex roots depending on the bifurcation parameter a of (2). If $a = a_s$ and $(a,b,c) \in \Omega$, from the above Proposition 3, we have $\lambda_1 = -a-b$ and $\lambda_{2,3} = \pm i\omega$ with $\omega = \sqrt{bc}$.

From (2) it follows that:

$$Re\left(\frac{d\lambda\,(a)}{da}\Big|_{a=a_s,\lambda=i\sqrt{bc}}\right) = Re\left(\frac{bc-4ab}{2bc-2i\sqrt{bc}\,(a+b)}\right) = \frac{bc-4ab}{2bc+2\,(a+b)^2} \neq 0$$

for $c \neq 8b$ because $c - 4a = \pm\sqrt{c^2-8bc}$, $(b,c>0)$.

In the following we compute the index number K from the Hopf bifurcation theorem [2], employing the central manifold theory.

Using the transformation $(x,y,z) \to (X_1,Y_1,Z_1)$,

$$x = X_1 + x_0,\ y = Y_1 + x_0,\ z = Z_1 + c/a - 1, \tag{3}$$

the system T leads to:

$$\dot{X}_1 = a(Y_1 - X_1),\ \dot{Y}_1 = -ax_0Z_1 - aX_1Z_1,\ \dot{Z}_1 = x_0(X_1+Y_1) - bZ_1 + X_1Y_1 \tag{4}$$

or, in the normal form:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{Z}_1 \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & -ax_0 \\ x_0 & x_0 & -b \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -aX_1Z_1 \\ X_1Y_1 \end{pmatrix}. \tag{5}$$

The eigenvalues of the Jacobian matrix J,

$$J = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & -ax_0 \\ x_0 & x_0 & -b \end{pmatrix}$$

are, respectively $\lambda_1 = -a-b < 0$, $\lambda_{2,3} = \pm i\omega, \omega = \sqrt{bc}$, $b > 0$, $c > 0$, with corresponding eigenvectors:

$$v_1\left(1, \frac{-c+2a}{c}, -\frac{2}{c}\sqrt{c-2a-b}\right), \quad v_2\left(-\frac{a}{\omega}qx_0 + iqx_0, -\frac{a}{\omega}x_0, i\right),$$

$$v_3\left(-\frac{a}{\omega}qx_0 - iqx_0, -\frac{a}{\omega}x_0, -i\right)$$

with $q = \dfrac{a^2}{\omega^2+a^2}$. Then the vectors $v_2' = \dfrac{v_2+v_3}{2} = \left(-\dfrac{a}{\omega}qx_0, -\dfrac{a}{\omega}x_0, 0\right)$, $v_3' = \dfrac{v_2-v_3}{2i} = (qx_0, 0, 1)$ and $v_1\left(1, \dfrac{-c+2a}{c}, -\dfrac{2}{c}\sqrt{c-2a-b}\right)$ lead us to the transformation:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where

$$P = \begin{pmatrix} 1 & -\frac{a}{\omega}qx_0 & qx_0 \\ \frac{-c+2a}{c} & -\frac{a}{\omega}x_0 & 0 \\ -\frac{2}{c}\sqrt{c-2a-b} & 0 & 1 \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P^{-1} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

where

$$P^{-1} = \frac{1}{d}\begin{pmatrix} c & -qc & -qx_0c \\ \frac{-c+2a}{ax_0}\omega & -\frac{c+2\sqrt{c-2a-b}qx_0}{ax_0}\omega & q\frac{c-2a}{a}\omega \\ 2\sqrt{c-2a-b} & -2q\sqrt{c-2a-b} & c+qc-2qa \end{pmatrix}$$

with $d = c + qc - 2qa + 2\sqrt{c-2a-b}qx_0$.

Then the system (5) reads:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = P^{-1}JP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + P^{-1}\begin{pmatrix} 0 \\ -aX_1Z_1 \\ X_1Y_1 \end{pmatrix}, \tag{6}$$

or, equivalently

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 2a\frac{-c+a}{c} & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \tag{7}$$

where

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = P^{-1}\begin{pmatrix} 0 \\ -a\left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(-\frac{2}{c}\sqrt{c-2a-b}X + Z\right) \\ \left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(\frac{-c+2a}{c}X - \frac{a}{\omega}x_0Y\right) \end{pmatrix}, \tag{8}$$

or

$$g_1 = \frac{1}{d} qca \left(X - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(-\frac{2}{c} \sqrt{c - 2a - b} X + Z \right) -$$
$$- \frac{1}{d} qx_0 c \left(aX - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(\frac{-c + 2a}{c} X - \frac{a}{\omega} x_0 Y \right),$$
$$g_2 = \frac{1}{d} \frac{c + 2\sqrt{c - 2a - b} qx_0}{x_0} \omega \left(X - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(-\frac{2}{c} \sqrt{c - 2a - b} X + Z \right) +$$
$$\frac{1}{d} q \frac{c - 2a}{a} \omega \left(aX - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(\frac{-c + 2a}{c} X - \frac{a}{\omega} x_0 Y \right),$$
$$g_3 = \frac{2}{d} q \sqrt{c - 2a - b} a \left(X - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(-\frac{2}{c} \sqrt{c - 2a - b} X + Z \right) +$$
$$+ \frac{1}{d} (c + qc - 2qa) \left(aX - \frac{a}{\omega} qx_0 Y + qx_0 Z \right) \left(\frac{-c + 2a}{c} X - \frac{a}{\omega} x_0 Y \right).$$

Consider the 2-dimensional central manifold at the origin given by:

$$X = h(Y, Z) = AY^2 + BYZ + CZ^2 + \ldots \tag{9}$$

Then, replacing (9) in (7) and taking into account that $\dot{X} = 2AY\dot{Y} + B\dot{Y}Z + BY\dot{Z} + 2CZ\dot{Z}$, obtained from (9), one gets:

$$\dot{X} = BZ^2\omega - BY^2\omega - 2C\omega YZ + 2A\omega ZY + \ldots \tag{10}$$

On the other hand, from (7) we have

$$\dot{X} = \alpha_1 Y^2 + \alpha_2 YZ + \alpha_3 Z^2 + \ldots \tag{11}$$

where

$$\alpha_1 = -2aA + 2\frac{a^2}{c} A - \frac{1}{d} q^2 x_0^3 c \frac{a^2}{\omega^2},$$
$$\alpha_2 = \frac{1}{d} q^2 x_0^3 c \frac{a}{\omega} - \frac{1}{d} q^2 ca^2 \frac{1}{\omega} x_0 - 2aB + 2\frac{a^2}{c} B,$$
$$\alpha_3 = 2\frac{a^2}{c} C + \frac{1}{d} q^2 cax_0 - 2aC.$$

Then, equalling the coefficients of the terms Y^2, Z^2, YZ in the above relations (10) and (11), one gets

$$Y^2 : -\omega B = -2aA + 2\frac{a^2}{c} A - \frac{1}{d} q^2 x_0^3 c \frac{a^2}{\omega^2},$$
$$YZ : 2\omega A - 2C\omega = \frac{1}{d} q^2 x_0^3 c \frac{a}{\omega} - \frac{1}{d} q^2 ca^2 \frac{1}{\omega} x_0 - 2aB + 2\frac{a^2}{c} B,$$
$$Z^2 : B\omega = 2\frac{a^2}{c} C + \frac{1}{d} q^2 cax_0 - 2aC.$$

Finally we get

$$A=-\frac{1}{4}q^2c^2x_0\frac{-\omega^2ca^3+\omega^2c^2a^2-\omega^4c^2+\omega^2cx_0^2a^2-4x_0^2a^4c+2x_0^2a^3c^2+2x_0^2a^5}{\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)d\omega^2\left(-a+c\right)},$$

$$B=\frac{1}{2}q^2x_0c^2\frac{2x_0^2ca+a^3-ca^2+\omega^2c-x_0^2a^2}{\omega d\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)}a,$$

$$C=-\frac{1}{4}q^2c^2x_0\frac{2x_0^2c^2a+5ca^3-3c^2a^2-\omega^2c^2-cx_0^2a^2-2a^4}{\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)d\left(-a+c\right)}.$$

Hence, the system (8) restricted to the central manifold is given by:

$$\begin{pmatrix}\dot{Y}\\ \dot{Z}\end{pmatrix}=\begin{pmatrix}0&\omega\\ -\omega&0\end{pmatrix}\begin{pmatrix}Y\\ Z\end{pmatrix}+\begin{pmatrix}g^2\left(Y,Z\right)\\ g^3\left(Y,Z\right)\end{pmatrix},$$

where $g^2\left(Y,Z\right)=g_2\left(h\left(Y,Z\right),Y,Z\right),\ \ g^3\left(Y,Z\right)=g_3\left(h\left(Y,Z\right),Y,Z\right).$

Now K can be computed as follows [2]:

$$K(Y,Z)=\frac{1}{16}\left[\frac{\partial^3g^2}{\partial Y^3}+\frac{\partial^3g^2}{\partial Y\partial Z^2}+\frac{\partial^3g^3}{\partial Y^2\partial Z}+\frac{\partial^3g^3}{\partial Z^3}\right]+$$
$$+\frac{1}{16\omega}\left[\frac{\partial^2g^2}{\partial Y\partial Z}\left(\frac{\partial^2g^2}{\partial Y^2}+\frac{\partial^2g^2}{\partial Z^2}\right)-\frac{\partial^2g^3}{\partial Y\partial Z}\left(\frac{\partial^2g^3}{\partial Y^2}+\frac{\partial^2g^3}{\partial Z^2}\right)-\right.$$
$$\left.-\frac{\partial^2g^2}{\partial Y^2}\frac{\partial^2g^3}{\partial Y^2}+\frac{\partial^2g^2}{\partial Z^2}\frac{\partial^2g^3}{\partial Z^2}\right].$$

Then

$$K=K(0,0)=-\frac{1}{8d}x_0caqC+\frac{1}{4d}x_0a^2qC-\frac{1}{4d}x_0cq^2C+\frac{1}{d}a\sqrt{c-2a-b}Cq+$$
$$+\frac{1}{d}\frac{x_0}{c}aCq^2b-\frac{1}{8d}cAqx_0-\frac{1}{4d\omega}a^2Bqx_0+\frac{1}{8d\omega}caBq^2x_0+\frac{1}{4d\omega}a^3Bqx_0-$$
$$-\frac{1}{2d\omega c}q^2a^2Bbx_0-\frac{1}{2d\omega c}q^2a^3Bx_0-\frac{1}{8d\omega}ca^2Bx_0-\frac{1}{8d\omega}ca^2Bqx_0-\frac{3}{8}\frac{C}{d}cqx_0+$$
$$+\frac{3}{4}\frac{C}{d}aqx_0-\frac{1}{8d^2\omega^4}c^2q^4x_0^4a^3-\frac{1}{2d^2\omega^4}q^4a^5x_0^4-\frac{1}{2d^2\omega^4}q^4\sqrt{c-2a-b}a^5x_0^3+$$
$$+\frac{1}{4d^2\omega^4}cq^4\sqrt{c-2a-b}a^4x_0^3+\frac{1}{4d^2\omega^4}cq^3\sqrt{c-2a-b}a^4x_0^3+$$
$$+\frac{1}{2d^2\omega^2}cq^4a^3x_0^2-\frac{1}{d^2\omega^2}q^4a^4x_0^2-\frac{1}{2d^2\omega^2}q^4a^3x_0^2b+\frac{1}{2d^2}x_0^2caq^4-\frac{1}{d^2}x_0^2a^2q^4-$$
$$-\frac{1}{2d^2}x_0^2aq^4b-\frac{1}{8d^2}x_0^2c^2a^2\frac{q^3}{\omega^2}+\frac{1}{4d^2}x_0^2ca^3\frac{q^3}{\omega^2}-\frac{1}{8d^2}c^2aq^2-\frac{1}{8d^2}x_0^4c^2a\frac{q^4}{\omega^2}+$$
$$+\frac{1}{2d^2}x_0^4ca^2\frac{q^4}{\omega^2}-\frac{1}{4d^2}x_0^3cq^4\sqrt{c-2a-b}-\frac{1}{8d^2}x_0^2c^2q^3-\frac{1}{2d^2}x_0^4a^3\frac{q^4}{\omega^2}+$$

$$+\frac{1}{2d^2}x_0^3aq^4\sqrt{c-2a-b}+\frac{1}{4d^2}x_0^2caq^3+\frac{1}{2d^2\omega^4}cq^4x_0^4a^4+$$
$$+\frac{1}{4d^2\omega^2}cq^3x_0^3a^2\sqrt{c-2a-b}+\frac{1}{8d^2\omega^4}c^2q^2x_0^4a^3+\frac{A}{d}\sqrt{c-2a-b}aq+\frac{1}{8dx_0}c\omega B+$$
$$+\frac{1}{d}\frac{x_0}{c}a^2q^2C-\frac{1}{8d}x_0\frac{c}{a}q^2\omega B-\frac{A}{d}\frac{x_0}{c}q^2a^2-\frac{A}{d}\frac{x_0}{c}q^2ab-\frac{3}{8}\frac{A}{d}x_0cqa+\frac{3}{4}\frac{A}{d}x_0qa^2+$$
$$+\frac{1}{4}\frac{A}{d}x_0cq^2+\frac{1}{2d}\frac{x_0}{c}aq^2\omega B+\frac{1}{2d}\frac{x_0}{c}\omega Bq^2b+\frac{1}{4d}Aaqx_0+\frac{1}{8d\omega}caBqx_0.$$

Concluding we have the theorem:

Theorem 3. *If* $a=a_s:=\dfrac{c\pm\sqrt{c^2-8bc}}{4}$, $c\neq 8b$, *the equation (2) has a negative solution* $\lambda_1=-a-b<0$ *together with two purely imaginary roots* $\lambda_{2,3}=\pm i\omega,\omega=\sqrt{bc}$ *such that* $R:=Re\left(\dfrac{d\lambda(a)}{da}\Big|_{a=a_s,\lambda=i\omega}\right)\neq 0$. *Consequently, if* $K\neq 0$, *the equilibrium* E_1 *of the system* T *undergoes a Hopf bifurcation. In addition, the periodic orbits that bifurcate from the equilibrium* E_1 *for* a *in the neighborhood of* a_s, *are stable if* $K<0$, *and unstable if* $K>0$. *The direction of bifurcation are above (bellow)* a_s *if* $RK<0$ $(RK>0)$.

Remark 1. In the particular case $a=2.1$, $b=1.806$, $c=30$ we have $K=-2.815\times 10^{-3}$, $R=0.28$. So the periodic orbits that bifurcate from the equilibria $E_{1,2}$ are stable and the bifurcation is above a_s.

In the following we show that the system T belongs to a class of cvasi-metriplectic systems.

Definition 1. Consider X a vector field on $\mathbb{R}^3$. The vector field X is called cvasi-metriplectic, if there exists a Poisson tensor field P, a cvasi-metric tensor field g, a Hamilton function H and a Casimir function S associated to P, such that:

$$X=P\nabla H+g\nabla S. \tag{12}$$

Consider the vector field $X=(x_1,x_2,x_3)$ given by:

$$x_1=-a_1x+a_1y,\ x_2=a_2x+a_3y-a_4xz+a_6,\ x_3=-a_5z+xy, \tag{13}$$

with $a_i, i=\overline{1,6}$, real numbers.

Remark 2. 1. If $a_1=a,a_2=c,a_3=-1,a_4=1,a_5=b,a_6=0$, the dynamical system associated to X is the Lorenz system.

2. If $a_1=a,a_2=c-a,a_3=c,a_4=1,a_5=b,a_6=0$, the dynamical system associated to X is the Chen (or Chen 1) system.

3. If $a_1=a,a_2=0,a_3=c,a_4=1,a_5=b,a_6=0$, the dynamical system associated to X is the Lü system.

4. If $a_1 = a, a_2 = c - a, a_3 = 0, a_4 = a, a_5 = b, a_6 = 0$, the dynamical system associated to X is the T system.

5. If $a_1 = a, a_2 = c - a, a_3 = c, a_4 = 1, a_5 = b, a_6 = m$, the dynamical system associated to X is the Chen 2 system.

Proposition 4. *The vector field X (13) has the cvasi-metriplectic representation given by:*

$$X = P\nabla H + g\nabla S \tag{14}$$

where

$$P = \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & -x \\ 0 & x & 0 \end{pmatrix}, g = \begin{pmatrix} -a_1 & 0 & 0 \\ 0 & \varepsilon & -\dfrac{a_3 y + a_6}{a_1} \\ 0 & -\dfrac{a_3 y + a_6}{a_1} & \dfrac{a_5 z}{a_1} \end{pmatrix} \tag{15}$$

$$H(x, y, z) = \frac{1}{2}(y^2 + a_4 z^2) - a_2 z, \quad S(x, y, z) = \frac{1}{2}x^2 - a_1 z \quad \textit{with} \quad \varepsilon \in \mathbb{R}.$$

The proof is immediately.

Observe that the Poisson tensor field P and the Casimir S are the same for the all above five systems, consequently, the systems belong to the same class of cvasi-metriplectic systems. In addition, for the system T, the tensor g is in diagonal form.

4 Conclusions

In this paper we further investigated a nonlinear differential system with three equilibrium points, origin and another two. In the origin, the system displays a pitchfork bifurcation and in the other two equilibrium points a Hopf bifurcation. The system belongs to a class of cvasi-metriplectic systems.

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