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The $GL(2,\mathbb{R})-$orbits of the homogeneous polynomial differential systems

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Abstract. In this work, we study the generic homogeneous polynomial differential system $\dot{x}_1 = P_k(x_1,x_2)$, $\dot{x}_2 = Q_k(x_1,x_2)$ under the action of the center-affine group of transformations of the phase space, $GL(2,\mathbb{R})$. We show that if the dimension of the $GL(2,\mathbb{R})-$ orbits of this system is smaller than four, then $deg(GCD(P_k,\, Q_k)) \geq k-1$.

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1 Center-affine transformations

We consider the system

$$\dot{x_1} = P_k(x_1,x_2),\ \dot{x_2} = Q_k(x_1,x_2), \tag{1}$$

where P_k, Q_k are homogeneous polynomials of degree k:

$$P_k = \sum_{i+j=k} a_{ij}x_1^i x_2^j,\ Q_k = \sum_{i+j=k} b_{ij}x_1^i x_2^j.$$

Denote by $\mathbb{E}$ the space of coefficients

$$\mathbf{e} = (\mathbf{a};\mathbf{b}) = (a_{k,0}, a_{k-1,1}, ..., a_{0k};\ b_{k,0}, b_{k-1,1}, ..., b_{0k})$$

of system (1) and by $GL(2,\mathbb{R})$ the group of center-affine transformations of the phase space $O\mathbf{x}$, $\mathbf{x} = (x_1, x_2)$.

Applying in (1) the transformations $\mathbf{X} = q\mathbf{x}$, where $\mathbf{X} = (X_1, X_2)$, $q \in GL(2,\mathbb{R})$, i.e.

$$q = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix};\ \alpha_{ij} \in \mathbb{R},\ det(q) \neq 0,\ q^{-1} = \frac{1}{det(q)} \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix},$$

we obtain the system

$$\dot{X}_1 = P_k^*(X_1,X_2),\ \dot{X}_2 = Q_k^*(X_1,X_2), \tag{2}$$

where

$$P_k^* = \alpha_{11}\cdot P_k(q^{-1}\mathbf{x}) + \alpha_{12}\cdot Q_k(q^{-1}\mathbf{x}) = \sum_{i=0}^{k} a_{k-i,i}^* X_1^{k-i} X_2^i,$$

$$Q_k^* = \alpha_{21}\cdot P_k(q^{-1}\mathbf{x}) + \alpha_{22}\cdot Q_k(q^{-1}\mathbf{x}) = \sum_{i=0}^{k} b_{k-i,i}^* X_1^{k-i} X_2^i.$$

The coefficients $\mathbf{e}^*$ of the system (2) can be expressed linearly by the coefficients of the system (1): $\mathbf{e}^* = \Lambda_{(q)}(\mathbf{e})$, $det\Lambda_{(q)} \neq 0$. The set $\Lambda = \{\Lambda_{(q)} | q \in GL(2,\mathbb{R})\}$ forms a 4-parameter linear group with the operation of composition. It is called the representation of the group $GL(2,\mathbb{R})$ in the space of coefficients $\mathbb{E}$ of system (1).

The set $O(\mathbf{e}) = \{\Lambda_{(q)}(\mathbf{e}) |\, q \in GL(2,\mathbb{R}\}$ is called the $GL(2,\mathbb{R})-$orbit of the point $\mathbf{e} \in \mathbb{E}$ or of the differential system (1) corresponding to this point.

Let

$$q_1^t = \begin{pmatrix} exp(t) & 0 \\ 0 & 1 \end{pmatrix},\ q_2^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ q_3^t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\ q_4^t = \begin{pmatrix} 1 & 0 \\ 0 & exp(t) \end{pmatrix}$$

and $G_l = \{q_l^t |\, t \in \mathbb{R}\} \subset GL(2,\mathbb{R})$, $l = \overline{1,4}$. Denote $g_l^t = \Lambda_{(q_l^t)}$. Il is obvious that $\Lambda_l = \{g_l^t\}$, $l = \overline{1,4}$, are the linear representations in $\mathbb{E}$ of the subgroups G_l respectively. Each of the pairs $(\mathbb{E}, \{g_l^t\})$, $l = \overline{1,4}$, corresponds to a flow defined in $\mathbb{E}$ by the following systems of linear equations:

$$\frac{d\mathbf{e}}{dt} = \left(\frac{dg_l^t(\mathbf{e})}{dt}\right)|_{t=0} = A^{(l)}\cdot\mathbf{e},\ l = \overline{1,4}. \tag{3}$$

If we represent the matrix $A^{(l)}$ of dimension $(2k+2)\times(2k+2)$ as four quadratic blocks of dimensions $(k+1)\times(k+1)$: $A^{(l)} = \begin{pmatrix} A_l & B_l \\ C_l & D_l \end{pmatrix}$ and if denote by O the matrix null, and by I the unity matrix, both of dimensions $(k+1)\times(k+1)$, we get :

$$A_1 = -diag(k-1, k-2, \dots, 1, 0, -1), B_1 = C_1 = O,$$
$$D_1 = -diag(k, k-1, \dots, 1, 0);$$

$$A_2 = -\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ k & 0 & 0 & \cdots & 0 & 0 \\ 0 & k-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & k-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$B_2 = I,\ C_2 = O,\ D_2 = A_2;$$

$$A_3 = -\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$B_3 = O,\ C_3 = I,\ D_3 = A_3;$$

$$A_4 = -diag(0, 1, 2, \ldots, k),\ B_4 = C_4 = O,\ D_4 = -diag(-1, 0, 1, 2, \ldots, k-1).$$

Let $\mathbf{v}_l$, $l = \overline{1,4}$, be the vector fields defined in $\mathbb{E}$ by the systems (3) and $L_{\mathbf{v}}$ be the derivative in the direction of the vector $\mathbf{v}$. Setting $\mathbf{w} = [\mathbf{u}, \mathbf{v}]$, where $L_{\mathbf{w}} = L_{\mathbf{u}} L_{\mathbf{v}} - L_{\mathbf{v}} L_{\mathbf{u}}$, it is easy to verify that the vector fields $\mathbf{v}_l$, $l = \overline{1,4}$, generate a Lie algebra. Following [1, 2] the dimension of the orbit $O(\mathbf{e})$ is equal to the dimension of this algebra applying to the element $\mathbf{e}$, i.e. to the rank of a matrix $M_k = \big(\mathbf{v}_l(\mathbf{e}) |\, l = \overline{1,4}\big)$ of the dimensions $4 \times (2k+2)$. The classification of some polynomial systems according to the dimensions of their $GL(2,\mathbb{R})-$orbits was done in [2–11].

Denote $\mathbf{v}_l(\mathbf{e}) = (A^{(l)}_{k0}, A^{(l)}_{k-1,1}, \ldots, A^{(l)}_{0k}; B^{(l)}_{k0}, B^{(l)}_{k-1,1}, \ldots, B^{(l)}_{0k})$, $l = \overline{1,4}$. Taking into account that $\mathbf{v}_l(\mathbf{e}) = A^{(l)} \cdot \mathbf{e}$, the coordinates of vectors $\mathbf{v}_l(\mathbf{e})$ can be represented by coefficients of the system (1) as follows:

$$A^{(1)}_{k-i,i} = -(k-i-1)a_{k-i,i},\ B^{(1)}_{k-i,i} = -(k-i)b_{k-i,i},\ i = \overline{0,k};$$

$$A^{(2)}_{k0} = b_{k0},\ A^{(2)}_{k-i,i} = b_{k-i,i} - (k-i+1)a_{k-i+1,i-1},$$

$$B^{(2)}_{k0} = 0,\ B^{(2)}_{k-i,i} = -(k-i+1)b_{k-i+1,i-1},\ i = \overline{1,k};$$

$$A^{(3)}_{k-i,i} = -(i+1)a_{k-i-1,i+1},\ A^{(3)}_{0k} = 0,$$

$$B^{(3)}_{k-i,i} = a_{k-i,i} - (i+1)b_{k-i-1,i+1},\ B^{(3)}_{0k} = a_{0k},\ i = \overline{0,k-1};$$

$$A^{(4)}_{k-i,i} = -i a_{k-i,i},\ B^{(4)}_{k-i,i} = -(i-1)b_{k-i,i},\ i = \overline{0,k}.$$

For $k = 0$ and $k = 1$ the matrix M_k becomes

$$M_0 = \begin{pmatrix} a_{00} & 0 \\ b_{00} & 0 \\ 0 & a_{00} \\ 0 & b_{00} \end{pmatrix},\quad M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01} - a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10} - b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}.$$

By direct calculations, we obtain the following two theorems:

Theorem 1. *Let $k = 0$ and d be the dimension of the $GL(2,\mathbb{R})-$orbit $O(\mathbf{e})$ of the system (1). Then,*
$d = 0$, *iff* $P_0 = Q_0 = 0$ *and*
$d = 2$ *in other cases.*

Theorem 2. *Let $k = 1$ and d be the dimension of the $GL(2,\mathbb{R})-$orbit $O(\mathbf{e})$ of the system (1). Then,*
$d = 0$, *iff* $a_{10} - b_{01} = a_{01} = b_{10} = 0$ *and*
$d = 2$ *in other cases.*

Let $GCD(P, Q)$ be the greatest common divisor of the polynomials P and Q. The main result of this paper is the following theorem.

Theorem 3. *If the dimension of the $GL(2,\mathbb{R})-$orbit of the differential system (1) is smaller than four, then $deg(GCD(P, Q)) \geq k - 1$.*

Next, in this work we will suppose that

$$k \geq 2 \text{ and } |P_k(x_1, x_2)| + |Q_k(x_1, x_2)| \not\equiv 0. \tag{4}$$

2 One lemma

Let $\tau \in \{0, 1, 2, ..., k\}$. Consider the polynomial

$$f = z_1 x^k + z_2 x^{k-1} + \ldots + z_{k+1},\ z_i \in \mathbb{C},\ i = \overline{1, k+1} \tag{5}$$

and the $(k + 1) \times (k + 1)$-matrix $\tilde{A}$ defined by :

$$\begin{array}{l} \tilde{a}_{i,i-1} = (k - i + 2)\xi_1\xi_2,\ i = \overline{2, k+1};\ \tilde{a}_{i,i+1} = -i,\ i = \overline{1, k}; \\ \tilde{a}_{ii} = (k - \tau - i + 1)\xi_1 + (\tau - i + 1)\xi_2,\ i = \overline{1, k+1}; \\ \tilde{a}_{il} = 0,\ |i - l| > 1, \end{array} \tag{6}$$

where ξ_1, ξ_2 are constant. It is easy to show that

$$k \leq rank(\tilde{A}) \leq k + 1. \tag{7}$$

Lemma 1. *If the vector*

$$Z = (z_1, z_2, ..., z_{k+1})^{tr} \tag{8}$$

is a solution of the equation

$$\tilde{A}Z = 0, \tag{9}$$

then (5) has the form

$$f = c \cdot (x + \xi_1)^{k-\tau}(x + \xi_2)^{\tau}, \tag{10}$$

where c is a constant.

Proof. Without loss of generality we can assume that $\tau \in \{0,\ 1,\ 2, ..., [k/2]\}$, where by $[k/2]$ we denoted the integer part of the number $k/2$.

Let $\tilde{R} = \tilde{A}\tilde{Z} = (\tilde{r}_1, \ldots, \tilde{r}_{k+1})^{tr}$, where $\tilde{Z} = (\tilde{z}_1, \ldots, \tilde{z}_{k+1})^{tr}$ and

$$\tilde{z}_i = \sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu-1} \xi_2^{\mu}, \tag{11}$$

if $1 \le i \le \tau + 1$;

$$\tilde{z}_i = \sum_{\mu=0}^{\tau} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu-1} \xi_2^{\mu}, \tag{12}$$

if $\tau + 1 < i \le k - \tau + 1$ and

$$\tilde{z}_i = \sum_{\mu=0}^{k-i+1} C_{k-\tau}^{i-\tau+\mu-1} C_{\tau}^{\tau-\mu} \xi_1^{i-\tau+\mu-1} \xi_2^{\tau-\mu}, \tag{13}$$

if $k - \tau + 1 < i \le k + 1$.

We will prove that the vector $\tilde{Z}$ with the coordinates (11)−(13) is a solution of the equation (9).

a) Let $1 \le i \le \tau$. Taking into consideration (6) and (11), we obtain:

$$\tilde{r}_1 = \tilde{a}_{12} \cdot \tilde{z}_2 + \tilde{a}_{11} \tilde{z}_1 = -((k-\tau)\xi_1 + \tau\xi_2) + ((k-\tau)\xi_1 + \tau\xi_2) \cdot 1 = 0;$$

$$\tilde{r}_i = \tilde{a}_{i,i+1}\tilde{z}_{i+1} + \tilde{a}_{i,i-1}\tilde{z}_{i-1} + \tilde{a}_{i,i}\tilde{z}_i = -i\tilde{z}_{i+1} + (k-i+2)\sum_{\mu=0}^{i-2} C_{k-\tau}^{i-\mu-2} C_{\tau}^{\mu} \xi_1^{i-\mu-1} \xi_2^{\mu+1} +$$

$$+(k-\tau-i+1)\sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu} \xi_2^{\mu} + (\tau-i+1)\sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu-1} \xi_2^{\mu+1} =$$

$$= -i\tilde{z}_{i+1} + (k-i+2)\sum_{\mu=1}^{i-1} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu-1} \xi_1^{i-\mu} \xi_2^{\mu} +$$

$$+(k-\tau-i+1)\sum_{\mu=1}^{i-1} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu} \xi_2^{\mu} + (k-\tau-i+1) C_{k-\tau}^{i-1} C_{\tau}^{0} \xi_1^{i} \xi_2^{0} +$$

$$+(\tau-i+1)\sum_{\mu=0}^{i-2} C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} \xi_1^{i-\mu-1} \xi_2^{\mu+1} + (\tau-i+1) C_{k-\tau}^{0} C_{\tau}^{i-1} \xi_1^{0} \xi_2^{i} =$$

$$= -i\tilde{z}_{i+1} + \sum_{\mu=1}^{i-1} [(k-i+2) C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu-1} + (k-\tau-i+1) C_{k-\tau}^{i-\mu-1} C_{\tau}^{\mu} +$$

$$+(\tau-i+1) C_{k-\tau}^{i-\mu} C_{\tau}^{\mu-1}] \xi_1^{i-\mu} \xi_2^{\mu} + i C_{k-\tau}^{k-\tau-i} C_{\tau}^{0} \xi_1^{i} \xi_2^{0} + i C_{k-\tau}^{0} C_{\tau}^{i} \xi_1^{0} \xi_2^{i} =$$

$$= -i\tilde{z}_{i+1} + \sum_{\mu=1}^{i-\mu} C_{k-\tau}^{i-\mu} C_{\tau}^{\mu} \left[(k-i+2) \cdot \frac{\mu(i-\mu)}{(k-\tau-i+\mu+1)(\tau-\mu+1)} + \right.$$

$$\left. +(k-\tau-i+1) \cdot \frac{i-\mu}{k-\tau-i+\mu+1} + (\tau-i+1) \cdot \frac{\mu}{\tau-\mu+1} \right] \xi_1^{i-\mu} \xi_2^{\mu} +$$

$$+iC^i_{k-\tau}C^0_\tau\xi_1^i\xi_2^0+iC^0_{k-\tau}C^i_\tau\xi_1^0\xi_2^i=-i\tilde{z}_{i+1}+i\Big(C^i_{k-\tau}C^0_\tau\xi_1^i\xi_2^0+\sum_{\mu=1}^{i-1}C^{i-\mu}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu+$$

$$+C^0_{k-\tau}C^i_\tau\xi_1^0\xi_2^i\Big)=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

b) $\boldsymbol{i=\tau+1}$**.** From formulae (6), (11) and (12) we get:

$$\tilde{r}_{\tau+1}=\tilde{a}_{\tau+1,\tau}\tilde{z}_\tau+\tilde{a}_{\tau+1,\tau+1}\tilde{z}_{\tau+1}-\tilde{a}_{\tau+1,\tau+2}\tilde{z}_{\tau+2}=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+$$

$$+(k-2\tau)\sum_{\mu=0}^{\tau}C^{\tau-\mu}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu-(\tau+1)\sum_{\mu=0}^{\tau}C^{\tau-\mu+1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu=$$

$$=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+(k-2\tau)\sum_{\mu=1}^{\tau}C^{\tau-\mu}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu-$$

$$-(\tau+1)\sum_{\mu=1}^{\tau}C^{\tau-\mu+1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+$$

$$+\xi_1\xi_2\sum_{\mu=0}^{\tau-1}\left[(k-2\tau)\frac{\tau-\mu}{\mu+1}-(\tau+1)\frac{k-2\tau+\mu+1}{\tau-\mu}\cdot\frac{\tau-\mu}{\mu+1}\right]C^{\tau-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu-1}\xi_2^\mu=$$

$$=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau-(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau=0.$$

c) $\boldsymbol{\tau+2\le i\le k-\tau}$**.** In this case the formulae (6) and (12) give us:

$$\tilde{r}_i=-i\tilde{z}_{i+1}+(k-i+2)\xi_1\xi_2\tilde{z}_{i-1}+[(k-\tau-i+1)\xi_1+(\tau-i+1)\xi_2]\tilde{z}_i=$$

$$=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=0}^{\tau}C^{i-\mu-2}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+$$

$$+(\tau-i+1)\sum_{\mu=0}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+(k-\tau-i+1)\sum_{\mu=0}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu=$$

$$=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=1}^{\tau}C^{i-\mu-2}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+(\tau-i+1)\sum_{\mu=0}^{\tau-1}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+$$

$$+(k-\tau-i+1)\sum_{\mu=1}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu+(k-\tau-i+1)C^{i-1}_{k-\tau}\xi_1^i=$$

$$-i\tilde{z}_{i+1}+iC^i_{k-\tau}\xi_1^i+\sum_{\mu=1}^{\tau}\left[(k-i+2)\frac{(i-\mu)\mu}{(k-\tau-i+\mu+1)(\tau-\mu+1)}+\right.$$

$$+(\tau-i+1)\frac{\mu}{\tau-\mu+1}+(k-\tau-i+1)\frac{i-\mu}{k-\tau-i+\mu+1}\Bigg]C_{k-\tau}^{i-\mu}C_\tau^\mu\xi_1^{i-\mu}\xi_2^\mu=$$
$$=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

d) $\boldsymbol{i=k-\tau+1}$. From (6), (12) and (13) we obtain:

$$\tilde{r}_{k-\tau+1}=(\tau+1)\sum_{\mu=0}^{\tau}C_{k-\tau}^{\mu+1}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-(k-2\tau)\sum_{\mu=0}^{\tau}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-$$
$$-(k-\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\tau-\mu-1}C_\tau^\mu\xi_1^{k-2\tau+\mu+1}\xi_2^{\tau-\mu}=(\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu+1}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-$$
$$-(k-2\tau)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-(k-\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^{\mu+1}\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}=$$
$$=\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}\left[(\tau+1)\frac{k-\tau-\mu}{\mu+1}-(k-2\tau)-(k-\tau+1)\frac{\tau-\mu}{\mu+1}\right]=0.$$

e) $\boldsymbol{k-\tau+2\le i\le k+1.}$

$$\tilde{r}_i=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=0}^{k-i+2}C_{k-\tau}^{i-\tau+\mu-2}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}+$$
$$+(k-\tau-i+1)\sum_{\mu=0}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}+$$
$$+(\tau-i+1)\sum_{\mu=0}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}=$$
$$-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=1}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-2}C_\tau^{\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}+$$
$$+(k-\tau-i+1)\sum_{\mu=0}^{k-i}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\mu}\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}+$$
$$+(\tau-i+1)\sum_{\mu=1}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}=-i\tilde{z}_{i+1}+$$
$$+\sum_{\mu=0}^{k-i}C_{k-\tau}^{i-\tau+\mu}C_\tau^{\mu}\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}\Bigg[(k-i+2)\frac{(i-\tau+\mu)(\tau-\mu)}{(k-i-\mu+1)(\mu+1)}+$$

$$+(k-\tau-i+1)\frac{i-\tau+\mu}{k-i-\mu+1}+(\tau-i+1)\frac{\tau-\mu}{\mu+1}\Bigg]=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

Hence, taking into account (7), the rank of the matrix $\tilde{A}$ is equal to k and therefore the general solution of the matrix equation (9) has the form $Z=\{c\tilde{Z}|c\in\mathbb{C}\}$. □

Corollary 1. *If* $\mathbf{Z}=\mathbf{a}$ $(\mathbf{Z}=\mathbf{b})$*, where*

$$\mathbf{a}=(a_{k0},\,a_{k-1,1},...,a_{0k})\ (\mathbf{b}=(b_{k0},\,b_{k-1,1},...,b_{0k})),\tag{14}$$

is a solution of the matrix equation (9) then the first (second) equation of (1) has the form

$$\dot{x}=c\cdot(x+\xi_1 y)^{k-\tau}(x+\xi_2 y)^{\tau},\ \ (\dot{y}=c\cdot(x+\xi_1 y)^{k-\tau}(x+\xi_2 y)^{\tau}).$$

3 Proof of Theorem 3

Applying to the system (1) the transposition of coordinates

$$x_1\to x_2,\ \ x_2\to x_1\tag{15}$$

we obtain

$$\dot{x}_1=Q_k(x_2,x_1),\ \ \dot{x}_2=P_k(x_2,x_1).\tag{16}$$

Denote by $\mathbf{v}_l^*$, $l=\overline{1,4}$, the vector fields associated to the differential system (16).

Remark 1. The equalities $\alpha\mathbf{v}_1+\beta\mathbf{v}_2+\gamma\mathbf{v}_3+\delta\mathbf{v}_4=0$ and $\delta\mathbf{v}_1^*+\gamma\mathbf{v}_2^*+\beta\mathbf{v}_3^*+\alpha\mathbf{v}_4^*=0$ are equivalent.

By Remark 1, in order to determine the orbits of dimension two and three it is sufficient to examine the following two cases:

$$\mathbf{v}_1-\delta\mathbf{v}_4=0,\quad \alpha\mathbf{v}_1+\beta\mathbf{v}_2-\mathbf{v}_3+\delta\mathbf{v}_4=0.$$

3.1 The case $\mathbf{v}_1-\delta\mathbf{v}_4=0$.

Let $\mathbf{v}_1(\mathbf{e})-\delta\mathbf{v}_4(\mathbf{e})=0$ or

$$(A^{(1)}-\delta A^{(4)})\mathbf{e}=0.\tag{17}$$

Because $\mathbf{e}\neq 0$ (see (4)) the equality (17) is realized for those δ for which $det(A^{(1)}-\delta A^{(4)})=0$, i.e.

$$-(k-1)^2(1+k\delta)(\delta+k)\prod_{\nu=2}^{k}[(\nu-1)\delta+\nu-k]^2=0.$$

By the assumption (4), $k\geq 2$. If $\delta=-1/k$ $(\delta=-k)$, then $det(D_1-\delta D_4)\neq 0$ $(det(A_1-\delta A_4)\neq 0)$, but the matrix $A_1-\delta A_4$ $(D_1-\delta D_4)$ has on the principal diagonal unique element equal to zero and this element is placed on $(k+1,k+1)$ $((1,1))$. In these cases equality (17) leads us to the systems

$$\dot{x_1} = a_{0k}x_2^k,\ \dot{x_2} = 0; \tag{18}$$

$$\dot{x_1} = 0,\ \dot{x_2} = b_{k0}x_1^k. \tag{19}$$

Let $\delta = (k-\nu)/(\nu-1)$. For this δ both matrixes $A_1 - \delta A_4$ and $D_1 - \delta D_4$ have on the principal diagonal only one element equal to zero: first on the cells $(\nu,\nu),\ \nu = \overline{2,k}$, and second on the cells $(\nu+1,\nu+1),\ \nu = \overline{2,k}$. Taking into account (17), we obtain the systems

$$\dot{x}_1 = a_{k-\nu+1,\nu-1}x_1 \cdot F,\ \dot{x}_2 = b_{k-\nu,\nu}x_2 \cdot F,\ F = x_1^{k-\nu}x_2^{\nu-1},\ \nu = \overline{2,k}. \tag{20}$$

Remark 2. Substitutions (15) reduce system (19) to one of the form (18).

3.2 The case $\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \delta\mathbf{v}_4$.

In this subsection we will determine the systems (1), $k \geq 2$, for which there exist numbers α, β and δ such that

$$\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \delta v_4. \tag{21}$$

Denote

$$M = A^{(3)} - \alpha A^{(1)} - \beta A^{(2)} - \delta A^{(4)},\ \mathbf{a} = (a_{k0},\dots,a_{0k}),\ \mathbf{b} = (b_{k0},\dots,b_{0k}).$$

Then

$$M = \begin{pmatrix} A & I \\ -\beta I & D \end{pmatrix},\ A = A_3 - \alpha A_1 - \beta A_2 - \delta A_4,\ D = A + (\alpha-\delta)I,\ \mathbf{e} = (\mathbf{a},\mathbf{b}).$$

We have to find α, β and δ such that the matrix equation

$$M\mathbf{e} = 0 \qquad \text{or} \qquad \begin{cases} A\mathbf{a} = -\mathbf{b}, \\ [A + (\alpha-\delta)I]\mathbf{b} = \beta\mathbf{a} \end{cases} \tag{22}$$

have nontrivial solutions with respect to $\mathbf{e}$.

From (22) it follows that $\mathbf{a}$ and $\mathbf{b}$ verify the same matrix equation:

$$SZ = 0, \tag{23}$$

where $S = A^2 + (\alpha-\delta)A + \beta I$, $dim S = (k+1)\times(k+1)$, and Z is the vector (8).

The matrix S has the following elements:

$$s_{11} = (k-1)(k\alpha^2 - \alpha\delta - \beta), \quad s_{12} = -2(k-1)\alpha, \quad s_{13} = 2;$$

$$s_{21} = 2k(k-1)\alpha\beta,\ s_{22} = (k-1)\left[(k-2)\alpha^2 + \alpha\delta - 3\beta\right],$$
$$s_{23} = -4\left[(k-2)\alpha + \delta\right],\ s_{24} = 6;$$

$$s_{i,i-2} = (k-i+2)(k-i+3)\beta^2,\ s_{i,i-1} = 2(k-i+2)[(k-i+1)\alpha + (i-2)\delta]\beta,$$
$$s_{i,i} = [(k-i)\alpha + (i-1)\delta] \cdot [(k-i+1)\alpha + (i-2)\delta] - [(2i-1)k - 2(i-1)^2 - 1]\beta,$$
$$s_{i,i+1} = -2i[(k-i)\alpha + (i-1)\delta],\ s_{i,i+2} = i(i+1),\ i = \overline{2, k-1};$$
$$s_{k,k-2} = 6\beta^2,\ s_{k,k-1} = 4[\alpha + (k-2)\delta] \cdot \beta,\ s_{k,k} = (k-1)[\alpha\delta + (k-2)\delta^2 - 3\beta],$$
$$s_{k,k+1} = -2k(k-1)\delta;$$
$$s_{k+1,k-1} = 2\beta^2,\ s_{k+1,k} = 2(k-1)\delta\beta,\ s_{k+1,k+1} = (k-1)(k\delta^2 - \alpha\delta - \beta);$$
$$s_{ij} = 0,\ i,j = \overline{1, k+1}, |i-j| > 2.$$

The rank of S verifies the inequalities $k-1 \le rank(S) \le k+1$ and the determinant ($\Delta = det(S)$) is equal to

$$\begin{aligned}&\Delta = (k-1)^4(\beta+\alpha\delta)^2[(k+1)^2\beta - (\alpha - k\delta)(k\alpha - \delta)]\times\\ &\prod_{j=0}^{m-2}[(2j+1)^2\beta + ((m+j)\alpha + (m-j-1)\delta)\times\\ &((m-j-1)\alpha + (m+j)\delta)]^2\end{aligned} \tag{24}$$

if $k = 2m$ and

$$\begin{aligned}&\Delta = 4m^4(\alpha+\delta)^2(\beta+\alpha\delta)^2[(k+1)^2\beta - (\alpha - k\delta)(k\alpha - \delta)]\times\\ &\prod_{j=1}^{m-1}[4j^2\beta + ((m+j)\alpha + (m-j)\delta)\times\\ &((m-j)\alpha + (m+j)\delta)]^2\end{aligned} \tag{25}$$

if $k = 2m+1$.

Denote

$$A_{1,2} = A - \frac{\delta - \alpha \pm \sqrt{(\delta-\alpha)^2 - 4\beta}}{2} I.$$

We have that $A_1A_2 = A_2A_1$, $k \le rank A_{1,2} \le k+1$ and in (23) that $S = A_2A_1$. Hence, every solution of the matrix equation

$$A_1 Z = 0 \tag{26}$$

or

$$A_2 Z = 0 \tag{27}$$

is also a solution of the equation (23).

Next we will analyse each of the cases when the determinant Δ of the matrix S is equal to zero and will indicate the systems (1) of which coefficients (14) verify the matrix equation (23), i.e. each of the vectors **a** and **b** verifies at least one of the equations (26), (27).

3.2.1. $\beta = (\alpha - k\delta)(k\alpha - \delta)/(k+1)^2$. Let

$$\xi_1 = (\alpha - k\delta)/(k+1), \quad \xi_2 = (k\alpha - \delta)/(k+1).$$

Then $\beta = \xi_1\xi_2$ and

$$A_1 = A + \xi_1 I, \quad A_2 = A + \xi_2 I. \tag{28}$$

Setting in (6) $\tau = k$, we obtain that $\tilde{A} = A_1$. Therefore, $det A_1 = 0$ and $ker A_1 = \{c\mathbf{Z_1} | c = const\}$, where $\mathbf{Z_1}$ has coordinates (11).

If $A_2 \neq A_1$, i.e. $\alpha + \delta \neq 0$, then from (24), (25) and $\Delta = det S = det A_1 \cdot det A_2$ it follows that $det A_2 \neq 0$. Thus, in this case, in order that the dimension of the $GL(2, \mathbb{R})-$orbit of the system (1) be smaller than four it is necessary that its coefficients (14) (**a** and **b**) verify the equation (26). By Lemma 1 $f = c(x + \xi_2)^k$ and by Corollary 1, we have

$$\begin{cases} \dot{x_1} = c_1 \cdot F(x_1, x_2),\ \dot{x_2} = c_2 \cdot F(x_1, x_2);\ c_1, c_2 = const, \\ F = [(k+1)x_1 + (k\alpha - \delta)x_2]^k . \end{cases} \tag{29}$$

3.2.2. $\beta = -\alpha\delta$. In this case, we put $\xi_1 = \alpha,\ \xi_2 = -\delta$. Then $A_1 = A + \alpha I,\ A_2 = A - \delta I$ and setting in (6) $\tau = 0$ ($\tau = 1$), we have that $A_1 = \tilde{A}$ ($A_2 = \tilde{A}$) and $f = c_1(x + \xi_1)^k$ ($f = c_2(x + \xi_1)^{k-1}(x + \xi_2)$). If $\tau = 0$ ($\tau = 1$) the vector $\mathbf{Z_1}$ ($\mathbf{Z_2}$) with the coordinates (12) ((12), (13)) is a solution of the equation (26) ((27)). The solutions $\mathbf{Z_1}$ and $\mathbf{Z_2}$ are linear independent and therefore $c_1\mathbf{Z_1} + c_2\mathbf{Z_2};\ c_1, c_2 = const$, is the general solution of (23). This implies (1) to have the form

$$\begin{cases} \dot{x_1} = (ax_1 + bx_2)F(x_1, x_2),\ \dot{x_2} = (cx_1 + dx_2)F(x_1, x_2); \\ a, b, c, d = const,\ F(x_1, x_2) = (x_1 + \alpha x_2)^{k-1}. \end{cases} \tag{30}$$

3.2.3. $k = 2m$,

$$\beta = -((m-j-1)\alpha + (m+j)\delta)((m+j)\alpha + (m-j-1)\delta)/(2j+1)^2,\ j = \overline{0, m-2}.$$

Let

$$\xi_1 = -[(m-j-1)\alpha + (m+j)\delta]/(2j+1),$$

$$\xi_2 = [(m+j)\alpha + (m-j-1)\delta]/(2j+1).$$

Then $\beta = \xi_1\xi_2$ and the equalities (28) hold. Setting in (6) $\tau = m + j$ ($\tau = m + j + 1$), we obtain that $\tilde{A} = a_1$ ($\tilde{A} = a_2$), and the relations (11)$-$(13) lead us to the polynomial

$$f = c(x + \xi_1)^{m-j}(x + \xi_2)^{m+j}\ \left(f = c(x + \xi_1)^{m-j-1}(x - \xi_2)^{m+j+1}\right).$$

Hence, for $\tau = m - j$ ($\tau = m - j - 1$) the vector Z_1 (Z_2) with the coordinates (11)$-$(13) is a solution of the equation (26) ((27)). The vectors Z_1 and Z_2 are linear independent which implies the differential system (1) to be written as:

$$\begin{cases} \dot{x_1} = (ax_1 + bx_2) \cdot F,\ \dot{x_2} = (cx_1 + dx_2) \cdot F, \\ F = [(2j+1)x_1 - ((m-j-1)\alpha + (m+j)\delta)x_2]^{m-j-1} \times \\ [(2j+1)x_1 + ((m+j)\alpha + (m-j-1)\delta)x_2]^{m+j},\ j = \overline{0, m-2}. \end{cases} \tag{31}$$

3.2.4. Let $k = 2m+1$ and $\beta = \xi_1\xi_2$, where

$$\xi_1 = -\left[(m-j)\alpha + (m+j)\delta\right]/(2j),$$

$$\xi_2 = \left[(m+j)\alpha + (m-j)\delta\right]/(2j),\ \ j = \overline{1, m-1}.$$

In these conditions equalities (28) hold. If $\tau = m+j$ ($\tau = m+j+1$), then the vector Z_1 (Z_2) with the coordinates (11)−(13) is a solution of the equation (26) ((27)) and the polynomial (5) looks as:

$$f = c(x+\xi_1)^{m-j+1}(x+\xi_2)^{m+j}\ \ (f = c(x+\xi_1)^{m-j}(x-\xi_2)^{m+j+1}).$$

The solutions $c_1Z_2 + c_2Z_2$; $c_1c_2 = const$ of the equation (23) lead us to the following system

$$\begin{cases} \dot{x_1} = (ax_1+bx_2)\cdot F,\ \dot{x_2} = (cx_1+dx_2)\cdot F, \\ F = \left[2jx_1 - ((m-j)\alpha + (m+j)\delta)x_2\right]^{m-j} \times \\ \left[2jx_1 + ((m+j)\alpha + (m-j)\delta)x_2\right]^{m+j},\ j = \overline{1, m-1}. \end{cases} \tag{32}$$

3.2.5. $\alpha + \delta = 0$. Let

$$\delta = -\alpha,\ \ \xi_1 = \alpha - \sqrt{\alpha^2 - \beta},\ \ \xi_2 = \alpha + \sqrt{\alpha^2-\beta}.$$

Substituting in (11)−(13) $\tau = m$ ($\tau = m+1$), we obtain that the vector Z_1 (Z_2) with these coordinates is a solution of the equation (26) ((27)), where A_1 and A_2 are given in (28). The polynomial f looks as:

$$f = c(x+\xi_1)^{m+1}(x+\xi_2)^m\ \ (f = c(x+\xi_1)^m(x+\xi_2)^{m+1}).$$

This case leads us to the following differential system

$$\begin{cases} \dot{x_1} = (ax_1+bx_2)\cdot F,\ \dot{x_2} = (cx_1+dx_2)\cdot F, \\ F = (x_1^2 + 2\alpha x_1x_2 + \beta x_2^2)^m. \end{cases} \tag{33}$$

Theorem 3 is proved.

From proving Theorem 3 follows

Theorem 4. *In order that the dimension of the $GL(2,\mathbb{R})-$orbit of the system (1) be smaller than four it is necessary (up to transformation (15)) that the system (1) have one of the forms (18),(20),(29)–(33).*

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