BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(58), 2008, Pages 34–43
ISSN 1024–7696

Ore extensions over 2-primal Noetherian rings

V. K. Bhat *

Abstract. Let R be a ring and σ an automorphism of R. We prove that if R is a 2-primal Noetherian ring, then the skew polynomial ring $R[x;\sigma]$ is 2-primal Noetherian. Let now δ be a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R. We prove that $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring if R is a Noetherian $\mathbb{Q}$-algebra, σ and δ are such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$, P being any minimal prime ideal of R. We use this to prove that if R is a Noetherian $\sigma(*)$-ring (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$), δ a σ-derivation of R such that R is a δ-ring and $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring.

Mathematics subject classification: Primary 16XX; secondary 16S36, 16N40, 16P40, 16W20, 16W25.
Keywords and phrases: 2-primal, minimal prime, prime radical, nil radical, automorphism, derivation.

1 Introduction

A ring R always means an associative ring. $\mathbb{Q}$ denotes the field of rational numbers. $Spec(R)$ denotes the set of prime ideals of R. $MinSpec(R)$ denotes the set of minimal prime ideals of R. $P(R)$ and $N(R)$ denote the prime radical and the set of nilpotent elements of R, respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J. Let I be an ideal of a ring R such that $\sigma^m(I) = I$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0.

This article concerns the study of Ore extensions in terms of 2-primal rings. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [18], G. Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R.

Let σ be an endomorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a homomorphism defined by

*The author would like to express sincere thanks to the referee for his suggestions to give the manuscript the present shape.

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \quad \text{for all} \quad r \in R.$$

Then δ is a σ-derivation of R.

Also let $R = K[x]$, K a field. Then the formal derivative d/dx is a derivation of R.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [15] and Shin in [20]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring R is called 2-primal if the set of nilpotent elements of R coincides with the prime radical of R (G. Marks [18]), or equivalently if its radical contains every nilpotent element of R, or if $P(R)$ is a completely semiprime ideal of R. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$.

We also note that a reduced ring (i. e. a ring with no nonzero nilpotent elements) is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [5, 11, 14, 15, 20].

Recall that $R[x; \sigma, \delta]$ is the skew polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x; \sigma, \delta]$ by $O(R)$. In case σ is the identity map, we denote the ring of differential operators $R[x; \delta]$ by $D(R)$, if δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$.

Recall that in Krempa [16], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [17], Kwak defines a $\sigma(*)$-ring R to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $S(R)$.

Remark 1. If R is a ring and σ an automorphism of R such that R is a $\sigma(*)$-ring, then R is 2-primal.

Proof. We will show that $P(R)$ is a completely semiprime ideal of R. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

In Theorem 12 of [17], Kwak has proved that if R is a $\sigma(*)$-ring such that $\sigma(P(R)) = P(R)$, then $R[x; \sigma]$ is 2-primal if and only if $P(R)[x; \sigma] = P(R[x; \sigma])$.

Hong, Kim and Kwak have proved in Corollary 2.8 of [13] that if R is a 2-primal ring and every simple singular left R-module is p-injective, then every prime ideal of R is maximal. In particular, every prime factor ring of R is a simple domain.

It is known (Theorem 1.2 of Bhat [5]) that if R is 2-primal Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $D(R)$ is 2-primal. We also note that if R is a Noetherian ring, then even $R[x]$ need not be 2-primal.

Example 1. Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.

Let now R be a 2-primal ring. Is $O(R)$ also a 2-primal ring? For the time being we are not able to answer this question, but towards this we have the following.

Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$. We note that a ring with identity is not a δ-ring. We ultimately prove the following:

1. Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian. This is proved in Theorem 2.

2. Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal Noetherian. This is proved in Theorem 6.

3. Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and R is a δ-ring. Then $R[x; \sigma, \delta]$ is 2-primal Noetherian.

Before proving (2) and (3) above, we find a relation between the minimal prime ideals of R and those of the Ore extension $O(R)$, where R is a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R. This is proved in Theorem 3.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 3, 4, 6–8, 12, 16, 17].

2 Skew polynomial ring $S(R)$

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also note that in a right Noetherian ring R, $MinSpec(R)$ is finite (Theorem 2.4 of Goodearl and Warfield [10]), and for any $P \in MinSpec(R)$, $\sigma^t(P) \in MinSpec(R)$ for all integers $t \geq 1$. Let $MinSpec(R) = \{P_1, P_2, \ldots, P_n\}$. Let $\sigma^{m_i}(P_i) = P_i$, for some positive integers m_i, $1 \leq i \leq n$, and $u = m_1.m_2...m_n$. Then $\sigma^u(P_i) = P_i$ for all $P_i \in MinSpec(R)$. We use same u henceforth, and as mentioned in introduction above, we denote $\cap_{i=1}^{u}\sigma^i(P)$ by P^0, P being any minimal prime ideal of R.

Proposition 1. *Let R be a right Noetherian ring. Let σ be an automorphism of R. Then $\sigma(N(R)) = N(R)$.*

Proof. Denote $N(R)$ by N. We have $\sigma(N) \subseteq N$ as R is right Noetherian, therefore, $\sigma(N)$ is a nilpotent ideal of R by Theorem 5.18 of Goodearl and Warfield [10]. Now let $n \in N$. Then σ being an automorphism of R implies that there exists $a \in R$ such that $n = \sigma(a)$. Now $I = \sigma^{-1}(N) = \{a \in R \text{ such that } \sigma(a) = n \in N\}$ is an ideal of R. Now I is nilpotent, so $I \subseteq \sigma(N)$, which implies that $N \subseteq \sigma(N)$. Hence $\sigma(N) = N$. □

Proposition 2. *Let R be a Noetherian ring and σ an automorphism of R. Then $S(N(R)) = N(S(R))$.*

Proof. It is easy to see that $S(N(R)) \subseteq N(S(R))$. We will show that $N(S(R)) \subseteq S(N(R))$. Let $f = \sum_{i=0}^{m} x^i a_i \in N(S(R))$. Then $f(S(R)) \subseteq N(S(R))$, and $f(R) \subseteq N(S(R))$. Let $(f(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get $(x^m a_m R)^k = 0$. This implies on simplification that

$$x^{km}\sigma^{(k-1)m}(a_m R)\cdot\sigma^{(k-2)m}(a_m R)\cdot\sigma^{(k-3)m}(a_m R)\ldots a_m R = 0.$$

Therefore,

$$\sigma^{(k-1)m}(a_m R)\cdot\sigma^{(k-2)m}(a_m R)\cdot\sigma^{(k-3)m}(a_m R)\ldots a_m R = 0 \subseteq P,$$

for all $P \in MinSpec(R)$. Now there are two cases:

1. $u \geq m$.

2. $m \geq u$.

If $u \geq m$, then we have

$$\sigma^{(k-1)u}(a_m R)\cdot\sigma^{(k-2)u}(a_m R)\cdot\sigma^{(k-3)u}(a_m R)\ldots a_m R \subseteq P.$$

This implies that $\sigma^{(k-j)u}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$, i.e. $a_m R \subseteq \sigma^{-(k-j)u}(P) = P$. So we have $a_m R \subseteq P$, for all $P \in MinSpec(R)$. Therefore, $a_m \in P(R) = N(R)$. Now $x^m a_m \in S(N(R)) \subseteq N(S(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(S(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore $f \in S(N(R))$. Hence $N(S(R)) \subseteq S(N(R))$ and the result follows. The other case is similar. □

Theorem 1. (Theorem 2.4, (2) of Bhat [4]) *Let R be a Noetherian ring and σ an automorphism of R. Then $P \in MinSpec(S(R))$ if and only if there exists $L \in MinSpec(R)$ such that $S(P \cap R) = P$ and $P \cap R = L^0$.*

Proof. Let $L \in MinSpec(R)$. Then $\sigma^u(L) = L$ for some integer $u \geq 1$. Then by Lemma 10.6.12 of McConnell and Robson [19] and by Theorem 7.27 of Goodearl and Warfield [10], $S(L^0) \in MinSpec(S(R))$.

Conversely suppose that $P \in MinSpec(S(R))$. Then $P \cap R = U^0$ for some $U \in Spec(R)$ and U contains a minimal prime ideal U_1. Now $P \supseteq S(R)U_1^0$, which is a prime ideal of $S(R)$. Hence $P = S(R)U_1^0$. □

We are now in a position to prove the main result of this section in the form of the following Theorem.

Theorem 2. *Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian.*

Proof. R is Noetherian implies $S(R)$ is Noetherian follows from Hilbert Basis Theorem, namely Theorem 1.12 of Goodearl and Warfield [10]. Now R is 2-primal implies $N(R) = P(R)$ and Proposition 1 implies that $\sigma(N(R)) = N(R)$. Therefore $S(N(R))$ and $S(P(R))$ are ideals of $S(R)$ and $S(N(R)) = S(P(R))$. Now by Proposition 2 $S(N(R)) = N(S(R))$.

We now show that $S(P(R)) = P(S(R))$. It is easy to see that $S(P(R)) \subseteq P(S(R))$. Now let g $= \sum_{i=0}^{t} x^i b_i \in P(S(R))$. Then $g \in P_i$, for all $P_i \in MinSpec(S(R))$. Now Theorem 1 implies that there exists $U_i \in MinSpec(R)$ such that $P_i = S((U_i)^0)$. Now it can be seen that P_i are distinct implies that U_i are distinct. Therefore $g \in S((U_i)^0)$. This implies that $b_i \in (U_i)^0 \subseteq U_i$. Thus we have $b_i \in U_i$, for all $U_i \in MinSpec(R)$. Therefore $b_i \in P(R)$, which implies that $g \in S(P(R))$. Therefore $P(S(R)) \subseteq S(P(R))$, and hence $S(P(R)) = P(S(R))$.

Thus we have $P(S(R)) = S(P(R)) = S(N(R)) = N(S(R))$. Hence $S(R)$ is 2-primal. □

Question 1. *Let R be a 2-primal ring. Is $S(R)$ 2-primal? The main difficulty is that Proposition 2 and Theorem 1 do not hold.*

3 Ore extension $O(R)$

We begin with the following definition:

Definition 1. Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. We say that R is a δ-ring if $\delta(a) \in P(R)$ implies $a \in P(R)$.

Recall that an ideal I of a ring R is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $O(I)$ is an ideal of $O(R)$ as for any $a \in I$, $\sigma^j(a) \in I$ and $\delta^j(a) \in I$ for all positive integers j.

Gabriel proved in Lemma 3.4 of [9] that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. We generalize this for σ-derivation δ of R and give a structure of minimal prime ideals of $O(R)$ in the following Theorem.

Theorem 3. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then:*

1. *$P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*
2. *$P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$.*

Proof. **(1)** Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Let $T = R[[t;\sigma]]$, the skew power series ring. We note that multiplication in $R[[t;\sigma]]$ is determined by the computation $ax = x\sigma(a)$ for all $a \in R$. Now we know that

$$e^{t\delta} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n$$

and it can be seen that $e^{t\delta}$ is an automorphism of T. Now $P_1T \in Spec(T)$. Suppose if possible that $P_1T \notin MinSpec(T)$ and $P_2 \subset P_1T$ be a minimal prime ideal of T. Then $P_2 \cap R \subset P_1T \cap R = P_1$, which is not possible as $P_1 \in MinSpec(R)$. Therefore $P_1T \in MinSpec(T)$. We also know that $(e^{t\delta})^k(P_1T) \in MinSpec(T)$ for all integers $k \geq 1$. Now T is Noetherian by Exercise (1ZA(c)) of Goodearl and Warfield [10], and therefore, Theorem 2.4 of Goodearl and Warfield [10] implies that $MinSpec(T)$ is finite. So there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(P_1T) = P_1T$, i.e. $(e^{nt\delta})(P_1T) = P_1T$. But R is a $\mathbb{Q}$-algebra, therefore, $e^{t\delta}(P_1T) = P_1T$. Now for any $a \in P_1$, $a \in P_1T$ also, and so $e^{t\delta}(a) \in P_1T$, i.e.

$$a + t\delta(a) + (t^2/2!)\delta^2(a) + \cdots \in P_1T,$$

which implies that $\delta(a) \in P_1$. Therefore $\delta(P_1) \subseteq P_1$.

Now on the same lines as in Theorem 2.22 of Goodearl and Warfield [10], it can be easily seen that $O(P_1) \in Spec(O(R))$. Suppose that $O(P_1) \notin MinSpec(O(R))$, and $P_2 \subset O(P_1)$ is a minimal prime ideal of $O(R)$. Then we have $P_2 = O(P_2 \cap R) \subset O(P_1) \in MinSpec(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in MinSpec(O(R))$.

(2) Let $P \in MinSpec(O(R))$ with $\sigma(P \cap R) = P \cap R$. Then on the same lines as in Theorem 2.22 of Goodearl and Warfield [10], it can be seen that $P \cap R \in Spec(R)$ and $O(P \cap R) \in Spec(O(R))$. Therefore $O(P \cap R) = P$. We now show that $P \cap R \in MinSpec(R)$. Suppose that $U \subset P \cap R$, and $U \in MinSpec(R)$. Then $O(U) \subset O(P \cap R) = P$. But $O(U) \in Spec(O(R))$ and, $O(U) \subset P$, which is not possible. Thus we have $P \cap R \in MinSpec(R)$. □

Recall that in Proposition 1.11 of Shin [20], it has been proved that a ring R is 2-primal if and only if each minimal prime ideal of R is a completely prime ideal.

Proposition 3. *Let R be a 2-primal ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. Let $P \in MinSpec(R)$. Now P is a completely prime ideal, therefore, for any $a \in P$, there exists $b \notin P$ such that $ab \in P(R)$ by Corollary 1.10 of Shin [20]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \in P(R)$; i.e. $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Now $\sigma(P) = P$ implies that $\sigma(b) \notin P$ and since P is completely prime in R, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$. □

Theorem 4. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\rho : R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that τ is an automorphism of R/P(R) and ρ is a τ-derivation of R/P(R). Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in R/P(R). Thus as in Proposition 5 of Hong, Kim and Kwak [12], R is a reduced ring and, therefore as mentioned in introduction, R is 2-primal. □

Proposition 4. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.*

2. *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Proof. **(1)** Let P be a completely prime ideal of R. Now let $f(x) = \sum_{i=0}^{n} x^i a_i \in O(R)$ and $g(x) = \sum_{j=0}^{m} x^j b_j \in O(R)$ be such that $f(x)g(x) \in O(P)$. Suppose $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. We use induction on n and m. For $n = m = 1$, the verification is easy. We check for $n = 2$ and $m = 1$. Let $f(x) = x^2a + xb + c$ and $g(x) = xu + v$. Now $f(x)g(x) \in O(P)$ with $f(x) \notin O(P)$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to P or all of them do not belong to P. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in O(P)$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in O(P)$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore a, $\delta(a) \in P$. Now $\delta(a)u+\sigma(b)u+av \in P$ implies that $\sigma(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $\delta(b)u+\sigma(c)u+bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in O(P)$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above a, $\delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i. e. $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in O(P)$.

Now suppose the result is true for k, $n = k > 2$ and $m = 1$. We will prove for $n = k + 1$. Let $f(x) = x^{k+1}a_{k+1} + x^k a_k + \cdots + xa_1 + a_0$, and $g(x) = xb_1 + b_0$ be such that $f(x)g(x) \in O(P)$, but $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. If $a_{k+1} \notin P$, then equating coefficients of x^{k+2}, we get $\sigma(a_{k+1})b_1 \in P$, which implies that $b_1 \in P$. Now equating coefficients of x^{k+1}, we get $\sigma(a_k)b_1 + a_{k+1}b_0 \in P$, which implies that $a_{k+1}b_0 \in P$, and therefore $b_0 \in P$. Hence $g(x) \in O(P)$.

If $a_j \notin P$, $0 \leq j \leq k$, then using induction hypothesis, we get that $g(x) \in O(P)$. Therefore the statement is true for all n. Now using the same process, it can be easily seen that the statement is true for all m also.

(2) Let U be a completely prime ideal of $O(R)$. Suppose $a, b \in R$ are such that $ab \in U \cap R$ with $a \notin U \cap R$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in U \cap R \subseteq U$, with $a \notin U$. Therefore we have $b \in U$, and thus $b \in U \cap R$. □

Corollary 1. *Let R be a ring and σ an automorphism of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$, $S(P)$ is a completely prime ideal of $S(R)$.*

2. *For any completely prime ideal U of $S(R)$, $U \cap R$ is a completely prime ideal of R.*

Corollary 2. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is moreover a δ-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Proof. R is 2-primal by Theorem 4, and so by Proposition 3 $\delta(P) \subseteq P$. Further more as mentioned in Proposition 3 above, P is a completely prime ideal of R. Now use Proposition 4, and the proof is complete. □

We now prove the following Theorem, which is crucial in proving Theorem 6.

Theorem 5. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.*

Proof. Let $O(R)$ be 2-primal. Now by Corollary 2 $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in O(P(R))$. Now R is a 2-primal subring of $O(R)$ by Theorem 4, which implies that a_j is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $O(P(R)) = P(O(R))$.

Conversely suppose $O(P(R)) = P(O(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in O(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(a_n)a_n \in P(R) \subseteq P$, for all $P \in MinSpec(R)$. Now $\sigma(P) = P$ and since R is 2-primal by Theorem 4, therefore, P is completely prime. Therefore we have $a_n \in P$, for all $P \in MinSpec(R)$; i.e. $a_n \in P(R)$. Now since $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$, we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $a_{n-1} \in P(R)$. With the same process in a finite number of steps we get $a_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $(g(x)) \in O(P(R))$, i.e. $(g(x)) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal. □

Theorem 6. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal.*

Proof. Let $P_1 \in MinSpec(R)$. Then it is given that $\sigma(P_1) = P_1$, and therefore Theorem 3 implies that $O(P_1) \in MinSpec(O(R))$. Similarly for any $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ Theorem 3 implies that $P \cap R \in MinSpec(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using Theorem 5. □

Corollary 3. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal.*

Proof. Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Then as in the proof of Theorem 3 $\delta(P_1) \subseteq P_1$, and therefore $\delta(P(R)) \subseteq P(R)$. Now the rest is obvious using Theorem 6. □

Theorem 7. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and R is a δ-ring. Then $R[x;\sigma,\delta]$ is 2-primal Noetherian.*

Proof. We show that $\sigma(U) = U$ for all $U \in MinSpec(R)$. Suppose $U = U_1$ is a minimal prime ideal of R such that $\sigma(U) \neq U$. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of R. Now $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$. Now the rest is obvious. □

We now have the following question:

Question 2. *If R is a Noetherian $\mathbb{Q}$-algebra (even commutative), σ is an automorphism of R and δ is a σ-derivation of R. Is $O(R)$ 2-primal? The main problem is to get Theorem 5 satisfied.*

References

[1] Annin S. *Associated primes over skew polynomial rings.* Comm. Algebra, 2002, **30(5)**, 2511–2528.

[2] Argac N., Groenewald N.J. *A generalization of 2-primal near rings.* Quest. Math., 2004, **27(4)**, 397–413.

[3] Bhat V.K. *A note on Krull dimension of skew polynomial rings.* Lobachevskii J. Math., 2006, **22**, 3–6.

[4] Bhat V.K. *Associated prime ideals of skew polynomial rings.* Beitrage Algebra Geom., 2008, **49/1**, 277–283.

[5] Bhat V.K. *Differential operator rings over 2-primal rings.* Ukr. Math. Bull., 2008, **5(2)**, 153–158.

[6] Blair W.D., Small L.W. *Embedding differential and skew-polynomial rings into Artinain rings.* Proc. Amer. Math. Soc., 1990, **109(4)**, 881–886.

[7] Cohn P.M. *Difference Algebra.* Interscience Publishers, Acad. Press, New York-London-Sydney, 1965.

[8] Cohn P.M. *Free rings and their relations.* Acad. Press, London-New York, 1971.

[9] Gabriel P. *Representations des Algebres de Lie Resoulubles.* D Apres J. Dixmier. In Seminaire Bourbaki, 1968–1969, 1–22 (Lecture Notes in Math., 1971, No. 179, Berlin, Springer Verlag).

[10] Goodearl K.R., Warfield R.B., Jr. *An introduction to non-commutative Noetherian rings.* Cambridge Uni. Press, 1989.

[11] Hong C.Y., Kwak T.K. *On minimal strongly prime ideals.* Comm. Algebra, 2000, **28(10)**, 4868–4878.

[12] Hong C.Y., Kim N.K., Kwak T.K. *Ore-extensions of Baer and p.p.-rings*. J. Pure Appl. Algebra, 2000, **151(3)**, 215–226.

[13] Hong C.Y., Kim N.K., Kwak T.K. *On rings whose prime ideals are maximal*. Bull. Korean Math. Soc., 2000, **37(1)**, 1–9.

[14] Hong C.Y., Kim N.K., Kwak T.K., Lee Y. *On weak -regularity of rings whose prime ideals are maximal*. J. Pure Appl. Algebra, 2000, **146(1)**, 35–44.

[15] Kim N.K., Kwak T.K. *Minimal prime ideals in 2-primal rings.* Math. Japonica, 1999, **50(3)**, 415–420.

[16] Krempa J. *Some examples of reduced rings.* Algebra Colloq., 1996, **3(4)**, 289–300.

[17] Kwak T.K. *Prime radicals of skew-polynomial rings*. Int. J. Math. Sci., 2003, **2(2)**, 219–227.

[18] Marks G. *On 2-primal Ore extensions*. Comm. Algebra, 2001, **29(5)**, 2113–2123.

[19] McConnell J.T., Robson J.C. *Noncommutative Noetherian Rings*. Wiley, 1987; revised edition: American Math. Society 2001.

[20] Shin G.Y. *Prime ideals and sheaf representations of a pseudo symmetric ring*. Trans. Amer. Math. Soc., 1973, **184**, 43–60.

School of Applied Physics and Mathematics
SMVD University, P/o Kakryal, Katra, J and K
India-182320

E-mail: *vijaykumarbhat2000@yahoo.com*

Received August 9, 2007