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Classification of $GL(2,\mathbb{R})$-orbit's dimensions for the differential system with cubic nonlinearities

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Abstract. Center-affine invariant conditions for $GL(2,\mathbb{R})$-orbit's dimensions are defined for two-dimensional autonomous system of differential polynomial equations with cubic nonlinearities.

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Consider two-dimensional differential system with cubic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \quad (j,\alpha,\beta,\gamma = \overline{1,2}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma}$ is symmetrical in lower indices in which the complete convolution holds.

Consider also the group of center-affine transformations $GL(2,\mathbb{R})$ given by the equalities

$$\bar{x}^1 = \alpha x^1 + \beta x^2, \quad \bar{x}^2 = \gamma x^1 + \delta x^2, \quad \Delta = det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0.$$

Further will use the notations

$$a^1_1 = c, \quad a^1_2 = d, \quad a^2_1 = e, \quad a^2_2 = f, \quad a^1_{111} = p, \quad a^1_{112} = q, \quad a^1_{122} = r,$$

$$a^1_{222} = s, \quad a^2_{111} = t, \quad a^2_{112} = u, \quad a^2_{122} = v, \quad a^2_{222} = w, \quad x^1 = x, \quad x^2 = y. \tag{2}$$

According to [1] and taking into consideration (2) the representation operators of the group $GL(2,\mathbb{R})$ in the space of coefficients of the system (1) will take the form

$$D_1 = -d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} - s\frac{\partial}{\partial s} + 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v};$$

$$D_2 = -e\frac{\partial}{\partial c} + (c-f)\frac{\partial}{\partial d} + e\frac{\partial}{\partial f} - t\frac{\partial}{\partial p} + (p-u)\frac{\partial}{\partial q} +$$

$$+(2q-v)\frac{\partial}{\partial r} + (3r-w)\frac{\partial}{\partial s} + t\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v} + 3v\frac{\partial}{\partial w};$$

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$$D_3 = d\frac{\partial}{\partial c} + (f-c)\frac{\partial}{\partial e} - d\frac{\partial}{\partial f} + 3q\frac{\partial}{\partial p} + 2r\frac{\partial}{\partial q} +$$

$$+ s\frac{\partial}{\partial r} + (3u-p)\frac{\partial}{\partial t} + (2v-q)\frac{\partial}{\partial u} + (w-r)\frac{\partial}{\partial v} - s\frac{\partial}{\partial w};$$

$$D_4 = d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} + q\frac{\partial}{\partial q} + 2r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w}. \quad (3)$$

The operators (3) form a four-dimensional reductive Lie algebra [1].

Let $\tilde{a} = (c, d, ..., w) \in E^{12}(\tilde{a})$, where $E^{12}(\tilde{a})$ is the Euclidean space of the coefficients of the right-hand sides of the system (1). Denote by $\tilde{a}(q)$ the point from $E^{12}(\tilde{a})$ that corresponds to the system, obtained from the system (1) with coefficients $\tilde{a}$ by a transformation $q \in GL(2,\mathbb{R})$.

Definition 1. Call the set $O(\tilde{a}) = \{\tilde{a}(q) | q \in GL(2,\mathbb{R})\}$ the $GL(2,\mathbb{R})$-orbit of the point $\tilde{a}$ for the system (1).

Definition 2. Call the set $M \subseteq E^{12}(\tilde{a})$ the $GL(2,\mathbb{R})$-invariant if for any point $\tilde{a} \in M$ its orbit $O(\tilde{a}) \subseteq M$.

It is known from [1] that

$$dim_{\mathbb{R}} O(\tilde{a}) = rank M_1, \quad (4)$$

where M_1 is the following matrix

$$M_1 = \begin{pmatrix} 0 & -d & e & 0 & 2p & q & 0 & -s & 3t & 2u & v & 0 \\ -e & c-f & 0 & e & -t & p-u & 2q-v & 3r-w & 0 & t & 2u & 3v \\ d & 0 & f-c & -d & 3q & 2r & s & 0 & 3u-p & 2v-q & w-r & -s \\ 0 & d & -e & 0 & 0 & q & 2r & 3s & -t & 0 & v & 2w \end{pmatrix}, \quad (5)$$

constructed on coordinate vectors of operators (3).

The following comitants and invariants for system (1) are known from [2]

$$P_1 = a^{\alpha}_{\alpha\beta\gamma} x^{\beta} x^{\gamma}, \quad P_2 = a^{p}_{\alpha\beta\gamma} x^{\alpha} x^{\beta} x^{\gamma} x^{q} \varepsilon_{pq},$$

$$P_3 = a^{\alpha}_{p\alpha\beta} a^{\beta}_{q\gamma\delta} x^{\gamma} x^{\delta} \varepsilon^{pq}, \quad P_4 = a^{\alpha}_{\alpha\beta\gamma} a^{\beta}_{\delta\mu\theta} x^{\gamma} x^{\delta} x^{\mu} x^{\theta}, \quad P_5 = a^{\alpha}_{\beta\gamma\delta} a^{\beta}_{\alpha\mu\theta} x^{\gamma} x^{\delta} x^{\mu} x^{\theta},$$

$$P_6 = a^{\alpha}_{\alpha pr} a^{\beta}_{\gamma\delta q} a^{\gamma}_{\beta\nu s} x^{\delta} x^{\nu} \varepsilon^{pq} \varepsilon^{rs}, \quad Q_1 = a^{p}_{\alpha} a^{q}_{\beta\gamma\delta} x^{\alpha} x^{\beta} x^{\gamma} x^{\delta} \varepsilon_{pq},$$

$$Q_2 = a^{\alpha}_{\beta} a^{\beta}_{\alpha\gamma\delta} x^{\gamma} x^{\delta}, \; Q_3 = a^{\alpha}_{\gamma} a^{\beta}_{\alpha\beta\delta} x^{\gamma} x^{\delta}, \quad Q_7 = a^{\alpha}_{\beta} a^{\beta}_{p\alpha\gamma} a^{\gamma}_{q\eta\mu} x^{\eta} x^{\mu} \varepsilon^{pq},$$

$$K_2 = a^{\alpha}_{\beta} x^{\beta} x^{\gamma} \varepsilon_{\alpha\gamma}, \quad I_1 = a^{\alpha}_{\alpha}, \quad I_2 = a^{\alpha}_{\beta} a^{\beta}_{\alpha},$$

$$J_1 = a^{\alpha}_{\alpha pr} a^{\beta}_{\beta qs} \varepsilon^{pq} \varepsilon^{rs}, \quad J_2 = a^{\alpha}_{\beta pr} a^{\beta}_{\alpha qs} \varepsilon^{pq} \varepsilon^{rs}, \quad J_4 = a^{\alpha}_{pru} a^{\beta}_{\gamma qs} a^{\gamma}_{\alpha\beta v} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{uv}. \quad (6)$$

Theorem 1. *The dimension of $GL(2,\mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $K_2P_1P_2[K_2^2(2P_1^2-6P_4+4P_5)+2K_2P_2(Q_3-Q_2)+$
$+2P_1P_2(4Q_1-3K_2P_1)+2P_2^2(2Q_3+J_1P_1)]\not\equiv 0,$ *or*
$P_1P_2(3P_1P_3-2J_1P_2)\not\equiv 0,\ K_2\equiv 0,$ *or*
$P_2(J_2P_5-J_4P_2)\not\equiv 0,\ K_2\equiv P_1\equiv 0,$ *or*
$K_2[P_2^2(I_1^2-I_2)+2Q_1^2+3K_2^2P_5+2P_2Q_1I_1]\not\equiv 0,\ P_1\equiv 0,$ *or*
$K_2P_1Q_7\not\equiv 0,\ P_2\equiv 0$ *;*

3 for $K_2P_1P_2\not\equiv 0,\ K_2^2(2P_1^2-6P_4+4P_5)+2K_2P_2(Q_3-Q_2)+$
$+2P_1P_2(4Q_1-3K_2P_1)+2P_2^2(2Q_3+J_1P_1)]\equiv 0,$ *or*
$P_2^2(I_1^2-I_2)+2Q_1^2+3K_2^2P_5+2P_2Q_1I_1\equiv P_1\equiv 0,\ K_2P_2\not\equiv 0,$ *or*
$J_2P_5-J_4P_2\equiv K_2\equiv P_1\equiv 0,\ P_2P_5\not\equiv 0,$ *or*
$P_1P_2\not\equiv 0,\ 3P_1P_3-2J_1P_2\equiv K_2\equiv 0,$ *or*
$P_2\equiv K_2\equiv 0,\ J_1\neq 0,$ *or*
$P_2\equiv Q_7\equiv 0,\ K_2P_1(P_1Q_1+P_6)\not\equiv 0;$

2 for $P_2\equiv Q_7\equiv P_1Q_1+P_6\equiv 0,\ P_1^2+K_2^2\not\equiv 0,$ *or*
$P_2\equiv P_1\equiv 0,\ K_2\not\equiv 0,$ *or*
$P_2\not\equiv 0,\ P_1\equiv P_5\equiv K_2\equiv J_2P_5-J_4P_2\equiv 0,$ *or*
$P_2\equiv K_2\equiv 0,\ J_1=0,\ P_1\not\equiv 0;$

0 for $P_1\equiv P_2\equiv K_2\equiv 0,$

where $P_1,\ P_2,\ P_3,\ P_4,\ P_5,\ P_6,\ K_2,\ Q_1,\ Q_2,\ Q_3,\ Q_7,\ I_1,\ I_2,\ J_1,\ J_2,\ J_4$ *are from* (6).

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