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Differentially prime, quasi-prime and $\Delta - MP$-modules

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Abstract. The notions of differentially prime, quasi-prime and $\Delta - MP$-modules introduced by the author are investigated. The results obtained are module analogues of the well-known facts on differentially prime and quasi-prime ideals of differential rings. The principal constructions are based on the concept of relative multiplicatively closed subset of the differential module.

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Throughout the paper, rings are assumed to be associative and differential with nonzero identity. The differential structure on the ring R is defined by the set $\Delta = \{\delta_1, \ldots, \delta_n\}$, comprising n pairwise commutative derivations of R. If the contrary is not specifically stated, at least one of the derivations from Δ is nontrivial.

All modules are assumed to be unitary left and differential with the set of structure derivations $D = \{d_1, \ldots, d_n\}$, where the derivations are pairwise commutative and agree with the corresponding derivations from the set Δ. We will use the terminology on differential algebra as in [1].

Remind that a nonempty subset S of the differential ring R is called a dm-system if for each $a, b \in S$ there exists $r \in R$ and $n \in \mathbb{N}$ such that $arb^{(n)} \in S$ [2]. If S is an arbitrary dm-system of the differential ring R and M is the left differential R-module, then the module analogue for the dm-system is defined in the natural way. Namely, a nonempty subset X of the differential module M will be called an *Sdm-system* of M if for each $s \in S$ and $x \in X$ there exists $r \in R$ and $n \in \mathbb{N}$ such that $srx^{(n)} \in X$. It follows that for every $s \in S$ and $x \in X$ the intersection $sR[x] \cap X$ is nonempty, where $[x]$ is a differential submodule of M generated by x. If, in addition, the condition $srx \in X$ implies $s \in S$ and $x \in X$ (for $s \in R$, $r \in R$ and $x \in M$) than the Sdm-system X will be called *saturated*. In the case of regular module $_RR$, these sets coincide with the classical multiplicatively closed sets of commutative rings and with m-systems of noncommutative rings. Note that, for modules over commutative rings without derivations, similar notions were introduced in [3].

A differential module M will be called *differentially prime* if the left annihilator of every its nonzero differential submodule coincides with the annihilator of the whole module. In other words, the left differentially prime differential module is the prime module over the ring of linear differential operators with the coefficients in the ring R. Note that prime modules were first studied by V. A. Andrunakievich in [4]. A differential submodule N of the left differential module M will be called *differentially prime* if the differential module M/N is differentially prime. If all the structure derivations of the ring R are trivial, then these notions give us the well-known notions of prime module and prime submodule of a given module (see, for example [3]).

Every differentially prime ideal is a differentially prime submodule of R. Every prime differential ideal is differentially prime as submodule. Every prime differential submodule is differentially prime. In particular, every maximal differential submodule is differentially prime. Factor modules of the differentially prime modules are differentially prime.

Proposition 1. *A differential submodule N of the differential module M is differentially prime if and only if its complement $M \backslash N$ is an Sdm-system in M for some dm-system S of ring R.*

Proof. Suppose that N is a differentially prime submodule of M. Then the differential module M/N is differentially prime. Let $0 \neq m \in M/N$. Owing to its differential primeness we have that $\wp = \mathrm{Ann}_l([m]) = \mathrm{Ann}_l(M)$ is a differentially prime ideal of R. Indeed, if it is not, $\wp \supseteq \mathcal{AB}$ for some differential ideals $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A} \nsubseteq \wp$ and $\mathcal{B} \nsubseteq \wp$. It follows that $\mathcal{B}(M/N) \neq 0$, but $\mathcal{AB}(M/N) = 0$. Hence $\mathcal{A} = \wp$, because of the primeness of M/N. Then the set $S = R \backslash \wp$ is a dm-system according to the result of [2]. Now show that the set $X = M \backslash N$ is an Sdm-system. Let $s \in S$ and $x \in X$, such that an element $srx^{(n)}$ does not belong to X for every $r \in R$ and every $n \in \mathbb{N}$. Then $srx^{(n)} \in N$ for any $r \in R$ and any $n \in \mathbb{N}$. It means that $s \in \mathrm{Ann}_l([x]) = \mathrm{Ann}_l(M/N)$, that contradicts $s \in S$. □

A differential submodule N of the left differential module M will be called *quasi-prime* if it is maximal among differential submodules of M disjoint from some Sm-system X of M. Every prime submodule which is differential is quasi-prime, since the complement of the prime submodule is an Sm-system, where the role of S is played by the set $\{1\}$. In the case of regular module, we obtain the notion of quasi-prime ideal investigated by Keigher in commutative case [5] and Verschoren in associative rings [6]. For differential ideals it is known that every maximal among differential ideals not meeting some m-system is quasi-prime. The following proposition establishes the analogue of this fact for differential modules:

Proposition 2. *Every maximal among differential submodules of an arbitrary differential module is quasi-prime.*

Proof. Let N be a maximal amongst differential submodules of M, $S = U(R)$ be the group of units of the ring R and $X = M \backslash N$. Then X is an Sm-system and N is a maximal amongst differential submodules disjoint from X. Hence N is a quasi-prime submodule. □

By analogy with the differentially prime radical of the differential ideal, it is natural to introduce the notion of *differentially prime radical* of the differential submodule of an arbitrary left differential module. Assume that $N_\Delta(K) = M$ if

K contains no differentially prime submodules of M. The submodule $N_{\Delta}(K)$ is a differential submodule of M.

Theorem 1. *For every differential submodule N of the differentially noetherian module M the following are equivalent:*

1. N is a differentially prime submodule;

2. N is a quasi-prime submodule;

3. $N = P_{\sharp}$ for some prime submodule P of M, where $P_{\sharp} = \{x \in M | x^{(i_1,i_2,\dots,i_n)} \in P\, for\, all\, i_1, i_2, ..., i_n, n \in \{0, 1, 2, ...\}\}$

Proof. 1) ⇒ 2). Let N be some differentially prime submodule of M. Then, by Proposition 1, the set $M \setminus N$ is a Sdm-system for some dm-system S of the ring R. Since N is maximally differential submodule disjoint from $M \setminus N$, then, by definition, it is quasi-prime.

2) ⇒ 3). Let N be a quasi-prime submodule of M, i. e. maximal among differential submodules disjoint from the Sdm-system X, and let K be maximal among ordinary submodules disjoint from X and containing N. Then K is prime submodule in M as each Sdm-system is Sm-system. Show that $N = K_{\sharp}$. Since N is differential submodule of M and , then $N \subseteq K_{\sharp}$. The converse inclusion implies due to maximality of the differential submodule N among those disjoint from X, because $K_{\sharp}$ is disjoint from X and it is differential submodule of M.

3) ⇒ 1). Let $N = P_{\sharp}$, for some prime submodule P of M. Then N is maximal amongst differential submodules of M contained in P. Let $T = M \setminus P$. By Proposition 1, assuming that all the derivations of the set Δ are trivial, we see that T is a Sm-system for some m-system of the ring R. Denote by K the intersection of all Sdm-systems of the module M, which contain T. Then K is the least Sdm-system of those containing T. Hence $M \setminus K$ is a differentially prime submodule of M because of 1. It remains to verify that $M \setminus K = N$. Since $(M \setminus K) \cap T = \varnothing$, then $N \setminus K \subseteq P$, and due to the fact that $M \setminus K$ is a differential submodule of M, we have the inclusion $M \setminus K \subseteq N$. Taking into consideration the minimality of the set K, we obtain that the set $M \setminus K$ is a maximal submodule among the differential submodules of N. Hereby $M \setminus K = N$. □

The intersection of all the differentially prime submodules of M containing N will be called a *differentially prime radical* of the differential submodule N of the differential module M.

Denote the radical by $r_d(N)$. It is evident that $N \subseteq r_d(M) \subseteq M$.

Proposition 3. *Every proper differential submodule of the differentially finitely generated left differential module has a maximal submodule. In particular every submodule of such module has its differentially prime radical.*

Proof. The proof of this fact is based upon the considerations adapted from ([7], Lemma 6.8.) for the case of differential modules. □

The following assertion is a module analogue of the appropriate statement from [8].

Theorem 2. *The following conditions are equivalent:*

(1) Every quasi-prime submodule N of the differential R-module M is prime.

(2) Every quasi-prime submodule N of the differential R-module M is radical.

(3) Every prime submodule N minimal over some differential submodule is differential submodule.

(4) The radical of any differential submodule is a differential submodule.

(5) For every prime submodule N of the module M the submodule $N_\sharp$ is prime.

The module will be called a $\Delta - MP$*-module* if it satisfies the equivalent conditions of the Theorem 7.

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