BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(58), 2008, Pages 108–111
ISSN 1024–7696

Four-dimensional Ricci-flat space defined by the KP-equation

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Abstract. Four-dimensional affinely connected Ricci-flat space depending of solutions of the Kadomtsev-Petviashvili equation is constructed. Conditions of metrizabilty of corresponding connection are investigated. Its properties are discussed.

Mathematics subject classification: 34C14.
Keywords and phrases: Riemann extensions, geodesic equations, Kadomtzev-Petviashvili equation.

1 The Ricci-flat 8-dim metrics

The notion of Riemann extensions of affinely-connected spaces plays an important role in differential geometry.

Here we give an example of four-dimensional Ricci-flat affinely connected space E^4 defined by solutions of the KP-equation which has the Ricci-flat Riemann extension also depending on solutions of the KP-equation.

Proposition 1. *Eight-dimensional Riemann space in local coordinates* (x, y, z, t, P, Q, U, V) *equipped with the metric*

$$\begin{aligned} ds^2 = &\left(-2\,PH11 - 2\,P\frac{\partial}{\partial t}F - 2\,H12Q - 2\Gamma^3_{11}\,U + 2\,H22\,V\right) dx^2 + \\ &+2\,\left(2\,H11(y,z,t)Q - 2\,H21V\right) dx\,dy + \\ &+2\left(-2\left(\frac{\partial}{\partial z}F\right)V + 2\left(\frac{\partial}{\partial t}F\right)U\right) dx\,dz + \\ &+2\,dx\,dP + \left(2\,H11U - 2\,H31V\right) dy^2 + 2\,dy\,dQ + 2\,dz\,dU + 2\,dt\,dV \end{aligned} \tag{1}$$

is Ricci-flat $R_{ij} = 0$ *if the conditions on the coefficients* $Hij(y,z,t)$, $F(y,z,t)$, $\Gamma^3_{11}(y,z,t)$

$$\begin{aligned} &\frac{\partial}{\partial y}H12\,(y,z,t) - \frac{\partial}{\partial t}H22\,(y,z,t) = 0, \\ &-\frac{\partial}{\partial y}H11\,(y,z,t) + \frac{\partial}{\partial t}H21\,(y,z,t) = 0, \\ &-\frac{\partial}{\partial z}H11\,(y,z,t) + \frac{\partial}{\partial t}H31\,(y,z,t) = 0 \end{aligned} \tag{2}$$

are valid.

Arbitrary functions $\Gamma^3_{11}(y,z,t)$ *and* $F(y,z,t)$ *satisfy the relation*

$$\frac{\partial}{\partial z}\Gamma^3_{11}(y,z,t) = 2\left(\frac{\partial}{\partial t}F(y,z,t)\right)^2 + 2\,H11(y,z,t)\frac{\partial}{\partial t}F(y,z,t) + 2\,(H11(y,z,t))^2$$

or

$$\Gamma^3_{11}(y,z,t) = \int 2\left(\frac{\partial}{\partial t}F(y,z,t)\right)^2 + 2\,H11(y,z,t)\frac{\partial}{\partial t}F(y,z,t) + +2\,(H11(y,z,t))^2\,dz + \_F1(y,t). \tag{3}$$

The metric (1) is an example of the Riemann extension of affinely-connected four-dimensional space with symmetrical connection $\Gamma^i_{jk}(x^l) = \Gamma^i_{kj}(x^l)$ depending of the local coordinates x^i.

In general case it is defined by the expression

$$ds^2 = -2\Gamma^i_{jk}\xi_i dx^j dx^k + 2d\xi_k dx^k, \tag{4}$$

where ξ_k are additional coordinates [1].

After the substitutions of the form

$$H11\,(y,z,t) = -1/2\,u\,(y,z,t)\,, \quad H12\,(y,z,t) = -1/3\,v\,(y,z,t)\,, \tag{5}$$

$$H21\,(y,z,t) = -2/3\,v\,(y,z,t) - 1/2\,\frac{\partial}{\partial t}u\,(y,z,t)\,,$$

$$H31\,(y,z,t) = -3/4\,w\,(y,z,t) + 3/8\,(u\,(y,z,t))^2 - \frac{\partial}{\partial t}v\,(y,z,t) - 1/2\,\frac{\partial^2}{\partial t^2}u\,(y,z,t)\,,$$

$$H22\,(y,z,t) = -1/2\,w\,(y,z,t) + 1/2\,(u\,(y,z,t))^2 - \frac{\partial}{\partial t}v\,(y,z,t) -$$

$$-1/2\,\frac{\partial^2}{\partial t^2}u\,(y,z,t) + 1/2\,\frac{\partial}{\partial y}u\,(y,z,t)$$

from the conditions (2) the famous KP-equation follows:

$$\frac{\partial}{\partial t}\left(\frac{\partial u\,(y,z,t)}{\partial z} - 3/2\,u\,(y,z,t)\,\frac{\partial}{\partial t}u\,(y,z,t) - 1/4\,\frac{\partial^3}{\partial t^3}u\,(y,z,t)\right) = 3/4\,\frac{\partial^2}{\partial y^2}u\,(y,z,t)\,.$$

Remark 1. The notion of the Riemann extensions of affinely-connected spaces (see [1]) was used by author for the study of geometrical problems in the theory of nonlinear dynamical systems and in General Relativity [2]–[5].

Remark 2. About presentation of the KP-equation in the form of system (2) see [6].

2 Four-dimensional subspace

Eight-dimensional metrics (4) is closely related with properties of four-dimensional spaces in local coordinates x^k.

Let us consider an example.

The full system of geodesics of the metric (4) decomposes into two parts.

The first part has the form of the linear system of equations for the coordinates ($\xi_k = P, Q, U, V$)

$$\frac{d^2\xi_k}{ds^2} + A(x^i)\frac{d\xi_k}{ds} + B(x^i)\xi_k = 0$$

and the second part for local coordinates $x^i = (x, y, z, t)$ is defined by the system of equations

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

In our case it takes the form

$$\begin{gathered}
\frac{d^2}{ds^2}x(s) + \left(\frac{d}{ds}x(s)\right)^2 H11(y,z,t) + \left(\frac{d}{ds}x(s)\right)^2 \frac{\partial}{\partial t}F(y,z,t) = 0,\\
\frac{d^2}{ds^2}y(s) + H12(y,z,t)\left(\frac{d}{ds}x(s)\right)^2 - 2\,H11(y,z,t)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) = 0,\\
\frac{d^2}{ds^2}z(s) + \Gamma^3_{11}(y,z,t)\left(\frac{d}{ds}x(s)\right)^2 - 2\left(\frac{\partial}{\partial t}F(y,z,t)\right)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s) -\\
-H11(y,z,t)\left(\frac{d}{ds}y(s)\right)^2 = 0,\\
\frac{d^2}{ds^2}t(s) - H22(y,z,t)\left(\frac{d}{ds}x(s)\right)^2 + 2\,H21(y,z,t)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) +\\
+2\left(\frac{\partial}{\partial z}F(y,z,t)\right)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s) + H31(y,z,t)\left(\frac{d}{ds}y(s)\right)^2 = 0.
\end{gathered} \tag{6}$$

From these relations we find the coefficients of affine connection Γ^i_{jk} of the 4-dimensional subspace in local coordinates ($x^i = x, y, z, t$)

$$\begin{gathered}
\Gamma^1_{11} = H11 + \frac{\partial F}{\partial t}, \quad \Gamma^2_{11} = H12, \quad \Gamma^2_{12} = -H11,\\
\Gamma^3_{11} = \Gamma^3_{11}, \quad \Gamma^3_{13} = -\frac{\partial F}{\partial t}, \quad \Gamma^3_{22} = -H11,\\
\Gamma^4_{11} = -H22, \quad \Gamma^4_{12} = H21, \quad \Gamma^4_{13} = \frac{\partial F}{\partial z}, \quad \Gamma^4_{22} = H31.
\end{gathered} \tag{7}$$

The corresponding Ricci tensor

$$R_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{im}\Gamma^m_{kj}$$

is symmetrical $R_{ij} = R_{ji}$ and it is equal to zero $R_{ik} = 0$ if the conditions (2), (3) hold.

Corresponding equations for the coordinates U, P, Q, V have more cumbersome form and may be omitted here.

The problem of metrizability of the connection Γ^i_{jk} defined by the expressions (7) is of great importance.

It is known that in order to obtain the metric tensor g_{ij} corresponding to the connection Γ^i_{jk} we have to solve the system of equations

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} = 0.$$

A necessary and sufficient condition of integrability of this system implies the study of the properties of the set of equations

$$R^k_{ijl} g_{km} + R^k_{ijm} g_{kl} = 0, \quad \nabla_s R^k_{ijl} g_{km} + \nabla_s R^k_{ijm} g_{kl} = 0, ...$$

where R^k_{ijl} are the components of Riemann tensor of the connection (7).

In the case being considered the number of components of the Riemann tensor is thirty-six and affine connection (7) obeys the condition $\Gamma^i_{ki} = 0$.

On this basis and taking in consideration the relation $\Gamma^i_{ki} = \dfrac{1}{2g}\dfrac{\partial g}{\partial x^k}$ it is follows that in the case of metrizability of the connection (7) the determinant of corresponding metric tensor must be constant.

3 Acknowledgements

The research of author was partially supported by Grant CSTD AS Moldova and Grant RFBR.

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Received October 15, 2008