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A virtual analog of Pollaczek-Khintchin transform equation *

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Abstract. The virtual queue length distribution for the queueing system $M|G|1$ is obtained. It is shown that these results can be viewed as generalization of Pollaczek-Khintchin transform equation.

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1 Introduction

The queueing system $M|G|1$ plays an important role in Queueing Analysis. This system is studied and described in most standard textbooks and monographs on Queueing Theory (see, for example, Allen 1978 [1], Kleinrok 1975 [2], Cooper 1981 [3], Takagi 1991 [4], Gnedenko and Kovalenko 2005 [5]). Various methods and techniques necessary for the evaluation of its characteristics have been developed. Many characteristics were obtained by pioneers and founders of Queueing Theory. Among such characteristics one can mention the stationary distribution of the number of messages in the system, or, in other words, the queue length distribution, first obtained by Pollaczek in 1961 [6] and independently by Khintchin in 1963 [7]. It is necessary to mention that many outstanding researchers repeated this classical result using new elaborated methods and approaches and referred to it in their papers and books. Although the $M|G|1$ system is well studied the impetuous development of contemporary technologies puts forward new problems requesting new approaches and results. Thus, it turned out that some results for queueing system $M|G|1$ can be used for the analysis of polling systems: mathematical models used as a theoretical approach for broad band WLAN (Wireless Local Area Networks). In this paper the queue length distribution for an arbitrary time t ($t \in (0, \infty)$) is obtained for mentioned $M|G|1$ system. In other words, the nonstationary or virtual distribution of the number of the messages in the system is obtained. We show below that this virtual distribution contains as a particular case, namely in the steady state, the mentioned Pollaczek-Khintchin transform equation. In this context we can consider the distribution obtained below as an analog of a well-known Pollaczek-Khintchin equation. Results were obtained using the method of "catastrophes" (Gnedenco et

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all 1973 [8], Klimov and Mishkoy 1979 [9]) and the approach based on regenerative processes with embedded periods.

2 Preliminary results. Pollaczek-Khintchin transform equation

Let's consider the queueing system $M|G|1$ with exhaustive service (messages are served continuously until there is a nonmessage in the system). Denote by λ the parameter of input Poisson flow, by B the length of service and by $B(x) = P\{B < x\}$ the distribution function of service. Let's also denote by $\beta(s) = \int_0^\infty e^{-sx} dB(x)$ the Laplace-Stieltjes transform of function $B(x)$, by $\beta_1 = \int_0^\infty x dB(x)$ its first moment.

Let's consider the random variable X – the number of messages in the queue. Obviously, X is a discrete random variable. Denote by P_k distribution of this variable

$$P_k = \{X = k\}$$

and let $P(z)$ be the generating function of probabilities P_k,

$$P(z) = \sum_k z^k P_k$$

where $0 \le z \le 1$.

The traffic intensity ρ is defined as follows

$$\rho = \frac{E(B)}{E(z_k)}$$

where

$$E(B) = \beta_1 = \int_0^\infty x dB(x), \quad B(x) = P\{B < x\},$$

$$E(z_k) = \int_0^\infty x dA(x), \quad A(x) = P\{z_k < x\} = 1 - e^{-\lambda x}.$$

Here z_k is the interarrival interval between t_{k-1} and t_k time moments. Using the traffic intensity definition for $M|G|1$ system we get $\rho = \lambda\beta_1$.

The following result is known as Pollaczek-Khintchin transform equation [6,7].

Theorem 1. *If $\rho < 1$, then the steady state generating function of the queue length distribution is given by expresion*

$$P(z) = \frac{\beta(\lambda - \lambda z)(z-1)(1-\lambda\beta_1)}{z - \beta(\lambda - \lambda z)}. \tag{1}$$

The proof of the formula (1) can be obtained employing the method of the embedded Markov chain and notion of irreductibility, aperidiocity and ergodicity of the Markov chains (Asumussen 1987 [10], Cohen 1982 [11], Takagi 1991 [4]).

Remark 1. From expression (1) we easily get the mean value formula

$$N = \sum_{k} kP_k = P^{'}(1) = \lambda\beta_1 + \frac{\lambda^2\beta_1}{2(1-\lambda\beta_1)}. \quad (2)$$

Consider the busy period. By the busy period we shall understand the time interval beginning with the arrival message in the free $M|G|1$ system and finishing with the next moment when the system becomes free. Obviously, the busy period is a random variable. Let's denote by $\Pi(x)$ the distribution function of the busy period, by $\pi(s) = \int\limits_0^\infty e^{-sx} d\Pi(x)$ – the Laplace-Stieltjes transform of $\Pi(x)$ and by $\pi_1 = \int\limits_0^\infty x d\Pi(x)$ – the first moment.

The distribution function of the busy period is given (in terms of Laplace-Stieltjes transform) by the following theorem.

Theorem 2. *The Laplace-Stieltjes transform $\pi(s)$ of the busy period is determined in the unique way from functional equation*

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \quad (3)$$

If $\rho < 1$, then

$$\pi_1 = \frac{\beta_1}{1-\lambda\beta_1}. \quad (4)$$

The functional equation (3) is known as Kendall-Takacs functional equation (Kendall [12], Takacs [13]). The multidimensional analog of the mentioned equation is presented in (Mishkoy 2006 [14] and Mishkoy 2007 [15,16]).

3 Nonstationary (virtual) analog of transform equation

According to the $M|G|1$ system we assume that there is an infinite buffer to store waiting messages and exhaustive service, those messages are served continuously until there is no message in the system. By the service discipline we assume LIFO (last in, first out; or reverse order of arrival) or FIFO (first in, first out, or order of arrival). For these disciplines, the order of service is not affected by the service time of waiting messages. The first objective of our analysis in this section is to find the nonsteady state distribution of the queue size, that is the number of messages present in the system at the moment of time t.

Denote by $P_m(t)$ probability that at the instant t there are m messages in the system, by $P(z,t)$ the generating function of probabilities $P_m(t)$,

$$P(z,t) = \sum_{m \geq o} P_m(t) z^m, \quad 0 \leq z \leq 1$$

and by

$$p(z,s)=\int\limits_0^\infty e^{-st}P(z,t)dt \tag{5}$$

its Laplace transform.

We shall assume that independently of the evolution of the system some events, called "catastrophes", which form a Poisson flow with parameter $s>0$ happen. We also assume that an arbitrary message will be coloured either in red with probability z or in blue with probability $1-z$, independently of the colour of the other messages. We shall multiply the both parts of the expression (5) by s. Then

$$sp(z,s)=s\int\limits_0^\infty e^{-st}P(z,t)dt$$

is the probability that the first "catastrophe" happens at the moment of time t when in the queueing system there are at least red messages. The profit from this probability means that we shall obtain the formulas to determine the function $p(z,s)$. We shall denote in addition by

$s\pi(z,s)$ – the probability that the first "catastrophe" happened during the busy period when in the queueing system there are at least red messages;

$s\beta(z,s)$ – the probability that the first "catastrophe" happened during a messages service time B when in the queueing system there are at least red messages.

Let's suppose that at any moment t there are n messages in the queue. The interval of time which starts with the service of one of the mentioned n messages and finishes as soon as the system becomes free, will be called Π^n–period. We shall denote the distribution function of this period by $\Pi^n(t)$ and its Laplace-Stieltjes transform by $\pi^n(s)$.

Obviously, $\pi^n(s)=[\pi(s)]^n$, where $\pi(s)$ is determined from the functional equation (3).

Let's consider a separate Π^n–period. Let's denote by $\overline{P}_m(t)$ the probability that at instant t in the queueing therre are m messages. Denote

$$\Pi^n(z,t)=\sum_{m\geq 0}\overline{P}_m(t)z^m, \quad 0\leq z\leq 1,$$

and let

$$\pi^n(z,s)=\int\limits_0^\infty e^{-st}\Pi^n(z,t)dt$$

be the Laplace transform of generating function $\Pi^n(z,t)$.

Similarly as we did above we conclude

$s\pi^n(z,s)$ – is the probability that the first "catastrophe" happens during a Π^n–period when in the queueing system there are at least red messages.

Then the following auxiliary result holds.

Lemma 1. *Laplace transform of the generating function of the queue length distribution on Π^n–period is given by*

$$\pi^n(z,s) = \pi(z,s)\frac{z^n - [\pi(s)]^n}{z - \pi(s)}$$

where $\pi(z,s)$ is Laplace transform of the generating function of the queue length distribution on the busy period which will be obtained below; function $\pi(s)$ is determined from (3).

Proof. First let's prove that

$$s\pi^n(z,s) = s\pi(z,s)z^{n-1} + \pi(s)s\pi(z,s)z^{n-2} + \tag{6}$$

$$+ \cdots + [\pi(s)]^{n-1}s\pi(z,s)$$

Really, assume that the first "catastrophe" happens during a Π^n–period when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$). For this it is necessary and sufficient that either the first "catastrophe" happen during the associated busy period with the first n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), the remaining $n-1$ messages are red (the probability of this event is z^{n-1});

or the first "catastrophe" happen during the associated busy period with the second n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), during the associated busy period with the first message "catastrophe" does not happen (the probability of this event is $\pi(s)$), the remaining $n-2$ messages are red (the probability of this event is z^{n-2}), etc. . . ,

or the first "catastrophe" happen during the associated busy period with the last n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), and the "catastrophe" does not happen during the associated busy period with $n-1$ initial messages (the probability of this event is $[\pi(s)]^{n-1}$).

We shall rewrite the expression (6) in the following way

$$s\pi^n(z,s) = s\pi(z,s)\{z^{n-1} + \pi(s)z^{n-2} + \cdots + [\pi(s)]^{n-1}\}$$

or

$$s\pi^n(z,s) = s\pi(z,s)\frac{z^n - [\pi(s)]^n}{z - \pi(s)}$$

and after reducing s we obtain the proof of Lemma 1. □

The following Lemma 2 gives the distribution of the queue size on the separate busy period.

Lemma 2. *The Laplace transform of the generating function of the queue size distribution on the busy period $\pi(z,s)$ is given by*

$$\pi(z,s)=\beta(z,s)\frac{z-\pi(s)}{z-\beta(s+\lambda-\lambda z)} \tag{7}$$

where $\pi(s)$ is determined from the functional equation (3), $\beta(s+\lambda-\lambda z)$ is the Laplace-Stieltjes transform of function $B(t)$ at point $s=s+\lambda-\lambda z$, $\beta(z,s)$ will be given below.

Proof. First we prove

$$s\pi(z,s)=s\beta(z,s)+\sum_{n\geq 1}s\pi^n(z,s)\int\limits_0^\infty e^{-st}\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}dB(t). \tag{8}$$

Really, let's suppose that the first "catastrophe" happens on the separate busy period when in the system there are at least red messages. As we have mentioned the probability of this event is $s\pi(z,s)$.

For this it is necessary and sufficient that either the first "catastrophe" happen during the service of the message that opens the busy period, when in the system there are at least red messages (this probability is $s\beta(z,s)$); or "catastrophe" does not happen during the service time of this message (the probability is e^{-st}), messages arrive $n\geq 1$ (the probability of this event is $\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}$) and the first "catastrophe" happens in the Π^n-period, when in the system there are at least red messages (the probability of this event is $s\pi^n(z,s)$).

Let's us denote by $\sum$ the second term in (8) and using Lemma 1 we have

$$\sum=\sum_{n\geq 1}\pi^n(z,s)\int\limits_0^\infty e^{-st}\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}dB(t)=$$

$$=\frac{\pi(z,s)}{z-\pi(s)}\sum_{n\geq 1}\int\limits_0^\infty[\frac{(\lambda zt)^n}{n!}-\frac{(\lambda\pi(s)t)^n}{n!}]e^{-(-s+\lambda)t}dB(t).$$

Observe that for $n=0$ the sum is equal to 0, hense letting $n\geq 1$ we obtain

$$\sum=\frac{\pi(z,s)}{z-\pi(s)}\int\limits_0^\infty[e^{\lambda zt}-e^{\lambda\pi(s)t}]e^{-(-s+\lambda)t}dB(t)=$$

$$=\frac{\pi(z,s)}{z-\pi(s)}[\beta(s+\lambda-\lambda z)-\beta(s+\lambda-\lambda\pi(s))].$$

According to Theorem 3 since $\pi(s)=\beta(s+\lambda-\lambda\pi(s))$ we receive

$$\sum=\frac{\pi(z,s)}{z-\pi(s)}[\beta(s+\lambda-\lambda z)-\pi(s)].$$

Substituting the obtained formula in (8) and reducing s we have

$$\pi(z,s) = \beta(z,s) + \frac{\pi(z,s)}{z-\pi(s)}[\beta(s+\lambda-\lambda z) - \pi(s)]$$

hence

$$\pi(z,s)\{1 - \frac{\beta(s+\lambda-\lambda z) - \pi(s)}{z-\pi(s)}\} = \beta(z,s)$$

or

$$\pi(z,s) = \frac{z-\beta(s+\lambda-\lambda z)}{z-\pi(s)}\beta(z,s).$$

This completes the proof of Lemma 2. □

Lemma 3. *The Laplace transform of generating function of the queue size distribution on service time B is given by*

$$\beta(z,s) = z\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \tag{9}$$

Proof. First we give the proof of equality

$$s\beta(z,s) = z\int_0^\infty [1-B(x)]se^{-st}e^{-\lambda(1-z)t}dt. \tag{10}$$

Really, let's suppose that the first "catastrophe" happened during service time B. Probability of this event is $s\beta(z,s)$. For this it is necessary and sufficient that the first "catastrophe" happen at the moment t (the probability of this event is se^{-st}), when the service is not finished yet (the probability of this event is $1-B(t)$) until the happened "catastrophe" does not arrive no red messages into the system (the probability of this event is $e^{-\lambda(1-z)t}$), the given messages are red (the probability of this event is z).

In this way from (10) after reducing s we have

$$\beta(z,s) = z\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}.$$

Thus, Lemma 3 is proved. □

Now we shall obtain the main result – the nonsteady state distribution of the queue length. The results are obtained in terms of Laplace transform, however we can easily get the moments of the size distribution and the stationary distribution. Besides, the results are well applicable for computer utilization.

Theorem 3. *The Laplace transform of the queue length distribution at an arbitrary time is given by*

$$p(z,s) = \frac{1+\lambda\pi(z,s)}{s+\lambda-\lambda\pi(s)} \tag{11}$$

where $\pi(z,s)$ is determined from Lemma 2, $\pi(s)$ – from functional equation (3).

Proof. We shall find the probability of the following event: "from two events: a) the "catastrophe" happened and b) the arrival of message, arrival of message occurs". The probability of the mentioned event, obviously, is

$$\frac{\lambda}{\lambda+s}.$$

Similarly, the probability of the event: "from two events: a) the "catastrophe" happened and b) the arrival or message – first "catastrophe" occurs", obviously, is

$$\frac{s}{\lambda+s}.$$

Now we show that the equality

$$sp(z,s)=\frac{s}{\lambda+s}+\frac{\lambda}{\lambda+s}sp(z,s)+\frac{\lambda}{\lambda+s}\pi(s)sp(z,s) \tag{12}$$

is fulfilled.

So, let's suppose that the first "catastrophe" happened at the moment when in the system there are at least red messages. The probability of this event is $sp(z,s)$. On the other hand, for this it is necessary and sufficient that
either the first "catastrophe" happen in the moment when the system is free (the probability of this event is $\frac{s}{\lambda+s}$);
or the first "catastrophe" happen during the busy period associated with the first message when in the system there are at least red messages (the probability of this event is $s\pi(z,s)$);
or during the busy period associated with the first message "catastrophe" do not happen, the first "catastrophe" happens after the completion of the busy period at the moment of time when in the system there are at least red messages (the probability of this event is $\frac{\lambda}{\lambda+s}\pi(s)sp(z,s)$).

Now from formula (12) we have

$$p(z,s)\left[1-\frac{\lambda\pi(s)}{\lambda+s}\right]=\frac{1+\lambda\pi(s)}{\lambda+s}$$

and Theorem 3 is proved. □

Note that the result of Theorem 3 allows us to receive the mean value of the queue size. Indeed, let's denote by $N(t)$ the mean value of the queue size at the moment t and by

$$n(s)=\int_0^\infty e^{-st}N(t)dt \tag{13}$$

the Laplace transform of function $N(t)$. Then from Theorem 3 we get

$$n(s)=\frac{\lambda}{s}\left[\frac{1}{s}-\frac{\beta(s)(1-\pi(s))}{(1-\beta(s))(s+\lambda-\lambda\pi(s))}\right]. \tag{14}$$

Formula (14) is obtained from (11) using the following algorithm

$$n(s) = \frac{\partial p(z,s)}{\partial z} \mid_{z=1} .$$

Remark 2. In the next section it will be shown that if $\rho < 1$, then Theorem 1 follows from Theorem 3, so the expression (11) given by Theorem 3 can be considered as virtual analog of formula (1). Also, the formula (14) can be considered to be a virtual analog of steady state mean value (2). Note that to get $N(t)$ it is necessary to invert $n(s)$, solving the integral equation (13).

4 Reduction of the virtual analog to Pollaczek-Khintchin equation

Using the method of embedded Markov chain we can prove that if the steady state condition $\rho = \lambda\beta_1 < 1$ is satisfied, then limits

$$\lim_{t\to\infty} P_m(t) = P_m, \quad \lim_{t\to\infty} P(z,t) = P(z)$$

exist.

Since $p(z,s)$ is the Laplace transform of generating function $P(z,t)$, applying the Tauber theorem we have

$$\lim_{t\to\infty} P(z,t) = \lim_{s\downarrow 0} sp(z,s)$$

or

$$P(z) = \lim_{s\downarrow 0} sp(z,s).$$

Thus, for $\lambda\beta_1 < 1$ substituing $p(z,s)$ given by Theorem 3 we obtain

$$P(z) = \lim_{s\downarrow 0} sp(z,s) = \lim_{s\downarrow 0} \frac{s(1+\lambda\pi(z,s))}{s+\lambda-\lambda\pi(s)}.$$

Since

$$\pi(0) = \int_0^\infty d\Pi(x) = 1, \tag{15}$$

applying the L'Hospital's rule we obtain

$$P(z) = \lim_{s\downarrow 0} \frac{[1+\lambda\pi(z,s)] + s\lambda\pi'(z,s)}{1-\lambda\pi'(s)},$$

or substituting $-\pi'(0)$ by

$$\pi_1 = \int_0^\infty x d\Pi(x) = -\pi'(0)$$

we obtain

$$P(z) = \frac{1 + \lambda\pi(z,0)}{1 - \lambda\pi_1}. \tag{16}$$

We get function $\pi(z,0)$ from Lemma 2 setting $s = 0$. We have

$$\pi(z,0) = \beta(z,0)\frac{z - \pi(0)}{z - \beta(\lambda - \lambda z)}. \tag{17}$$

We get function $\beta(z,0)$ from Lemma 3 setting $s = 0$. We have

$$\beta(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda - \lambda z}.$$

Find $\pi(z,0)$. According (17) and taking into consideration (15) we get

$$\pi(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda(1 - z)} \frac{z - 1}{z - \beta(\lambda - \lambda z)}$$

or

$$\pi(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda(\beta(\lambda - \lambda z)) - z}.$$

Substituting the obtained result in (16) and in accordance with Theorem 2

$$\pi_1 = \frac{\beta_1}{1 - \lambda\beta_1}$$

we finally obtain

$$P(z) = \frac{\beta(\lambda - \lambda z)(z - 1)(1 - \lambda\beta_1)}{z - \beta(\lambda - \lambda z)}$$

that exactly coincides with Pollaczek-Khintchin transform equation (Theorem 1).

Remark 3. From the above–presented we can conclude that Theorem 3 gives us the distribution of queue size for an arbitrary t, what can be considered as a virtual analog of the well–known Pollaczek-Khintchin equation.

Remark 4. Moreover, Lemmas 1-3 allows to get the steady state distribution of the queue size on a separate Π^n–period, a separate busy period Π, and a separate service time B. Indeed, let $P_1(z)$, $P_2(z)$, $P_3(z)$ be the generating functions of the mentioned intervals. If $\lambda\beta_1 < 1$, then the mentioned generating function can be obtained using the above procedure, namely

$$P_1(z) = \lim_{s\downarrow 0} s\pi^n(z,s),$$

$$P_2(z) = \lim_{s\downarrow 0} s\pi(z,s),$$

$$P_3(z) = \lim_{s\downarrow 0} s\beta(z,s).$$

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