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Moments of the Markovian random evolutions in two and four dimensions

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Abstract. Closed-form expressions for the mixed moments of the Markovian random evolutions in the spaces $\mathbb{R}^2$ and $\mathbb{R}^4$, are obtained. The moments of the Euclidean distance from the origin at any time $t > 0$ are also presented.

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1 Introduction

The symmetrical Markovian random evolution $\mathbf{X}(t)$ in the Euclidean spaces $\mathbb{R}^m$ of the lower dimensions $m = 2$, $m = 3$ and $m = 4$ have thoroughly been studied in [3–8]. In these works the distributions of $\mathbf{X}(t)$, $t \geq 0$, were explicitly obtained by different methods. The most difficult case $m = 3$ was examined in [8]. The distribution obtained has a very complicated integral form which seemingly cannot be expressed in terms of elementary functions.

In contrast to the three-dimensional case, the distributions of both the two- and four-dimensional random evolutions have fairly simple analytical forms (see [3, 5–7] for the planar random evolution, and [4] for the four-dimensional case). The reason of such a considerable difference in the forms of the distributions in different dimensions is not clear at all. A general method of studying the multidimensional random evolutions has recently been suggested in [2], however the closed-form expressions for the transition density of the motion cannot, apparently, been obtained in arbitrary higher dimension.

However, despite the fact that the distributions of random evolutions in the spaces $\mathbb{R}^2$ and $\mathbb{R}^4$ were obtained in the explicit forms, such an interesting and useful characteristic of the processes as their moments was not studied so far.

In this paper we obtain the closed-form expressions for the moments of the symmetrical Markovian random evolution $\mathbf{X}(t)$ in the dimensions $m = 2$ and $m = 4$. The moments of the Euclidean distance from the origin $\|\mathbf{X}(t)\|$ are also presented. We note that these moments are expressed in terms of special functions, namely, the Bessel and Struve functions in the planar case and the degenerated hypergeometric function and incomplete gamma-function for the four-dimensional random evolution.

2 Moments of the Planar Random Evolution

Consider the symmetrical planar random evolution performed by a particle that starts from the origin $\mathbf{0} = (0,0)$ of the plane $\mathbb{R}^2$ at time $t = 0$ and moves with constant finite speed c. The initial direction is a two-dimensional random vector with uniform distribution on the unit circumference

$$S_1^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = 1 \right\}.$$

The particle changes its direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1^2, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t))$ denote the particle's position at an arbitrary time $t \ge 0$. At any time $t > 0$ the particle, with probability 1, is located in the planar disc of radius ct

$$\mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \le c^2t^2 \right\}.$$

Consider the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}$, $\mathbf{x} \in \mathbf{B}_{ct}^2$, $t \ge 0$, where $d\mathbf{x}$ is the infinitesimal area in the plane $\mathbb{R}^2$. This distribution consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0,t)$ and is concentrated on the circumference

$$S_{ct}^2 = \partial \mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = c^2t^2 \right\}.$$

In this case the particle is located on S_{ct}^2 and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in S_{ct}^2\right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the disc $\mathbf{B}_{ct}^2$, and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in \operatorname{int} \mathbf{B}_{ct}^2\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}$ corresponding to this case is concentrated in the interior

$$\operatorname{int} \mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < c^2t^2 \right\},$$

and forms its absolutely continuous component. Therefore there exists the density of the absolutely continuous component of the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}$.

The principal known result states that the complete density $f(\mathbf{x}, t)$ of $\mathbf{X}(t)$ has the following form (see [5], formula (21)):

$$f(\mathbf{x}, t) = \frac{e^{-\lambda t}}{2\pi ct}\, \delta(c^2t^2 - \|\mathbf{x}\|^2) + \frac{\lambda}{2\pi c} \frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x}\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x}\|^2}}\, \Theta(ct - \|\mathbf{x}\|), \quad (1)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}^2_{ct}, \qquad \|\mathbf{x}\|^2 = x_1^2 + x_2^2, \qquad t \geq 0,$$

where $\delta(x)$ is the Dirac delta-function and $\Theta(x)$ is the Heaviside function. The first term in (1) represents the singular part, whereas the second term gives the absolutely continuous part of the density.

Let $\mathbf{q} = (q_1, q_2)$ be the two-multi-index. In this section we are interested in the mixed moments of the process $\mathbf{X}(t)$:

$$\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \mathrm{E}X_1^{q_1}(t) X_2^{q_2}(t), \qquad q_1 \geq 1, \; q_2 \geq 1.$$

The mixed moments of $\mathbf{X}(t)$ are given by the following theorem.

Theorem 1. *For any $q_1, q_2 \geq 1$ the following formula holds*

$$E\mathbf{X}^{\mathbf{q}}(t) = \begin{cases} \dfrac{e^{-\lambda t}}{\pi} \, (ct)^{q_1+q_2} \, B\left(\dfrac{q_1+1}{2}, \dfrac{q_2+1}{2}\right) + \\ + \dfrac{e^{-\lambda t}}{\sqrt{\pi}} \left(\dfrac{2}{\lambda t}\right)^{(q_1+q_2-1)/2} (ct)^{q_1+q_2} \, \Gamma\left(\dfrac{q_1+1}{2}\right) \Gamma\left(\dfrac{q_2+1}{2}\right) \times \\ \times \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t) \right], \\ \qquad\qquad \text{if } q_1 \text{ and } q_2 \text{ are even}, \\ 0, \qquad\qquad \text{otherwise}, \end{cases} \tag{2}$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k+\nu} \tag{3}$$

is the Bessel function of order ν with imaginary argument,

$$\mathbf{L}_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k + \frac{3}{2}\right) \, \Gamma\left(\nu + k + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k+\nu+1} \tag{4}$$

is the Struve function of order ν and $B(x, y)$ is the beta-function.

Proof. According to (1) we have

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{2\pi ct} \iint\limits_{x_1^2+x_2^2=c^2t^2} x_1^{q_1}\, x_2^{q_2}\, dx_1\, dx_2 + \\
&+ \frac{\lambda e^{-\lambda t}}{2\pi c} \iint\limits_{x_1^2+x_2^2\le c^2t^2} x_1^{q_1}\, x_2^{q_2}\, \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-(x_1^2+x_2^2)}\right)}{\sqrt{c^2t^2-(x_1^2+x_2^2)}}\, dx_1\, dx_2 = \\
&= \frac{e^{-\lambda t}}{2\pi} (ct)^{q_1+q_2} \int_0^{2\pi} (\cos\theta)^{q_1} (\sin\theta)^{q_2}\, d\theta + \\
&+ \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{ct} \int_0^{2\pi} (r\cos\theta)^{q_1} (r\sin\theta)^{q_2} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\, r\, dr\, d\theta = \\
&= \frac{e^{-\lambda t}}{2\pi} (ct)^{q_1+q_2} \int_0^{2\pi} (\cos\theta)^{q_1} (\sin\theta)^{q_2}\, d\theta + \\
&+ \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{ct} r^{q_1+q_2+1} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\, dr \int_0^{2\pi} (\cos\theta)^{q_1} (\sin\theta)^{q_2}\, d\theta.
\end{aligned}$$

Taking into account that

$$\int_0^{2\pi} (\cos\theta)^{q_1} (\sin\theta)^{q_2}\, d\theta = \begin{cases} 2B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right), & \text{if } q_1 \text{ and } q_2 \text{ are even,} \\ 0, & \text{otherwise,} \end{cases}$$

we obtain for even q_1 and q_2:

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{\pi} (ct)^{q_1+q_2} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{\lambda e^{-\lambda t}}{\pi c} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) \int_0^{ct} r^{q_1+q_2+1} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\, dr = \\
&= \frac{e^{-\lambda t}}{\pi} (ct)^{q_1+q_2} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{\lambda e^{-\lambda t}}{\pi c} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) (ct)^{q_1+q_2+1} \int_0^1 \xi^{q_1+q_2+1} \frac{e^{\lambda t\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}\, d\xi.
\end{aligned}$$

The substitution $z = \sqrt{1-\xi^2}$ in the last integral reduces this expression to

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{\pi} (ct)^{q_1+q_2} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{\lambda e^{-\lambda t}}{\pi c} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) (ct)^{q_1+q_2+1} \int_0^1 (1-z^2)^{(q_1+q_2)/2} e^{\lambda t z}\, dz.
\end{aligned}$$

Applying now [1], Formula 3.387(5), to the integral on the right-hand side of this equality we obtain

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{\pi} (ct)^{q_1+q_2} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{\lambda e^{-\lambda t}}{\pi c} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) (ct)^{q_1+q_2+1} \frac{\sqrt{\pi}}{2} \left(\frac{2}{\lambda t}\right)^{(q_1+q_2+1)/2} \times \\
&\times \Gamma\left(\frac{q_1+q_2}{2}+1\right) \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t)\right] = \\
&= \frac{e^{-\lambda t}}{\pi} (ct)^{q_1+q_2} B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{e^{-\lambda t}}{\sqrt{\pi}} \left(\frac{2}{\lambda t}\right)^{(q_1+q_2-1)/2} (ct)^{q_1+q_2} \Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \times \\
&\times \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t)\right]
\end{aligned}$$

The theorem is proved. □

Consider now the one-dimensional stochastic process

$$R(t) = \|\mathbf{X}(t)\| = \sqrt{X_1^2(t) + X_2^2(t)},$$

representing the Euclidean distance of the moving particle from the origin **0**. Clearly, $0 \le R(t) \le ct$ and, according to [5], Remark 2, the absolutely continuous part of the distribution of $R(t)$ has the form:

$$\begin{aligned}
\Pr\{R(t) < r\} &= \Pr\left\{\mathbf{X}(t) \in \mathbf{B}_r^2\right\} \\
&= 1 - \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r^2}\right), \qquad 0 \le r < ct.
\end{aligned}$$

Therefore, the complete density of $R(t)$ in the interval $0 \le r \le ct$ is given by

$$f(r,t) = \frac{re^{-\lambda t}}{ct} \delta(ct-r) + \frac{\lambda}{c} \frac{r}{\sqrt{c^2t^2-r^2}} \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right) \Theta(ct-r). \quad (5)$$

In the following theorem we present an explicit formula for the moments of the process $R(t)$.

Theorem 2. *For any $q \ge 1$ the following formula holds*

$$\begin{aligned}
ER^q(t) &= (ct)^q e^{-\lambda t} + \\
&+ e^{-\lambda t}\sqrt{\pi} \left(\frac{2}{\lambda t}\right)^{(q-1)/2} (ct)^q \, \Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right],
\end{aligned} \quad (6)$$

where $I_\nu(x)$ and $\mathbf{L}_\nu(x)$ are given by (3) and (4), respectively.

Proof. According to (5) we have

$$\begin{aligned}
\mathrm{E}R^q(t) &= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c} \int_0^{ct} \frac{r^{q+1}}{\sqrt{c^2t^2-r^2}}\, e^{\frac{\lambda}{c}\sqrt{c^2t^2-r^2}}\, dr = \\
&= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c} (ct)^{q+1} \int_0^1 \xi^{q+1}\, (1-\xi^2)^{-1/2}\, e^{\lambda t\sqrt{1-\xi^2}}\, d\xi.
\end{aligned}$$

Making the substitution $z = \sqrt{1-\xi^2}$ in the last integral, we obtain

$$\begin{aligned}
\mathrm{E}R^q(t) &= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c} (ct)^{q+1} \int_0^1 (1-z^2)^{q/2}\, e^{\lambda t z}\, dz = \\
&= (ct)^q e^{-\lambda t} + \\
&+ \frac{\lambda e^{-\lambda t}\sqrt{\pi}}{2c} \left(\frac{2}{\lambda t}\right)^{(q+1)/2} (ct)^{q+1}\Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right] = \\
&= (ct)^q e^{-\lambda t} + \\
&+ e^{-\lambda t}\sqrt{\pi} \left(\frac{2}{\lambda t}\right)^{(q-1)/2} (ct)^q\, \Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right].
\end{aligned}$$

where we have used again [1], Formula 3.387(5). The theorem is proved. □

Remark 1. From (6) we can extract the formulae concerning two the most important moments, namely, the expectation and variance of the process $R(t)$:

$$\begin{aligned}
\mathrm{E}R(t) &= ct\, e^{-\lambda t} \left\{1 + \frac{\pi}{2}\, \left[I_1(\lambda t) + \mathbf{L}_1(\lambda t)\right]\right\}, \\
\mathrm{E}R^2(t) &= (ct)^2\, e^{-\lambda t} \left\{1 + \sqrt{\frac{2\pi}{\lambda t}}\, \left[I_{3/2}(\lambda t) + \mathbf{L}_{3/2}(\lambda t)\right]\right\}.
\end{aligned}$$

3 Moments of the Four-Dimensional Random Evolution

We consider now the similar symmetrical random evolution of a particle moving at constant finite speed c in the space $\mathbb{R}^4$ and subject to the control of a homogeneous Poisson process of rate $\lambda > 0$ in the manner described above.

Both the initial and every new direction are taken on according to the uniform law on the unit sphere

$$S_1^4 = \left\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}.$$

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t)),\ t > 0$, be the position process. At any time $t > 0$ the particle, with probability 1, is located in the four-dimensional ball of radius ct

$$\mathbf{B}_{ct}^4 = \left\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^2 :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \le c^2t^2\right\}.$$

Similarly to the planar case, we consider the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$, $\mathbf{x} \in \mathbf{B}^4_{ct}$, $t \geq 0$, where $d\mathbf{x}$ is the infinitesimal volume in the space $\mathbb{R}^4$. This distribution consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S^4_{ct} = \partial \mathbf{B}^4_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^2t^2 \right\}.$$

In this case the particle is located on S^4_{ct} and the probability of this event is

$$\Pr\left\{ \mathbf{X}(t) \in S^4_{ct} \right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the ball $\mathbf{B}^4_{ct}$, and the probability of this event is

$$\Pr\left\{ \mathbf{X}(t) \in \operatorname{int} \mathbf{B}^4_{ct} \right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior of the ball

$$\operatorname{int} \mathbf{B}^4_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 < c^2t^2 \right\},$$

and forms the absolutely continuous component of this distribution.

It is known that the density of $\mathbf{X}(t)$ has the form (see [4], formula (19) therein):

$$\begin{aligned} f(\mathbf{x}, t) = & \frac{e^{-\lambda t}}{2\pi^2 (ct)^3} \, \delta(c^2t^2 - \|\mathbf{x}\|^2) + \\ & + \frac{\lambda t}{\pi^2 (ct)^4} \left[2 + \lambda t \left(1 - \frac{\|\mathbf{x}\|^2}{c^2t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} \|\mathbf{x}\|^2 \right) \, \Theta(ct - \|\mathbf{x}\|), \end{aligned} \tag{7}$$

$$\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{B}^4_{ct}, \qquad \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \qquad t \geq 0.$$

Let $\mathbf{q} = (q_1, q_2, q_3, q_4)$ be the four-multi-index. We are interested in the mixed moments of the process $\mathbf{X}(t)$:

$$\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \mathrm{E} X_1^{q_1}(t) X_2^{q_2}(t) X_3^{q_3}(t) X_4^{q_4}(t), \qquad q_1 \geq 1, \; q_2 \geq 1, \; q_3 \geq 1, \; q_4 \geq 1.$$

The mixed moments of $\mathbf{X}(t)$ are given by the following theorem.

Theorem 3. *For any $q_1, q_2, q_3, q_4 \geq 1$ the following formula holds*

$$E\mathbf{X}^{\mathbf{q}}(t) = \begin{cases} \frac{e^{-\lambda t}}{\pi^2}(ct)^{q_1+q_2+q_3+q_4} \; B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+1}{2}\right) \times \\ \times \, B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+1}{2}\right) \; B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2}\right) + \\ + \frac{2\lambda t}{\pi^2} \; (ct)^{q_1+q_2+q_3+q_4} \; \frac{\Gamma\left(\frac{q_1+1}{2}\right) \; \Gamma\left(\frac{q_2+1}{2}\right) \; \Gamma\left(\frac{q_3+1}{2}\right) \; \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\ \times \left[(\lambda t)^{-(q_1+q_2+q_3+q_4+4)/2} \; \gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}, \; \lambda t\right) + \right. \\ + \frac{\lambda t}{2} \; \frac{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+8}{2}\right)} \times \\ \left. \times \; {}_1F_1\left(\frac{q_1+q_2+q_3+q_4+4}{2}; \frac{q_1+q_2+q_3+q_4+8}{2}; -\lambda t\right)\right], \\ \qquad\qquad\qquad \text{if all } \; q_1, q_2, q_3, q_4 \; \text{ are even}, \\ 0, \qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{8}$$

where

$$\gamma(\alpha, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\alpha+k)} \; x^{\alpha+k} \tag{9}$$

is the incomplete gamma-function, and

$${}_1F_1(\xi; \eta; z) = \Phi(\xi, \eta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k}{(\eta)_k} \frac{z^k}{k!} \tag{10}$$

is the degenerated hypergeometric function.

Proof. We consider separately the singular and the absolutely continuous parts of the density (7). According to (7), for the singular part of the distribution of the process we have:

$$\begin{aligned} \mathrm{E}\mathbf{X}_s^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{2\pi^2(ct)^3} \iiiint\limits_{x_1^2+x_2^2+x_3^2+x_4^2=c^2t^2} x_1^{q_1} \; x_2^{q_2} \; x_3^{q_3} \; x_4^{q_4} \; dx_1 \; dx_2 \; dx_3 \; dx_4 = \\ &= \frac{e^{-\lambda t}}{2\pi^2} \; (ct)^{q_1+q_2+q_3+q_4} \int_0^{\pi} (\cos\theta_1)^{q_1} \; (\sin\theta_1)^{q_2+q_3+q_4} \; d\theta_1 \times \\ &\times \int_0^{\pi} (\cos\theta_2)^{q_2} \; (\sin\theta_2)^{q_3+q_4} \; d\theta_2 \; \times \int_0^{2\pi} (\cos\theta_3)^{q_3} \; (\sin\theta_3)^{q_4} \; d\theta_3. \end{aligned} \tag{11}$$

Computing these integrals we have

$$\int_0^{2\pi} (\cos\theta_3)^{q_3} \, (\sin\theta_3)^{q_4} \, d\theta_3 = \begin{cases} 2B\left(\dfrac{q_3+1}{2}, \dfrac{q_4+1}{2}\right), & \text{if } q_3 \text{ and } q_4 \text{ are even}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\int_0^{\pi} (\cos\theta_2)^{q_2} \, (\sin\theta_2)^{q_3+q_4} \, d\theta_2 = \begin{cases} B\left(\dfrac{q_2+1}{2}, \dfrac{q_3+q_4+1}{2}\right), & \text{if } q_2 \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

$$\int_0^{\pi} (\cos\theta_1)^{q_1} \, (\sin\theta_1)^{q_2+q_3+q_4} \, d\theta_1 = \begin{cases} B\left(\dfrac{q_1+1}{2}, \dfrac{q_2+q_3+q_4+1}{2}\right), & \text{if } q_1 \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$

Substituting these values (12) into (11) we obtain for even q_1, q_2, q_3, q_4:

$$\begin{aligned} \mathrm{E}\mathbf{X}_s^{\mathbf{q}}(t) = \frac{e^{-\lambda t}}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \, B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+1}{2}\right) \times \\ \times B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+1}{2}\right) \, B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2}\right). \end{aligned} \tag{13}$$

Let us evaluate now the moments of the absolutely continuous part of the distribution of the process. By passing to four-dimensional polar coordinates, we have:

$$\begin{aligned} &\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \iiiint\limits_{x_1^2+x_2^2+x_3^2+x_4^2 \le c^2t^2} \prod_{i=1}^4 x_i^{q_i} \, \left[2 + \lambda t \left(1 - \frac{\|\mathbf{x}\|^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\|\mathbf{x}\|^2\right) \prod_{i=1}^4 dx_i = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} dr \int_0^{\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{2\pi} d\theta_3 \times \\ &\times \Big\{ (r\cos\theta_1)^{q_1} \, (r\sin\theta_1\cos\theta_2)^{q_2} \, (r\sin\theta_1\sin\theta_2\cos\theta_3)^{q_3} \, (r\sin\theta_1\sin\theta_2\sin\theta_3)^{q_4} \times \\ &\times \left[2 + \lambda t\left(1 - \frac{r^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2t} r^2\right) \, r^3 \, (\sin\theta_1)^2 \sin\theta_2 \Big\} = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} r^{q_1+q_2+q_3+q_4+3} \left[2 + \lambda t\left(1 - \frac{r^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2t} r^2\right) \, dr \times \\ &\times \int_0^{\pi} (\cos\theta_1)^{q_1} \, (\sin\theta_1)^{q_2+q_3+q_4+2} d\theta_1 \, \int_0^{\pi} (\cos\theta_2)^{q_2} \, (\sin\theta_2)^{q_3+q_4+1} d\theta_2 \times \\ &\times \int_0^{2\pi} (\cos\theta_3)^{q_3} \, (\sin\theta_3)^{q_4} \, d\theta_3. \end{aligned}$$

Taking into account (12), we can rewrite this equality for even q_1, q_2, q_3, q_4 as follows:

$$\begin{aligned}
&\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\
&= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} r^{q_1+q_2+q_3+q_4+3} \left[2 + \lambda t \left(1 - \frac{r^2}{c^2t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} r^2 \right) \, dr \times \\
&\times 2B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+3}{2} \right) B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+2}{2} \right) B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2} \right) = \\
&= \frac{\lambda t}{\pi^2 (ct)^4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \int_0^{ct} (r^2)^{(q_1+q_2+q_3+q_4+2)/2} \left[2 + \lambda t \left(1 - \frac{r^2}{c^2t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} r^2 \right) \, d(r^2) = \\
&= \frac{\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \, (2 + \lambda t(1-z)) \, e^{-\lambda t z} \, dz = \\
&= \frac{\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \left[2 \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \, e^{-\lambda t z} \, dz + \lambda t \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \, (1-z) \, e^{-\lambda t z} \, dz \right].
\end{aligned}$$

Applying now [1], Formula 3.381(1) and Formula 3.383(1) to the first and the second integrals of this equality, respectively, we obtain

$$\begin{aligned}
&\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\
&= \frac{2\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \left[(\lambda t)^{-(q_1+q_2+q_3+q_4+4)/2} \, \gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}, \, \lambda t \right) + \right. \\
&\left. + \frac{\lambda t}{2} \frac{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+8}{2}\right)} \, {}_1F_1\left(\frac{q_1+q_2+q_3+q_4+4}{2}; \frac{q_1+q_2+q_3+q_4+8}{2}; -\lambda t \right) \right].
\end{aligned}$$

Now, by adding to this expression the moments of the singular part of the distribution given by (13), we finally obtain (8). The theorem is thus completely proved. □

Consider now the following one-dimensional stochastic process

$$R(t) = \|\mathbf{X}(t)\| = \sqrt{X_1^2(t) + X_2^2(t) + X_3^2(t) + X_4^2(t)},$$

representing the Euclidean distance of the moving particle from the origin $\mathbf{0}$ of the space $\mathbb{R}^4$. Clearly, $0 \le R(t) \le ct$ and, according to [4], formula (18), the absolutely continuous part of the distribution of $R(t)$ has the form:

$$\begin{aligned}\Pr\{R(t) < r\} &= \Pr\left\{\mathbf{X}(t) \in \mathbf{B}_r^4\right\} = \\ &= 1 - \left(1 + \frac{\lambda}{c^2 t}\, r^2 - \frac{\lambda}{c^4 t^3}\, r^4\right) \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right). \qquad 0 \le r < ct,\end{aligned}$$

Therefore, the complete density of $R(t)$ in the interval $0 \le r \le ct$ is given by

$$\begin{aligned} f(r,t) &= \frac{r^3 e^{-\lambda t}}{(ct)^3}\, \delta(ct - r) + \\ &+ \left[\left(\frac{4\lambda}{c^4 t^3} + \frac{2\lambda^2}{c^4 t^2}\right) r^3 - \frac{2\lambda^2}{c^6 t^4} r^5\right] \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) \Theta(ct - r). \end{aligned} \tag{14}$$

In the following theorem we present an explicit formula for the moments of $R(t)$.

Theorem 4. *For any $q \ge 1$ the following formula holds*

$$ER^q(t) = (ct)^q \left\{ e^{-\lambda t} + (\lambda t)^{-(q+2)/2} \left[(2 + \lambda t)\, \gamma\left(\frac{q}{2} + 2, \lambda t\right) - \gamma\left(\frac{q}{2} + 3, \lambda t\right)\right]\right\}, \tag{15}$$

where $\gamma(\alpha, x)$ is the incomplete gamma-function given by (9).

Proof. According to (14) we have

$$\begin{aligned}\mathrm{E}R^q(t) = (ct)^q e^{-\lambda t} &+ \left(\frac{4\lambda}{c^4 t^3} + \frac{2\lambda^2}{c^4 t^2}\right) \int_0^{ct} r^{q+3} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) dr - \\ &- \frac{2\lambda^2}{c^6 t^4} \int_0^{ct} r^{q+5} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) dr.\end{aligned}$$

Making the substitution $\xi = r^2$ in both integrals, we obtain

$$\begin{aligned}\mathrm{E}R^q(t) = (ct)^q e^{-\lambda t} &+ \left(\frac{2\lambda}{c^4 t^3} + \frac{\lambda^2}{c^4 t^2}\right) \int_0^{c^2 t^2} \xi^{(q+2)/2} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) d\xi - \\ &- \frac{\lambda^2}{c^6 t^4} \int_0^{c^2 t^2} \xi^{(q+4)/2} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) d\xi = \end{aligned}$$

$$= (ct)^q e^{-\lambda t} + \left(\frac{2\lambda}{c^4 t^3} + \frac{\lambda^2}{c^4 t^2} \right) \left(\frac{\lambda}{c^2 t} \right)^{-(q+4)/2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) -$$
$$- \frac{\lambda^2}{c^6 t^4} \left(\frac{\lambda}{c^2 t} \right)^{-(q+6)/2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) =$$
$$= (ct)^q e^{-\lambda t} + \frac{2 + \lambda t}{c^2 t^2} \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) -$$
$$- \frac{1}{c^2 t^2} \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) =$$
$$= (ct)^q e^{-\lambda t} + \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \left[\frac{2 + \lambda t}{c^2 t^2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \frac{1}{c^2 t^2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) \right] =$$
$$= (ct)^q \left\{ e^{-\lambda t} + (\lambda t)^{-(q+2)/2} \left[(2 + \lambda t) \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \gamma \left(\frac{q}{2} + 3, \lambda t \right) \right] \right\},$$

where in the second step we have used [1], Formula 3.381(1). The theorem is proved. □

Remark 2. From (15) we can extract the formulae for the mean value and variance of the process $R(t)$:

$$\begin{aligned} \mathrm{E}R(t) &= ct \left\{ e^{-\lambda t} + (\lambda t)^{-3/2} \left[(2 + \lambda t) \gamma \left(\frac{5}{2}, \lambda t \right) - \gamma \left(\frac{7}{2}, \lambda t \right) \right] \right\}, \\ \mathrm{E}R^2(t) &= \frac{2c^2}{\lambda^2} \left(e^{-\lambda t} + \lambda t - 1 \right). \end{aligned} \tag{16}$$

The first formula in (16) immediately follows from (15) for $q = 1$. Let us now prove the second formula in (16). For $q = 2$ formula (16) yields:

$$\mathrm{E}R^2(t) = (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(2 + \lambda t) \gamma (3, \lambda t) - \gamma (4, \lambda t) \right] \right\} =$$

([1], Formula 8.356(1))

$$= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(2 + \lambda t) \gamma (3, \lambda t) - 3\gamma (3, \lambda t) + (\lambda t)^3 e^{-\lambda t} \right] \right\} =$$
$$= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(\lambda t - 1) \gamma (3, \lambda t) + (\lambda t)^3 e^{-\lambda t} \right] \right\} =$$

([1], Formula 8.352(1))

$$= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[2(\lambda t - 1) \left(1 - e^{-\lambda t} \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} \right) \right) + (\lambda t)^3 e^{-\lambda t} \right] \right\} =$$
$$= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[2\lambda t - 2 - e^{-\lambda t} \left((\lambda t)^2 - 2 \right) \right] \right\} =$$
$$= (ct)^2 \left\{ e^{-\lambda t} + \frac{2}{\lambda t} - \frac{2}{(\lambda t)^2} - \left(1 - \frac{2}{(\lambda t)^2} \right) e^{-\lambda t} \right\} =$$
$$= (ct)^2 \left\{ \frac{2}{\lambda t} - \frac{2}{(\lambda t)^2} (1 - e^{-\lambda t}) \right\} = \frac{2c^2}{\lambda^2} \left(e^{-\lambda t} + \lambda t - 1 \right),$$

and the second formula in (16) is proved.

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