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Measure of quasistability of a vector integer linear programming problem with generalized principle of optimality in the Helder metric

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Abstract. A vector integer linear programming problem is considered, principle of optimality of which is defined by a partitioning of partial criteria into groups with Pareto preference relation within each group and the lexicographic preference relation between them. Quasistability of the problem is investigated. This type of stability is a discrete analog of Hausdorff lower semicontinuity of the many-valued mapping that defines the choice function. A formula of quasistability radius is derived for the case of metric $l_p, 1 \le p \le \infty$, defined in the space of parameters of the vector criterion. Similar formulae had been obtained before only for combinatorial (boolean) problems with various kinds of parametrization of the principles of optimality in the cases of l_1 and l_∞ metrics [1–4], and for some game theory problems [5–7].

Mathematics subject classification: 90C10, 90C29, 90C31.
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1 Basic Definitions and Properties

Let us consider m-criteria integer linear programming problem with n variables:

$$Cx = (C_1x, C_2x, \dots, C_mx)^T \to \min_{x\in X},$$

where $C = [c_{ij}]_{m\times n} \in \mathbf{R}^{m\times n}$, $m, n \in \mathbf{N}$, C_i denotes the i-th row of matrix C, $i \in N_m = \{1, 2, \dots, m\}$, X is the finite set of solutions from $\mathbf{Z}^n$, $|X| > 1$, $x = (x_1, x_2, \dots, x_n)^T$.

We put for this problem parametric principle of optimality.

Let $s \in N_m$, $N_m = \bigcup_{r\in N_s} I_r$ be the partitioning of the set N_m into s nonempty disjoint subsets (groups), i. e. $I_r \ne \emptyset$, $r \in N_s$; $p \ne q \Rightarrow I_p \cap I_q = \emptyset$. In the criteria space $\mathbf{R}^m$ we put the binary relation of the strict preference $\Omega^m(I_1, I_2, \dots, I_s)$ between different vectors $y = (y_1, y_2, \dots, y_m)$ and $y' = (y'_1, y'_2, \dots, y'_m)$ in correspondence to any partition $(I_1, I_2, \dots, I_s)$, as follows:

$$y\ \Omega^m(I_1, I_2, \dots, I_s)\ y' \quad \Leftrightarrow \quad y_{I_k} \underset{P}{\succ} y'_{I_k},$$

where $k = \min\{i \in N_s : \ y_{I_i} \ne y'_{I_i}\}$; y_{I_k} and y'_{I_k} are projections of the vectors y and y' onto axises of the $\mathbf{R}^n$ space with indexes from group I_k; $\underset{P}{\succ}$ be the relation which generates Pareto principle of optimality:

$$y_{I_k} \underset{P}{\succ} y'_{I_k} \Leftrightarrow y_{I_k} \geq y'_{I_k} \;\&\; y_{I_k} \neq y'_{I_k}.$$

The introduced binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$ sets such order of groups of criteria in which any previous group is more important than all the following ones. Consequently this relation generates one set of $(I_1, I_2, \ldots, I_s)$-effective (or, otherwise, generalized effective) solutions according to the rule

$$G^m(C, I_1, I_2, \ldots, I_s) = \{x \in X : \forall x' \in X \quad (Cx \; \overline{\Omega^m(I_1, I_2, \ldots, I_s)} \; Cx')\},$$

where $\overline{\Omega^m(I_1, I_2, \ldots, I_s)}$, as usual, means the negation of the binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$.

It is obvious that the set $G^m(C, N_m)$ $(s = 1)$ of the N_m-effective solutions is Pareto set, i. e. the set of effective solutions

$$P^m(C) = \{x \in X : \; \forall x' \in X \; (Cx \underset{P}{\overline{\succ}} Cx')\}.$$

It is easy to understand that the set of $(\{1\}, \{2\}, \ldots, \{m\})$-effective solutions $G^m(C, \{1\}, \{2\}, \ldots, \{m\})$ $(s = m)$ is equal to the set of lexicographic optima

$$L^m(C) = \{x \in X : \; \forall x' \in X \; (Cx \underset{L}{\overline{\succ}} Cx')\}.$$

Here $\underset{L}{\succ}$ is a binary relation which sets lexicographic order:

$$y \underset{L}{\succ} y' \Leftrightarrow y_k > y'_k,$$

where $k = \min\{i \in N_m : \; y_i \neq y'_i\}$, $y = (y_1, y_2, \ldots, y_m)$, $y' = (y'_1, y'_2, \ldots, y'_m)$.

Thus in this case by the parametrization of the principle of optimality we mean introducing a characteristic of binary relation of preference that allows us to relate the well-known lexicographic and Pareto principles of optimality.

It is easy to show that the binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$ is antireflexive, asymmetric, transitive, and hence it is cyclic. And since the set X is finite, the set $G^m(C, I_1, I_2, \ldots, I_s)$ is non-empty for any matrix C and any partitioning $(I_1, I_2, \ldots, I_s), s \in N_m$, of the set N_m.

By $Z^m(C, I_1, I_2, \ldots, I_s)$ problem we understand the problem of finding the set $G^m(C, I_1, I_2, \ldots, I_s)$.

The following properties directly follow from the introduced definitions.

Property 1. $G^m(C, I_1, I_2, \ldots, I_s) \subseteq P_1(C) \subseteq X$, *where*

$$P_1(C) = \{x \in X : \; \forall x' \in X \; (C_{I_1} x \underset{P}{\overline{\succ}} C_{I_1} x')\}.$$

Let us define C_{I_1} as a submatrix of the matrix C, consisting of the rows of matrix C with numbers from the group I_1.

Property 2. *If* $C_{I_1}x \underset{P}{\succ} C_{I_1}x'$, *then* $Cx\ \Omega^m(I_1, I_2, \dots, I_s)\ Cx'$.

Property 3. *If* $Cx\ \Omega^m(I_1, I_2, \dots, I_s)\ Cx'$, *then* $C_{I_1}x \geq C_{I_1}x'$.

Property 4. *The solution* $x \notin G^m(C, I_1, I_2, \dots, I_s)$ *if and only if there exists such solution* x' *that* $Cx\ \Omega^m(I_1, I_2, \dots, I_s)\ Cx'$.

Property 5. *The solution* $x \in G^m(C, I_1, I_2, \dots, I_s)$ *if and only if for any solution* x' *the relation* $Cx\ \overline{\Omega^m(I_1, I_2, \dots, I_s)}\ Cx'$ *is true.*

Property 6. $S_1(C) \subseteq G^m(C, I_1, I_2, \dots, I_s)$, *where*

$$S_1(C) = \{x \in P_1(C) :\ \forall x' \in X \setminus \{x\} \quad \big(C_{I_1}x \neq C_{I_1}x'\big)\}.$$

Indeed, let $x \in S_1(C)$ and $x \notin G^m(C, I_1, I_2, \dots, I_s)$. Then according to Property 4 there exists a solution x' such that

$$Cx\,\Omega^m(I_1, I_2, \dots, I_s)\,Cx'.$$

Hence due to Property 3 we have $C_{I_1}x \geq C_{I_1}x'$. Taking into account the inclusion $x \in P_1(C)$, we obtain $C_{I_1}x = C_{I_1}x'$, i. e. $x \notin S_1(C)$, which contradicts the assumption.

It is obvious that the set $S_1(C)$ is nonempty. Directly from the definition of sets $P_1(C)$ and S_1 we obtain

Property 7. *For all* $x \in S_1(C)$ *for all* $x' \in X \setminus \{x\}$ *there exists such* $i \in I_1$ *that* $\big(C_ix' > C_ix\big)$.

For all natural number k in the real space $\mathbf{R}^k$ we define a Helder metric (l_p)

$$||y||_p = \Big(\sum_{j \in N_k} |y_j|^p\Big)^{\frac{1}{p}}, 1 \leq p < \infty.$$

Let us also use l_∞ metric:

$$||y||_\infty = \max\{|y_j| :\ j \in N_k\}.$$

It is known that l_p metric in the $\mathbf{R}^k$ and l_q metric in the conjugate space $(\mathbf{R}^k)^*$ are connected by the equality

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where $1 < p < \infty$; in addition, $q = 1$ if $p = \infty$, and $q = \infty$ if $p = 1$. We suppose that the range of variation of p and q is $[1, \infty]$, and numbers p and q are connected by the above conditions. Then according to the Helder inequality for any index $i \in N_m$ is fair the inequality

$$C_ix \leq ||C_i||_p||x||_q. \tag{1}$$

Let us define an operator of projection of the vector $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$ onto nonnegative orthant:

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_n^+),$$

where $a_i^+ = [a_i]^+ = \max\{0, a_i\}$. Then sign "+" over vector means a vector with positive coordinates and zero instead of negative coordinates.

Property 8. *If for some row $i \in N_m$ of the matrices $C, C' \in \mathbf{R}^{m\times n}$ the inequality*

$$(C_i + C_i')(x' - x) \le 0, \tag{2}$$

is satisfied then for any number $p \in [1, \infty]$ the inequality

$$[C_i(x' - x)]^+ \le ||C_i'||_p ||x' - x||_q \tag{3}$$

is fair.

Really, with $C_i(x' - x) \le 0$ the inequality (3) is evident. If $C_i(x' - x) > 0$, then from the condition (2) and Helder inequality (1) it follows

$$[C_i(x' - x)]^+ = C_i(x' - x) \le -C_i'(x' - x) \le ||C_i'||_p ||x' - x||_q.$$

Property 9. *If $p > 1$ and vectors $y, y' \in \mathbf{R}^m$ are such that $y_j = {y'}_j^{q-1}$, $j \in N_m$, then*

$$||y||_p = ||y'||_q^{q-1}.$$

Indeed, according to $p = \infty$ $(q = 1)$ Property 9 is trivial, taking into account $1 < p < \infty$ obtain

$$||y||_p = \Big(\sum_{j\in N_m} |y_j'|^{p(q-1)}\Big)^{\frac{1}{p}} = \Big(\sum_{j\in N_m} |y_j'|^{q}\Big)^{\frac{1}{p}} = ||y'||_q^{\frac{q}{p}} = ||y'||_q^{q-1}.$$

By analogy with [1–5] the problem $Z^m(C, I_1, I_2, \dots, I_s)$, $n \ge 1$, is called quasistabile if

$$\Xi_p := \{\varepsilon > 0 : \forall\, C' \in \Psi_p(\varepsilon)\ (G^m(C, I_1, I_2, \dots, I_s) \subseteq G^m(C + C', I_1, I_2, \dots, I_s))\} \ne \emptyset,$$

where

$$\Psi_p(\varepsilon) = \{C' \in \mathbf{R}^{m\times n} :\ ||C'||_p < \varepsilon\}$$

is perturbing matrices set. Under matrix metric we understand metric of the vector that consist from its elements.

Thereby, quasistability of the $Z^m(C, I_1, I_2, \dots, I_s)$ problem is the discrete analog of Hausdorff lower semicontinuity at the point C (for fixed p and partitioning method of the N_m into groups) of the many-valued mapping

$$G^m : \mathbf{R}^{m\times n} \to 2^X,$$

which puts into correspondence to any matrix C the $G^m(C, I_1, I_2, \dots, I_s)$ set. It is evident that the quasistability property is invariant relative to l_p metric, because all metrics in a finite linear space are equivalent ([6], p. 166).

According to above, let's define the quasistability radius of the problem $Z^m(C, I_1, I_2, \dots, I_s)$ by the next number in the space determined by the metric l_p:

$$\rho_p^m(C,\mathcal{I}) = \begin{cases} \sup \Xi_p(C,\mathcal{I}), & \text{if } \Xi_p(C,\mathcal{I}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the quasistability radius of the problem $Z^m(C, I_1, I_2, \dots, I_s)$ specifies the limit of the element perturbations by matrix C in the space $\mathbf{R}^{m\times n}$ with l_p metric, such that the set of generalized effective solutions is preserved.

2 Lemmas

For any different solutions x and x' we define the fraction:

$$\gamma(x,x') = \frac{||[C_{I_1}(x'-x)]^+||_p}{||x'-x||_q}.$$

Lemma 1. *If*

$$\gamma(x,x') \geq \varphi > 0, \tag{4}$$

then the following relation holds for any perturbing matrix $C' \in \Psi_p(\varphi)$

$$(C+C')x \ \overline{\Omega^m(I_1,\, I_2, \dots,\, I_s)} \ (C+C')x'.$$

Proof. Let exist such matrix $C' \in \Psi_p(\varphi)$ that $(C+C')x\, \Omega^m(I_1, I_2, \dots, I_s)\, (C+C')x'$. Then by virtue of Property 3 for any index $i \in I_1$ inequality (2) holds. Therefore due to Property 8 for any index $i \in I_1$ and any $p \in [1,\infty]$ inequality (3) holds. Hence, when $1 \leq p < \infty$, we derive

$$||[C_{I_1}(x'-x)]^+||_p = \Big(\sum_{i\in I_1}([C_i(x'-x)]^+)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{i\in I_1}||C'_i||_p^p||x'-x||_q^p\Big)^{\frac{1}{p}} =$$

$$= \Big(\sum_{i\in I_1}||C'_i||_p^p\Big)^{\frac{1}{p}}||x'-x||_q = ||C'_{I_1}||_p||x'-x||_q < \varphi||x'-x||_q,$$

and when $p=\infty$, we derive

$$||[C_{I_1}(x'-x)]^+||_\infty = \max_{i\in I_1}[C_i(x'-x)]^+ \leq ||x'-x||_1 \max_{i\in I_1}||C'_i||_\infty =$$

$$= ||C'_{I_1}||_\infty||x'-x||_1 < \varphi||x'-x||_1.$$

Inequalities from above contradict (4) and Lemma 1 holds.

Lemma 2. *Let $x \in G^m(C, I_1, I_2, \dots, I_s)$, $x' \in X \setminus x$ and components of vector $b = (b_1, b_2, \dots, b_m)$ satisfy the following:*

$$b_i||x'-x||_q > [C_i(x'-x)]^+,\ i \in I_1, \tag{5}$$

$$b_i = 0,\ i \in N_m \setminus I_1.$$

Then for any number $\varepsilon > ||b||_p$ there exists such matrix $C^* \in \Psi_p(\varepsilon)$ that

$$(C+C^*)x\ \Omega^m(I_1,\, I_2, \dots,\, I_s)\ (C+C^*)x'. \tag{6}$$

Proof. If $p > 1$, then let's define elements of matrix $C^* = [c^*_{ij} \in \mathbf{R}^{m\times n}]$ by the formula

$$c^*_{ij} = \begin{cases} b_i \ \mathrm{sign}(x_j - x'_j)\dfrac{|x'_j - x_j|^{q-1}}{||x' - x||_q^{q-1}}, & \text{if } \ i \in I_1, \ \ j \in N_n, \\ 0, & \text{if } \ i \in N_m \setminus I_1, \ \ j \in N_n, \end{cases}$$

Else if $p = 1$, then let's fix the index $s = \arg\max\{|x'_j - x_j| : \ j \in I_1\}$ and define elements of matrix C^* by formula

$$c^*_{ij} = \begin{cases} b_i \ \mathrm{sign}(x_j - x'_j), & \text{if } i \in I_1, j = s, \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that $||C^*||_1 = ||b||_1$. In accordance with Property 9 it is easy to show that $||C^*||_p = ||b||_p$ when $p > 1$. Thus, $C^* \in \Psi_p(\varepsilon)$. By the construction of the matrix C^* for any index $i \in I_1$ the equality

$$C^*_i(x' - x) = \begin{cases} -b_i||x' - x||_\infty, & \text{if } p = 1, \\ -b_i||x' - x||_q^{1-q} \sum_{j\in I_1} |x'_j - x_j|^q, & \text{if } 1 < p \leq \infty, \end{cases}$$

holds. Then

$$C^*_i(x' - x) = -b_i||x' - x||_q, \qquad i \in I_1.$$

Therefore under (5) the relations

$$(C_i + C^*_i)(x' - x) = C_i(x' - x) + C^*_i(x' - x) =$$

$$= C_i(x' - x) - b_i||x' - x||_q \leq [C_i(x' - x)]^+ - b_i||x' - x||_q < 0, \qquad i \in I_1$$

are correct, and thereby

$$(C_{I_1} + C^*_{I_1})x \underset{P}{\succ} (C_{I_1} + C^*_{I_1})x'.$$

Hence in accordance with Property 2 relation (6) is true.

Lemma 2 is proved.

3 Theorem

Theorem 1. *For any $1 \leq s \leq m$, $1 \leq p \leq \infty$ and any partitioning of the set N_m into s subsets the quasistability radius $\rho^m_p(C, I_1, I_2, \dots, I_s)$ of a vector integer linear programming problem $Z^n(C, I_1, I_2, \dots, I_s)$ is expressed by the formula*

$$\rho^m_p(C, I_1, I_2, \dots, I_s) = \min_{x\in G^m(C, I_1, I_2, \dots, I_s)} \ \min_{x'\in X\setminus\{x\}} \frac{||[C_{I_1}(x' - x)]^+||_p}{||x' - x||_q}. \qquad (7)$$

Proof. The right side of formula (7) we define by φ.

At first we prove the inequality

$$\rho_p^m(C, I_1, I_2, \ldots, I_s) \geq \varphi. \tag{8}$$

Without loss of generality assume that $\varphi > 0$ (otherwise inequality (8) is obvious). From the definition of the number $\gamma(x, x')$ it follows that for any solutions $x \in G^m(C, I_1, I_2, \ldots, I_s)$ and $x' \neq x$ the inequality (4) holds. Taking into account Lemma 1 we obtain $\forall C' \in \Psi_p(\varphi)\ \ \forall x \in G^m(C, I_1, I_2, \ldots, I_s),\ \ \forall x' \in X$

$$(\,(C + C')x\ \overline{\Omega^m(I_1, I_2, \ldots, I_s)}\ (C + C')x'\,),$$

Therefore by virtue of Property 5 any solution $x \in G^m(C, I_1, I_2, \ldots, I_s)$ belongs to the set $G^m(C + C', I_1, I_2, \ldots, I_s)$. Thus we conclude

$$\forall C' \in \Psi_p(\varphi)\ \ (G^m(C, I_1, I_2, \ldots, I_s) \subseteq G^m(C + C', I_1, I_2, \ldots, I_s)),$$

this formula proves (8).

It remained to prove that $\rho_p^m(C, I_1, I_2, \ldots, I_s) \leq \varphi$. Let $\varepsilon > \varphi$ and solutions $x \in G^m(C, I_1, I_2, \ldots, I_s)$ and $x' \neq x$ would be in accordance with (7) such that

$$\varphi||x' - x||_q = ||[C_{I_1}(x' - x)]^+||_p.$$

Then, taking into account continuous dependence of vector metric on its components, we derive that there exists such vector $b = (b_1, b_2, \ldots, b_m)$ with components

$$b_i > [C_i(x' - x)]^+||x' - x||_q^{-1},\ \ i \in I_1,$$

$$b_i = 0,\ \ i \in N_m \setminus I_1,$$

that $\varepsilon > ||b||_p > \varphi$. Then according to Lemma 2 the matrix $C^* \in \Psi_p(\varepsilon)$ exists and condition (6) holds. Hence taking into account (4) $x \notin G^m(C + C^*, I_1, I_2, \ldots, I_s)$. Then

$$\forall \varepsilon > \varphi \quad \exists C^* \in \Psi_p(\varepsilon) \quad (\,G^m(C, I_1, I_2, \ldots, I_s) \not\subseteq G^m(C + C^*, I_1, I_2, \ldots, I_s)\,),$$

which proves $\rho_p^m(C, I_1, I_2, \ldots, I_s) \leq \varphi$.

Note that before similar to (7) formulae had been obtained only for combinatorial (boolean) problems with various kinds of parametrization of the optimality maxima in the cases of l_1 and l_∞ metrics [1–3, 5, 7], and for some game theory problems [8–11].

4 Corollaries

The theorem implies several of corollaries.

If $s = 1$, then the theorem transforms to the following corollary

Corollary 1. *The quasistability radius of a vector integer linear programming problem $Z^m(C, N_m)$, $m \geq 1$, of finding Pareto set $P^m(C)$ is expressed by the formula*

$$\rho_p^m(C, N_m) = \min_{x \in P^m(C)} \min_{x' \in X \setminus \{x\}} \frac{||[C(x'-x)]^+||_p}{||x'-x||_q}, \;\; 1 \leq p \leq \infty.$$

This formula easy transforms into quasistability radius formula of a vector integer linear programming problem in the metric l_∞ [12].

When $s = m$ the theorem transforms to the following corollary

Corollary 2. *For any $m \geq 1$ and $1 \leq p \leq \infty$ the quasistability radius of a vector integer linear programming problem of finding lexicographic optima set $L^m(C)$ is expressed by the formula*

$$\rho_p^m(C, \{1\}, \{2\}, \dots, \{m\}) = \min_{x \in L^m(C)} \min_{x' \in X \setminus \{x\}} \frac{C_1(x'-x)}{||x'-x||_q}.$$

Particular case of this formula is well-known formula of quasistability radius of a vector integer linear programming problem with lexicographic principle of optimality in the case of metric l_∞ [13].

Corollary 3. *For any partitioning $(I_1, I_2, \dots, I_s)$ of the set N_m, into s subsets, $s \in N_m$, the following statements are equivalent for a problem $Z^m(C, I_1, I_2, \dots, I_s))$, $m \geq 1$:*

(i) *the problem $Z^m(C, I_1, I_2, \dots, I_s)$ is quasistable,*

(ii) $\forall\, x \in G^m(C, I_1, I_2, \dots, I_s) \quad \forall\, x' \in X \setminus \{x\} \quad \exists\, i \in I_1 \quad \big(C_i x' > C_i x\big)$,

(iii) $G^m(C, I_1, I_2, \dots, I_s) = S_1(C)$.

Proof. The equivalence of statements (i) and (ii) follows directly from the theorem.

The implication (ii) $\Rightarrow$ (iii) is proved by contradiction. Suppose that (ii) holds but (iii) does not.

From Properties 1 and 6 we obtain

$$S_1(C) \subseteq G^m(C, I_1, I_2, \dots, I_s) \subseteq P_1(A).$$

Then (since $G^m(C, I_1, I_2, \dots, I_s) \neq S_1(C)$ is assumed) there exists such solution $x \in G^m(C, I_1, I_2, \dots, I_s)) \subseteq P_1(A)$ that $x \notin S_1(C)$. It indicates that there exists a solution $x' \in P_1(C)$ such that

$$x' \neq x, \quad C_{I_1} x = C_{I_1} x',$$

which contradicts the statement (ii).

The implication (iii) $\Rightarrow$ (i) is obvious by virtue of Property 7.

Corollary 3 is correct.

From Corollary 3 we easily obtain the following attendant results (see, for example, [14, 15]).

Corollary 4. *The problem $Z^m(C, N_m)$, $m \geq 1$ of finding Pareto set $P^m(C)$ is quasistable if and only if $S^m(C)$ and $P^m(C)$ coincide.*

Here $S^m(C)$ is Smale set [16], i. e. set of strictly efficient solutions:

$$S^m(C) = \{x \in P^m(C) : \forall x' \in X \setminus x \ \ (Cx \neq Cx')\}.$$

From Corollary 3 we also obtain

Corollary 5. *[13]. The problem $Z^m(C, \{1\}, \{2\}, \ldots, \{m\})$, $m \geq 1$, of finding the lexicographically optimal solutions set $L^m(C)$ is quasistable if and only if*

$$|L^m(C)| = |Arg\min\{C_1 x : x \in X\}| = 1.$$

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