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Discrete Optimal Control Problem with Varying Time of States Transactions of Dynamical System and Algorithm for its solving

Dmitrii Lozovanu, Alexandru Lazari

Abstract. We consider time-discrete systems with finite set of states. The starting and the final states of dynamical system are given. The discrete optimal control problem with integral-time cost criterion by a trajectory is studied. An algorithm for solving the problem with varying time of states transactions is proposed. The running time of the proposed algorithm is estimated.

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1 Introduction and Problem Formulation

In this paper we study the discrete optimal control problem with varying time of states transaction of the dynamical system. This problem generalizes the classical optimal control problem with unit time of states transactions [1,2].

The statement of the problem is the following.

Let L be a time discrete system with a finite set of states $X \subseteq R^n$, where at every discrete moment of time $t = 0,\ 1,\ 2,\ \dots$ the state of L is $x(t) \in X$. The starting state $x_0 = x(0)$ and the final state x_f are fixed. Assume that the dynamical system should reach the final state x_f at the time moment $T(x_f)$ such that

$$T_1 \leq T(x_f) \leq T_2$$

where T_1 and T_2 are given. The control of the time-discrete system L at each time-moment $t = 0,\ 1,\ 2,\ \dots$ for an arbitrary state $x(t)$ is made by using the vector of control parameter $u(t)$ for which a feasible set $U_t(x(t))$ is given, i.e. $u(t) \in U_t(x(t))$. In addition we assume that for arbitrary t and $x(t)$ on $U_t(x(t))$ is defined an integer function

$$\tau : U_t(x(t)) \to N$$

which gives to each control $u(t) \in U_t(x(t))$ an integer value $\tau(u(t))$. This value represents the time of system's passage from the state $x(t)$ to the state $x(t+\tau(u(t)))$ if the control $u(t) \in U_t(x(t))$ has been applied at the moment t for given state $x(t)$.

Assume that the dynamics of the system is described by the following system of difference equations

$$\begin{cases} t_{j+1} & = & t_j + \tau(u(t_j)), \\ x(t_{j+1}) & = & g_{t_j}(x(t_j),\ u(t_j)), \\ & & u(t_j) \in U_{t_j}(x(t_j)), \\ & & j = 0,\ 1,\ 2,\ \dots\ , \end{cases} \tag{1}$$

where

$$t_0 = 0,\ x(t_0) = 0 \tag{2}$$

is a starting representation of the dynamical system L.

We suppose that the functions g_t and τ in (1) are known and t_{j+1} and $x(t_{j+1})$ are determined uniquely by $x(t_j)$ and $u(t_j)$ at every step $j = 0,\ 1,\ 2,\ \dots$.

Let $u(t_j),\ j = 0,\ 1,\ 2,\ \dots,$ be a control, which generates the trajectory

$$x(0),\ x(t_1),\ x(t_2),\ \dots x(t_k),\ \dots\ .$$

Then either this trajectory passes trough the final state x_f and $T(x_f) = t_k$ represents the time-moment when the final state x_f is reached or this trajectory does not pass trough x_f.

For an arbitrary control we define the quantity

$$F_{x_0,\ x_f}(u(t)) = \sum_{j=0}^{k-1} c_{t_j}(x(t_j),\ g_{t_j}(x(t_j),\ u(t_j))) \tag{3}$$

if the trajectory

$$x(0),\ x(t_1),\ x(t_2),\ \dots x(t_k),\ \dots$$

passes through the final state x_f i.e. $T(x_f) = t_k$; otherwise we put

$$F_{x_0,\ x_f}(u(t)) = \infty.$$

Here $c_{t_j}(x(t_j),\ g_{t_j}(x(t_j),\ u(t_j))) = c_{t_j}(x(t_j),\ x(t_{j+1}))$ represents the cost of system's passage from the state $x(t_j)$ to the state $x(t_{j+1})$ at the stage $[j,\ j+1]$.

We consider the following control problem:

Problem 1. To find time-moments $t_0 = 0,\ t_1,\ t_2,\ \dots,\ t_k$ and vectors of control parameters $u(t_0),\ u(t_1),\ u(t_2),\ \dots,\ u(t_{k-1})$ which satisfy conditions (1), (2) and minimize functional (3).

In the following we develop a mathematical tool for solving this problem. We show that a simple modification of time expanded network method from [3–5] allows to elaborate efficient algorithm for solving the considered problem.

2 Algorithm for solving the problem based on Dynamic Programming and Time-Expanded Network method

Here we develop the dynamic programming algorithm for solving Problem 1 in the case when T is fixed, i.e. $T_1 = T_2 = T$. The proposed algorithm can be argued in the same way as the algorithm from Section 1.

We denote by $F^*_{x_0,x(t_k)}$ the minimal integral-time cost of system's passage from the starting state $x_0 = x(0)$ to the state $x = x(t_k) \in X$ by using exactly t_k units of time. So,

$$F^*_{x_0,x(t_k)} = \sum_{j=0}^{k-1} c_{t_j}(x^*(t_j),\ g_{t_j}(x^*(t_j),\ u^*(t_j)))$$

where

$$x(0) = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_{k-1}),\ x^*(t_k)$$

is the optimal trajectory from $x_0 = x^*(0)$ to $x^*(t_k)$, generated by optimal control

$$u^*(0),\ u^*(t_1),\ u^*(t_2),\ \ldots,\ u^*(t_{k-1})$$

where

$$t_0 = 0;$$

$$t_{j+1} = t_j + \tau(u^*(t_j)),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1.$$

If for given $x \in X$ there is no trajectory from x_0 to $x = x(t_k)$ such that x may be reached by using t_k units of time then we put $F^*_{x_0,\ x(t_k)} = \infty$.

For $F^*_{x_0,\ x(t_k)}$ the following recursive formula can be gained:

$$F^*_{x_0x(t_j)} = \begin{cases} \min\limits_{x(t_{j-1}) \in X^-(x(t_j))} \left\{ F^*_{x_0x(t_{j-1})} + c_{t_{j-1}}(x(t_{j-1}),\ x(t_j)) \right\} & \text{if } X^-(x(t_j)) \neq \emptyset, \\ \infty & \text{if } X^-(x(t_j)) = \emptyset, \end{cases}$$

$$j = 1,\ 2,\ \ldots,$$

where

$$t_0 = 0,$$

$$F^*_{x_0x(0)} = \begin{cases} 0 & if\ x(0) = x_0, \\ \infty & if\ x(0) \neq x_0 \end{cases}$$

and

$$X^-(x(t_j)) = \{x(t_{j-1}) \in X \mid x(t_j) = g_{t_{j-1}}(x(t_{j-1}),\ u(t_{j-1})),$$

$$t_j = t_{j-1} + \tau(u(t_{j-1})),\ u(t_{j-1}) \in U_{t_{j-1}}(x(t_j))\}.$$

If $F^*_{x_0x(t)}$, $t = 0,\ 1,\ 2,\ \ldots,\ T$, are known then the optimal control

$$u^*(0),\ u^*(t_1),\ u^*(t_2),\ \ldots,\ u^*(t_{k-1})$$

and the optimal trajectory

$$x(0) = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_{k-1}),\ x(t_k) = x(T)$$

from x_0 to x_f can be found in the following way.

Find $t_{k-1},\ u^*(t_{k-1})$ and $x^*(t_{k-1}) \in X^-(x(t_k))$ such that

$$F^*_{x_0 x^*(t_k)} = F^*_{x_0 x^*(t_{k-1})} + c_{t_{k-1}}(x^*(t_{k-1}),\ g_{t_{k-1}}(x^*(t_{k-1}),\ u^*(t_{k-1}))),$$

where $t_k = t_{k-1} + \tau(u^*(t_{k-1}))$.

After that find $t_{k-2},\ u^*(t_{k-2})$ and $x^*(t_{k-2}) \in X^-(x_{t_{k-1}})$ such that

$$F^*_{x_0 x^*(t_{k-1})} = F^*_{x_0 x^*(t_{k-2})} + c_{t_{k-2}}(x^*(t_{k-2}),\ g_{t_{k-2}}(x^*(t_{k-2}),\ u^*(t_{k-2}))),$$

where $t_{k-1} = t_{k-2} + \tau(u^*(t_{k-2}))$.

Using $k-1$ steps we find the optimal control $u^*(0),\ u^*(t_1),\ u^*(t_2),\ \ldots,\ u^*(t_{k-1})$ and the trajectory $x(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_{k-1}),\ x(t_k) = x(T)$.

In order to argue the algorithm we shall use time-expanded network with a simple modification. First we ground the algorithm when $T_2 = T_1 = T$ and then we show that the general case of the problem with $T_2 > T_1$ can be reduced to the case with fixed T.

Assume that $T_2 = T_1 = T$ and construct a time-expanded network with the structure of acyclic directed graph $\overline{G} = (Y,\ \overline{E})$ where Y consists of $T+1$ copies of the set of states X corresponding to the time moments $t = 0,\ 1,\ 2,\ \ldots,\ T$. So,

$$Y = Y^0 \cup Y^1 \cup Y^2 \cup \ldots \cup Y^T \quad (Y^t \cap Y^l = \varnothing,\ t \neq l),$$

where $Y^t = (X,\ t)$ corresponds to the set of states of dynamical system at the time moment $t = 0,\ 1,\ 2,\ \ldots,\ T$. This means that $Y^t = \{(x,\ t) \mid x \in X\},\ t = 0,\ 1,\ 2,\ \ldots,\ T$ the graph $\overline{G}$ is represented in Fig. 1, where at each moment of time $t = 0,\ 1,\ 2,\ \ldots,\ T$ we can see all copies of vertex set X.

We define the set of edges $\overline{E}$ of the graph $\overline{G}$ in the following way.

If at given moment of time $t_j \in [0,\ T]$ for given state $x = x(t_j)$ of dynamical system there exists a vector of control parameters $u(t_j) \in U_{t_j}(x(t_j))$ such that

$$z = x(t_{j+1}) = g_{t_j}(x(t_j),\ u(t_j)),$$

where

$$t_{j+1} = t_j + \tau(u(t_j)),$$

then $((x,\ t_j),\ (z,\ t_{j+1})) \in \overline{E}$, i.e. in $\overline{G}$ we connect the vertex $y_j = (x,\ t_j) \in Y^{t_j}$ with the vertex $y_{j+1} = (z,\ t_{j+1})$ (see Fig. 1). To this edge $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1}))$ we associate in $\overline{G}$ a cost $c_{\overline{e}} = c_{t_j}(x(t_j),\ x(t_{j+1}))$.

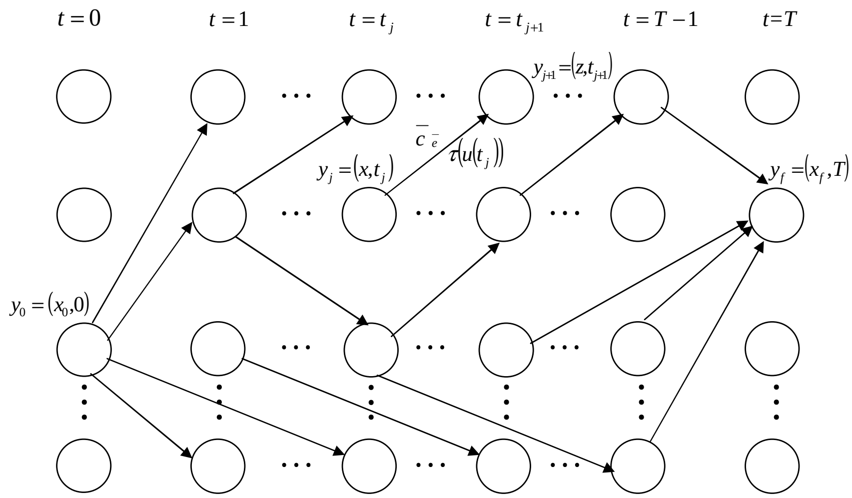

Figure 1.

The following lemma holds

Lemma 1. *Let* $u(t_0),\ u(t_1),\ u(t_2),\ \dots,\ u(t_{k-1})$ *be a control of the dynamical system in Problem 1, which generates a trajectory*

$$x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \dots,\ x(t_k) = x_f$$

from x_0 *to* x_f*, where*

$$t_0 = 0,\ t_{j+1} = t_j + \tau(u(t_j)),\ j = 0,\ 1,\ 2,\ \dots,\ k-1;$$
$$u(t_j) \in U_t(x(t_j)),\ j = 0,\ 1,\ 2,\ \dots,\ k-1;$$
$$t_k = T.$$

Then in $\overline{G}$ *there exists a directed path*

$$P_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x_1,\ t_1),\ (x_2,\ t_2),\ \dots,\ (x_k,\ T) = y_f\}$$

from y_0 *to* y_f*, where*

$$x_j = x(t_j),\ j = 0,\ 1,\ 2,\ \dots,\ k;$$

and $x(t_k) = x_f$*, i.e.* $t(x_f) = t_k = T$*. So, between the set of states of the trajectory* $x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \dots,\ x(t_k) = x_f$ *and the set of vertices of directed path* $P_{\overline{G}}(y_0,\ y_f)$ *there exists a bijective mapping*

$$(x_j,\ t_j) \Leftrightarrow x(t_j),\ j = 0,\ 1,\ 2,\ \dots,\ k,$$

such that $x_j = x(t_j), j = 0,\ 1,\ 2,\ \dots,\ k$*, and*

$$\sum_{j=0}^{k-1} c_{t_j}(x(t_j),\ x(t_{j+1})) = \sum_{j=0}^{k-1} \overline{c}_{(x_j,\ t_j),\ (x_{j+1},\ t_{j+1})}(t_j),$$

where $t_0 = 0,\ x_0 = x(t_0)$*, and* $x_f = x(t_k),\ t_k = T$*.*

Proof. In Problem 1 an arbitrary control $u(t_j)$ for given state $x(t_j) \in U_{t_j}(x(t_j))$ at given moment of time t_j uniquely determines the next state $x(t_{j+1})$. So, $u(t_j)$ can be identified with a unique passage $(x(t_j),\ x(t_{j+1}))$ from the state $x(t_j)$ to the state $x(t_{j+1})$. In $\overline{G} = (Y,\ \overline{E})$ this passage corresponds to a unique directed edge $((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))$ which connects vertices $(x_j,\ t_j)$ and $(x_{j+1},\ t_{j+1})$; the cost of this edge is $\overline{c}((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))(t_j) = c_{t_j}(x(t_j),\ x(t_{j+1}))$. This one-to-one correspondence between the control $u(t_j)$ at given moment of time and the directed edge $\overline{e} = ((x_j,\ t_j),\ (x_{j+1},\ t_{j+1})) \in \overline{E}$ implies the existence of bijective mapping between the set of trajectories from the starting state x_0 to the final state x_f in Problem 1 and the set of directed paths from y_0 to y_f in $\overline{G}$, which preserves the integral-time costs. □

Corollary. *If $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$ is the optimal control of the dynamical system in Problem 1, which generates a trajectory*

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f$$

from x_0 to x_f, then in $\overline{G}$ the corresponding directed path

$$P^*_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x_1^*,\ t_1),\ (x_2^*,\ t_2),\ \ldots,\ (x_k^*,\ t_k) = y_T\}$$

is the minimal integral cost directed path from y_0 to y_f and vice-versa.

On the basis of the results mentioned above we can propose the following algorithm for solving Problem 1.

Algorithm. Determining the optimal solution to Problem 1 based on the time-expanded network method

1. We construct the auxiliary time-expanded network consisting of directed acyclic graph $\overline{G} = (Y,\ \overline{E})$, cost function $\overline{c}:\ \overline{E} \to R^1$ and given starting and final vertices y_0 and y_f.
2. Find in G the directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from starting vertex y_0 to final vertex y_f with minimal sum of edge's costs.
3. We determine the control $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$, which corresponds to directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from y_0 to y_f. Then $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$, is a solution to Problem 1.

This algorithm finds the solution to the control problem with fixed time $T(x_f) = T$ of system's passage from starting state to final one. In the case $T_1 \leq T(x_f) \leq T_2$ $(T_2 > T_1)$ Problem 1 can be solved by its reducing to $T_2 - T_1 + 1$ problems with $T = T_1,\ T_1 + 1,\ \ldots,\ T_2$ and finding the best solution to these problems.

In general, if we construct the auxiliary acyclic directed graph $\overline{G} = (Y,\ \overline{E})$ with $T = T_2$ then in $\overline{G}$ the tree of optimal path from starting vertex $y_0 = (x_0,\ 0)$ to an arbitrary vertex $y = (x,\ t) \in Y$ can be found. This tree allows us to find the solution to the control problem with given starting state and an arbitrary state $x = x(t)$ with $t = 0,\ 1,\ 2,\ \dots,\ T_2$; in particular the solution to Problem 1 with $T_1 \le T(x_f) \le T_2$ can be obtained.

Denote by $GT^*_{y_0} = (Y^*,\ E^*_{y_0})$ the tree of optimal directed paths with root vertex $y_0 = (x_0,\ 0)$, which gives all optimal directed paths from y_0 to an arbitrary attainable directed vertex $y = (x,\ t) \in Y$. As we have noted this tree allows us to find in the control problem all optimal trajectory from starting state $x_0 = x(0)$ to an arbitrary reachable state $x = x(t)$ at given moment of time $t \in [0,\ T]$.

In G we can also find the tree of optimal directed paths $GT^0_{y_f} = (Y^0,\ E^0_{y_0})$ with sink vertex $y_f = (x_f,\ T)$, which gives all possible optimal directed paths from an arbitrary $y = (x,\ t) \in Y$ to sink vertex $y_f = (x_f,\ T)$. This mean that in the control problem we can find all possible optimal trajectories with starting state $x = x(t)$ at given moment of time $t \in [0,\ T]$ to the final state $x_f = x(T)$.

If the trees $GT^*_{y_0} = (Y^*,\ E^*_{y_0})$ and $GT^0_{y_f} = (Y^0,\ E^0_{y_f})$ are known then we can solve the following control problem:

To find an optimal trajectory from starting state $x_0 = x(0)$ to final state $x_f = x(T)$ such that the trajectory passes at the given moment of time $t \in [0,\ T]$ trough the state $x = x(t)$.

Finally we note that Algorithm can be simplified if we delete from $\overline{G}$ all vertices $y \in Y$ which are not attainable from y_0 and vertices $y \in Y$ for which does not exist a directed path from y to y_f. So, we should solve the auxiliary problem on a new graph $\overline{G}^0 = (\overline{Y}^0,\ \overline{E}^0)$ which is a subgraph of $\overline{G} = (Y,\ E)$.

3 The Discrete Control Problem with Cost Function of System's Passages that Depend on Transit-Time of States Transactions

In the control model from Section 1 the cost function

$$c_t(x(t),\ g_t(x(t),\ u(t))) = c_t(x(t),\ x(t+1))$$

of system's passage from the state $x = x(t)$ depends on the vector of control parameters $u(t)$. In general we may consider that the cost function of system's passage from the state $x(t)$ to state $x(t+1)$ depends also on transit-time $\tau(t)$, i.e. the cost function $c_t(x(t),\ g_t(x(t),\ u(t)),\ \tau(t)) = c_\tau(x(t),\ x(t+1),\ \tau(t))$ depends on $t,\ x(t),\ u(t)$ and $\tau(t)$.

It is easy to observe that the problem in such a general form can be solved in analogous way as the problem from Section 1 by using Algorithm with a simple modification. In the auxiliary time-expanded network the cost functions $\overline{c}_{\overline{e}}$ on edges $\overline{e}$ should be defined as follows:

$$\overline{c}_{\overline{e}} = c_{t_j}(x(t_j),\ x(t_{j+1}),\ \tau(u(t_j)))$$

So, the problem with cost functions of system's passage that depend on transit-time of states transactions can be solved by using Algorithm with the cost functions on time-expanded network defined above.

4 The Control Problem on Network with Transit-Time Functions on Edges

We extend the control problem on network from Section 4.1 by introducing the transit-time functions of states transaction on edges.

4.1 Problem Formulation

Let be given the dynamical system L with finite set of states X and given starting point $x_0 = x(0)$. Assume that system L should be transferred into the state x_f at the time moment T such that $T_1 \le T(x_f) \le T_2$, where T_1 and T_2 are given. We consider the control problem for which the dynamics of the system is described by directed graph $G = (X,\ E)$, where the vertices $x \in X$ correspond to the states and an arbitrary edge $e = (x,\ y) \in E$ means the possibility of the system to pass from the state x to the state y at every moment of time t. To each edge $e = (x,\ y) \in E$ is associated a transit function $\tau_e(t)$ of system's passage from the state $x = x(t)$ to the state y. This means that if at the time-moment t the system L starts to pass from the state $x = x(t)$ trough an edge $e = (x,\ y)$ then the state y is reached at the time-moment $t + \tau_e(t)$, i.e. $y = x(t + \tau_e(t))$. In addition to each edge $e(x,\ y) \in E$ is associated a cost function $c_e(t)$ that depends on time and which expresses the cost system's passage from the state $x = x(t)$ to the state $y = x(t + \tau_e(t))$.

The control on G with given transit-time functions τ_e on edges $e \in E$ is made in the following way.

For given starting state x_0 we fix $t_0 = 0$. Then select an directed edge $e_0 = (x_0,\ x_1)$ through which we transfer the system L from the state $x_0 = x(t_0)$ to the state $x_1 = x(t_1)$ at the moment of time t_1, where $t_1 = t_0 + \tau_{e_0}(0)$. If $x_1 = x_f$ then stop; otherwise we select an edge $e_1 = (x_1,\ x_2)$ and transfer the system L from the state $x_1 = x(t_1)$ at the moment of time t_1 to the state $x_2 = x(t_2)$ at the time moment $t_2 = t_1 + \tau_{e_1}(t_1)$. If $x_2 = x_f$ then stop; otherwise select an edge $e_2 = (x_2,\ x_3)$ and so on. In general, at the time moment t_{k-1} we select an edge $e_{k-1} = (x_{k-1},\ x_k)$ and transfer the system L from the state $x_{k-1} = x(t_{k-1})$ to the state $x_k = x(t_k)$ at the time-moment $t_k = t_{k-1} + \tau_{e_k}$. If $x_k = x_f$ then the integral-time cost of system passage from x_0 to x_f is

$$F_{x_0 x_k}(t_k) = \sum_{j=0}^{k-1} c_{(x(t_j),\ x(t_{j+1}))}(t_j).$$

So, at the time moment t_k the system L is transferred in the state $x_k = x_f$ with the integral-time cost $F_{x_0 x_f}(t_k)$. If $T \le t_k \le T_2$, we obtain an admissible control with $t_k = T(x_f)$ and integral-time cost $F_{x_0 x_f}(T(x_f))$.

We consider the following problem:

Problem 2. To find a sequence of system's transactions

$$(x_j,\ x_{j+1}) = (x(t_j),\ x(t_{j+1})), \quad t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j), \quad j = 0,\ 1,\ 2,\ \dots,\ k-1,$$

which transfer the system L from starting vertex (state) $x_0 = x(t_0),\ t_0 = 0$, to final vertex (state) $x_f = x_k = x(t_k)$ such that

$$T \le t_k \le T_2$$

and the integral-time cost

$$F_{x_0 x_f}(t_k) = \sum_{j=0}^{k-1} c_{(x_j,\ x_{j+1})}(t_j)$$

of system's transactions by a trajectory

$$x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \ldots,\ x(t_k) = x_f$$

is minimal.

4.2 Algorithm for Solving the Problem on Network with Transit-Time Functions on Edges

The algorithm from Section 2 can be specified for solving the control problem on the network with transit-time functions on the edges. Assume that $T_2 = T_1 = T$ and describe the details of the algorithm for the control problem on G.

We denote by $F^*_{x_0 x}(t_k)$ the minimal integral-time cost of system transactions from the starting state x_0 to the final state $x = x^*(t_k)$ by using t_k units of time, i.e.

$$F^*_{x_0 x}(t_k) = \sum_{j=0}^{k-1} c_{(x(t_j),\ x(t_{j+1}))}(t_j),$$

where

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f$$

is an optimal trajectory from x_0 to x_f, where

$$t_{j+1} = t_j + \tau_{(x(t_j),\ x(t_{j+1}))}(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1.$$

It is easy to observe that the following recursive formula for $F^*_{x_0 x}(t_k)$ holds:

$$F^*_{x_0 x(t_j)}(t_j) = \min_{x(t_{j-1}) \in X^-_G(x(t_j))} \{F^*_{x_0 x(t_{j-1})}(t_{j-1}) + c_{(x(t_{j-1}),\ x(t_j))}(t_{j-1})\},$$

where

$$X^-_G(x(t_j)) = \{x = x(t_{j-1}) \mid (x(t_{j-1}),\ x(t_j)) \in E,\ t_j = t_{j-1} + \tau_{(x(t_{j-1}),\ x(t_j))}(t_{j-1})\}.$$

This means that if we start with $F^*_{x_0 x(0)}(0) = 0,\ F^*_{x_0 x(t)}(t) = \infty,\ t = 1,\ 2,\ \ldots,\ t_k$, then on the basis of the recursive formula given above we can find $F^*_{x_0 x(t)}(t)$ for $t = 0,\ 1,\ 2,\ \ldots,\ t_k$ for an arbitrary vertex $x = x(t)$. After that the optimal trajectory

$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f$ from x_0 to x_f can be found in the following way.

Fix the vertex $x^*_{k-1} = x^*(t_{k-1})$ for which

$$F^*_{x_0 x^*(t_{k-1})}(t_{k-1}) + c_{(x(t_{k-1}),\ x^*(t_k))}(t_{k-1}) =$$

$$= \min_{x(t_{k-1}) \in X^-_G(x^*(t_k))} \{F^*_{x_0 x(t_{k-1})}(t_{k-1}) + c_{(x(t_{k-1}),\ x^*(t_k))}(t_{k-1})\}.$$

Then we find the vertex $x^*(t_{k-2})$ for which

$$F^*_{x_0 x^*(t_{k-2})}(t_{k-2}) + c_{(x(t_{k-2}),\ x^*(t_{k-1}))}(t_{k-2}) =$$

$$= \min_{x(t_{k-2}) \in X^-_G(x^*(t_{k-1}))} \{F^*_{x_0 x(t_{k-2})}(t_{k-2}) + c_{(x(t_{k-2}),\ x^*(t_{k-1}))}(t_{k-2})\}.$$

After that we fix the vertex $x^*(t_{k-3})$ for which

$$F^*_{x_0 x^*(t_{k-3})}(t_{k-3}) + c_{(x(t_{k-3}),\ x^*(t_{k-2}))}(t_{k-3}) =$$

$$= \min_{x(t_{k-3}) \in X^-_G(x^*(t_{k-2}))} \{F^*_{x_0 x(t_{k-3})}(t_{k-3}) + c_{(x(t_{k-3}),\ x^*(t_{k-2}))}(t_{k-3})\}$$

and so on.

Finally we find the optimal trajectory

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f.$$

This algorithm also can be grounded on the basis of the time-expanded network method.

We give the construction which allows to reduce our problem to auxiliary one on time-expanded network. The structure of this time-expanded network corresponds to an directed graph $\overline{G} = (Y,\ \overline{E})$ without directed cycles. The set of vertices Y of $\overline{G}$ consists of $T+1$ copies of the set of vertices (states) X of the graph G corresponding to time-moments $t = 0,\ 1,\ 2,\ \ldots,\ T$, i.e.

$$Y = Y^0 \cup Y^1 \cup Y^2 \cup\ \ldots \cup Y^T \quad (Y^t \cap Y^l = \varnothing,\ t \neq l),$$

where $Y^t = (X,\ t)$. So, $Y^t = \{(x,\ t) \mid x \in X\}, \quad t = 0,\ 1,\ 2,\ \ldots,\ T$.

We define the set of edges $\overline{E}$ of the graph $\overline{G}$ as follows: $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1})) \in \overline{E}$ if only if in G there exists a directed edge $e = (x,\ y) \in E$, where $x = x(t_j),\ z = x(t_{j+1})$, $t_{j+1} = t_j + \tau_e(t_j)$. So, in $\overline{G}$ we connect vertices $(x,\ t_j)$ and $(z,\ t_{j+1})$ with directed edge $(x(t_j),\ (z,\ t_{j+1})) \in \overline{G}$; to edge $\overline{e}$ we associate the cost $\overline{c}_{\overline{e}} = c_{(x,z)S}(t_j)$, i.e. $\overline{c}_{((x,\ t_j),\ (z,\ t_{j+1}))} = c_{(x,\ z)}(t_j)$.

On $\overline{G}$ we consider the problem of finding the directed path from $y_0 = (x_0,\ 0)$ to $y_f = (x_f,\ T)$ with minimum sum of edges cost. Basing on results from Section 2 we obtain the following result.

Lemma 2. *Let*

$$(x_j,\ x_{j+1}) = (x(t_j),\ x(t_{j+1})),\quad t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$$

be a sequence of system's transactions from the state $x_0 = x(t_0),\ t_0 = 0,$ *to the state* $x_f = x_k = x(t_k),\ t_k = T$. *Then in* $\overline{G} = (Y,\ \overline{E})$ *there exists the directed path*

$$P_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x_1,\ t_1),\ (x_2,\ t_2),\ \ldots,\ (x_k,\ T) = y_f\},$$

from y_0 *to* y_f, *where*

$$x_j = x(t_j),\quad j = 0,\ 1,\ 2,\ \ldots,\ k\quad (t_k = T).$$

So, between the set of vertices $\{x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \ldots,\ x(t_k) = x_f\}$ *and the set of vertices of directed path* $P_{\overline{G}}(y_0,\ y_f)$ *there exists a bijective mapping*

$$(x_j,\ t_j) \Leftrightarrow x(t_j) = x_j,\quad j = 0,\ 1,\ 2,\ \ldots,\ k,$$

such that $x_j = x(t_j),\quad j = 0,\ 1,\ 2,\ \ldots,\ k,$ *and*

$$\sum_{j=0}^{k-1} c_{(x_j,\ x_{j+1})}(t_j) = \sum_{j=0}^{k-1} \overline{c}_{((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))},$$

where $t_0 = 0,\ t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1.$

This lemma follows as a corollary from Lemma 1.

The algorithm for solving the control problem on G is similar to Algorithm from Section 2. So, the control problem on G can be solved in the following way.

Algorithm

1. We construct the network consisting of an auxiliary graph $\overline{G} = (Y,\ \overline{E})$, cost function $\overline{c}:\ \overline{E} \to R$ and given starting and final states $y_0 = (x_0,\ 0),\ y_f = (x_f,\ t)$.

2. Find in G the directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from y_0 to y_f with minimal sum of edges cost.

3. Determine the vertices $x_j = x(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k$, which correspond to vertices $(x_j,\ t_j)$ of a directed path $P^*_G(y_0,\ y_f)$ from y_0 to y_f. Then $x_0 = x(0),\ x_1 = x(t_1),\ x_2 = x(t_2),\ \ldots,\ x_k = x(t_k) = x_f$ represent the optimal trajectory from x_0 to x_f in the control problem G.

Remark 1. Algorithm can be modified for solving the optimal control problem on a network when the cost function on edges $e \in E$ depends not only on time t but also depends on transit-time $\tau_e(t)$ of system's passage trough the edge $e = (x(t),\ x(t + \tau_e(t)))$. So, Algorithm can be used for solving the control problem when to each edge $e = (x,\ z) \in E$ is given a cost function $c_{(x,\ z)}(t,\ \tau_{(x,\ z)}(t))$ that depends on time t and on transit-time $\tau_{(x,\ z)}(t)$. The modification of the algorithm for solving the control problem on network in such general form can be made in the same way as the modification of Algorithm 1 for the problem from Section 3 This means that the cost functions $\overline{c}_e$ on the edges $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1}))$ of the graph $\overline{G}$ should be defined as follows:

$$\overline{c}_{((x,\ t_j),\ (z,\ t_{j+1}))} = c_{(x,\ z)}(t_j,\ \tau_{(x,\ z)}(t_j)).$$

Remark 2. Algorithm can be simplified if we delete from $\overline{G}$ all vertices $y \in Y$ which are not attainable from y_0 and all vertices for which there is no directed path from y to x_f. So, the problem may be solved on a simplified graph $\overline{G}^0 = (\overline{Y}^0,\ \overline{E}^0)$.

The proposed approach for studying and solving discrete optimal control problems can be developed also for the multi-objective control problems from [3–6].

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, Chişinău MD−2028
Moldova

E-mail: *lozovanu@math.md*, *lazarialexandru@mail.md*

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