BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(57), 2008, Pages 35–45
ISSN 1024–7696

Non-fundamental 2-isohedral tilings of the sphere *

Elizaveta Zamorzaeva

Abstract. The investigation of 2-isohedral tilings of the 2-dimensional sphere is continued. In previous works all the fundamental 2-isohedral tilings of the sphere have been enumerated. Here non-fundamental 2-isohedral tilings of the sphere are obtained from the fundamental ones using the method of gluing disks. For all the 7 countable series of isometry groups of the sphere the classification of normal non-fundamental tilings is given in a table of pictures. For non-normal tilings only numerical results are given. For the 7 separate isometry groups of the sphere numerical results are also shown.

Mathematics subject classification: 52C20.
Keywords and phrases: Tilings of the sphere, 2-isohedral tilings, non-fundamental tilings.

1 Introduction

There are a lot of works where tilings of the two-dimensional sphere with transitivity properties are investigated. For survey of earlier results and the classification of isohedral, isogonal and isotoxal tilings of the sphere see [1]. In [2] 2-isotoxal tilings of the sphere have been researched. General methods of finding k-isohedral ($k \geq 2$) tilings of a two-dimensional constant curvature space (i. e. the Euclidean plane, the sphere and the hyperbolic plane) that use the known isohedral tilings have been developed by the author in [3]. Applying these methods the author has obtained the complete classification of 2-isohedral tilings of the sphere in some tables of pictures. A part of these results (and namely, normal fundamental 2-isohedral tilings of the sphere) can be found in [4, 5]. Note that the same methods using Delaney–Dress symbols have been implemented [6] in algorithms and computer programmes, the numerical results of [6] concerning 2-isohedral tilings of the sphere coincide with the ours. In the present paper we show the classification of normal non-fundamental tilings of the sphere for all 7 infinite series of isometry groups of the 2-dimensional sphere.

Consider a tiling W of the 2-dimensional sphere by topological disks and a discrete isometry group G of the sphere. The tiling W is called k-isohedral with respect to the group G if G maps W onto itself and all the tiles of G form exactly k transitivity classes under the group G. Two pairs (W, G) and (W', G') are said to be of the same Delone class (or equivariant type [6]) if there exists a homeomorphism φ of the

*The work was partially supported by bilateral grant RFBR – ASM of HCSTD ASM 06.34 CRF.

sphere with $\varphi(W) = W'$ and $G' = \varphi G \varphi^{-1}$. The above and some below concepts and definitions hold for all three 2-dimensional spaces of constant curvature (see [3, 7]).

In a tiling of the sphere by disks a vertex (an edge) is defined as a connected component of the intersection of two or more different disks which is (is not) a single point. A Delone class (W, G) is called $(h_1, h_2, \ldots, h_k)$-transitive if the group G acts h_i times transitively on the i-th class of tiles of the tiling W, $i = 1, 2, \ldots, k$. If $h_1 = h_2 = \cdots = h_k = 1$, the Delone class (W, G) is called fundamental, otherwise non-fundamental.

For the description of discrete isometry groups of the 2-dimensional sphere we use here the Conway's orbifold symbol, which is equivalent to the Macbeath's group signature. Remind the explanation of the orbifold symbol as it is given in [8]. Let X be one of the three 2-dimensional spaces of constant curvature and G be an isometry group of X with a compact fundamental domain. Consider the quotient $M = X/G$, which is a compact 2-dimensional manifold, maybe with boundary. Any point $x \in X$ with the non-trivial stabilizer group $G_x = \{g \in G | g(x) = x\}$ gives rise to a cone point $(\bar{x}, v)$ of degree v if G_x is a rotation group of order v, or to a corner point of degree v if G_x is a dihedral group generated by a v-fold rotation and a reflection. Here $\bar{x}$ denotes the equivalence class of points containing x. The orbifold symbol can be obtained by specifying the following four items:

(O1) The number of handles h if M is orientable, or the number of cross-caps k otherwise.

(O2) The system of branching numbers for all cone points.

(O3) The number of boundary components q.

(O4) For each boundary component B one must list the branching numbers of all corner points lying on B in a cyclic order. If M is orientable one must list the corner points of each boundary component in the order induced by a fixed orientation of the underlying manifold.

The rules for writing down the orbifold symbol are the following: First, if M is orientable with h handles one writes h small circles: $\circ \circ \circ \cdots$. Second, the branching numbers for all cone points are listed. Next, each boundary component is indicated by a star: $*$, followed by the list of branching numbers encountered while going around the boundary component. Finally, if M is non-orientable with k cross-caps one writes k crosses: $\times \times \times \cdots$.

There are 7 countable series of isometry groups of the sphere given by the following orbifold symbols: nn, $n\times$, $n*$, $*nn$, $22n$, $2*n$, $*22n$ where $n = 1, 2, \ldots$. Also there are 7 separate isometry groups of the sphere with the following orbifold symbols: 322, $3*2$, $*332$, 432, $*432$, 532, $*532$.

A tiling of the sphere is called normal if it satisfies the following conditions [1]:

SN1. Each tile is a topological disk.

SN2. The intersection of any sets of tiles is a connected (possibly empty) set.

SN3. Each edge of the tiling has two endpoints which are vertices of the tiling.

In a normal tiling every tile contains at least three edges on its boundary and the valence of each vertex is at least three. The works [4, 5] contain the complete enumeration of fundamental 2-isohedral tilings on the sphere, where the normal tilings are shown in figures, for the rest of tilings (both without digons and containing digons) the numerical results are given.

Now all the fundamental Delone classes of 2-isohedral tilings on the sphere by disks are known. The method of finding non-fundamental Delone classes is the same as the method proposed in [9] for finding non-fundamental Delone classes of isohedral tilings on the Euclidean plane.

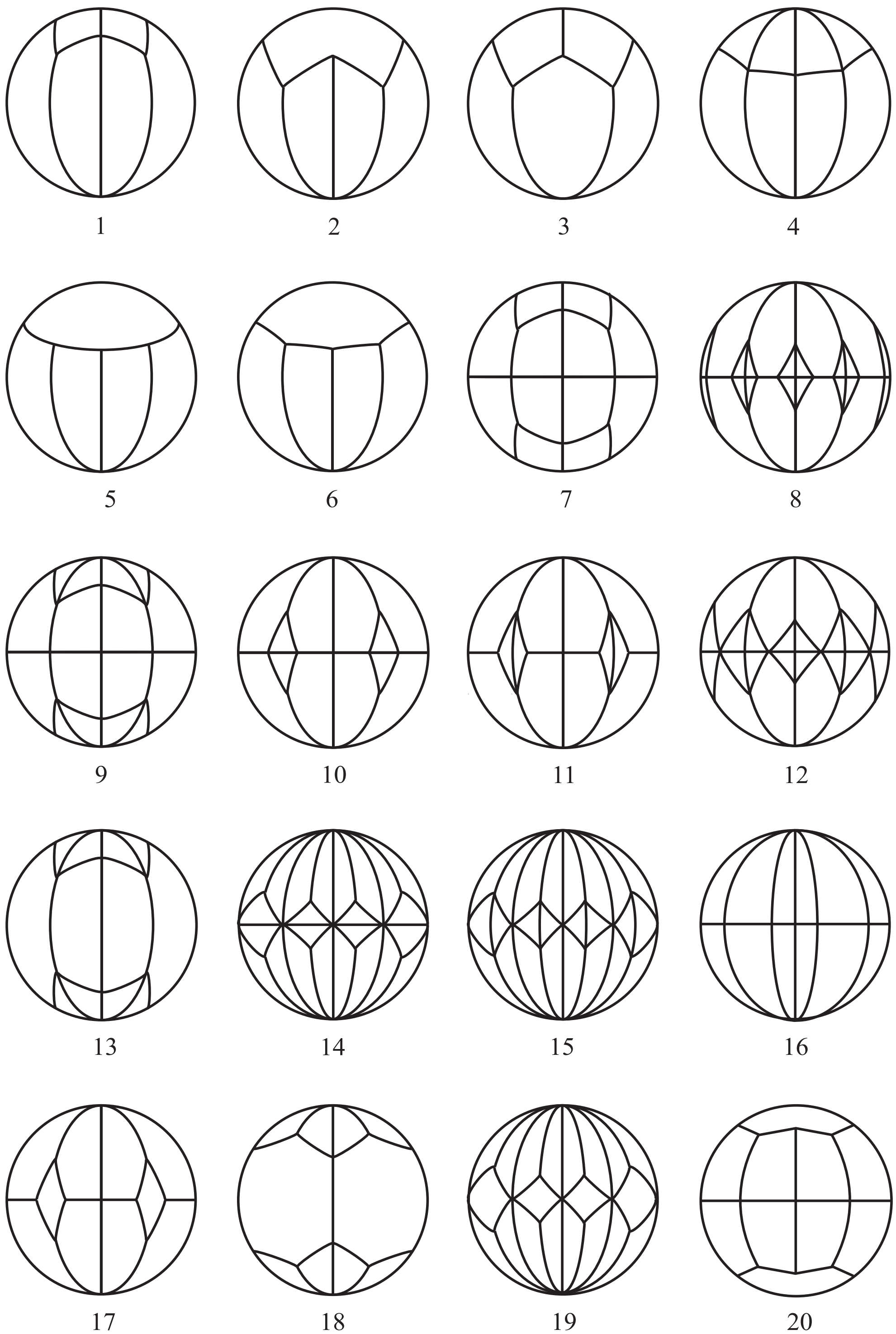
Let W be a tiling of the sphere by disks which is 2-isohedral with respect to a fundamental isometry group G. Let O be a vertex or the midpoint of an edge of the tiling W. If the order h of the stabilizer group $G_0 \subset G$ coincides with the number of tiles from W that contain O, we say the point O is good for gluing. Then glue (unite) all these tiles yielding a new disk. Do such a gluing at each point from the orbit $\{O_G\}$. As a result we obtain a new 2-isohedral tiling W' of the sphere by disks, the group G acts h times transitively on the set of new (glued) disks, so the Delone class of the pair (W', G) is non-fundamental.

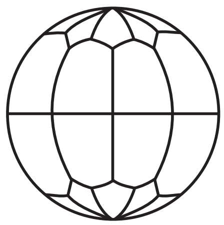
Applying the gluing method to all the fundamental Delone classes of 2-isohedral tilings on the sphere by disks the author has obtained all the possible non-fundamental Delone classes of 2-isohedral tilings on the sphere by disks. Because of a large number of the resulted Delone classes, here we give pictures for normal tilings and numerical data for non-normal tilings. Besides, in the present paper we show pictures only for 7 infinite series of isometry groups, one representative tiling from each series of Delone classes is drawn (for either $n = 4$ or $n = 8$). As to the 7 separate isometry groups of the sphere, here we give numerical results and plan to publish the pictures in a further paper.

For the series of groups $*nn$ there is 1 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings of the sphere (Fig. 1) and 5 series of (1,2)-transitive tilings containing digonal disks, 1 series of (1,$2n$)-transitive Delone classes of normal tilings (Fig. 2), 2 series of 2-transitive normal tilings (Fig. 3, 4) and 3 series of 2-transitive tilings containing digons, 1 series of (2,$2n$)-transitive normal tilings (Fig. 5), 1 series of $2n$-transitive tilings, each consisting of two disks; altogether there are 14 series of tilings (including 5 series of normal ones).

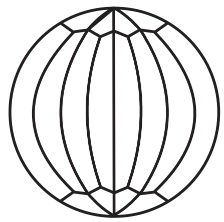
For the series of groups nn there is 1 series of (1,n)-transitive Delone classes of normal tilings (Fig. 6) and 1 series of n-transitive tilings, each consisting of two disks; altogether there are 2 series of tilings.

For the series of groups $*22n$ there are 10 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 7–16), 4 series of (1,2)-transitive non-normal tilings without digonal disks and 19 series of (1,2)-transitive tilings containing digons; 3 series of (1,4)-transitive normal tilings (Fig. 17–19), 2 series of (1,4)-transitive non-normal tilings without digonal disks and 4 series of (1,4)-transitive tilings containing digons; 1 series of (1,$2n$)-transitive normal tilings (Fig. 20), 2 series of (1,$2n$)-transitive non-normal tilings without digonal disks and 3 series of (1,$2n$)-transitive tilings containing digons; 8 series of 2-transitive normal tilings (Fig. 21–28), 4 series of 2-transitive non-normal tilings without digonal disks and 16 series of 2-transitive tilings containing digons; 4 series of (2,4)-transitive normal tilings (Fig. 29–32), 2 series of (2,4)-transitive non-normal tilings without digonal disks and 9 series of

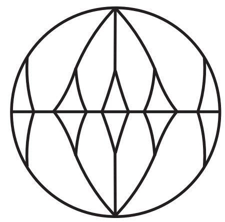

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20



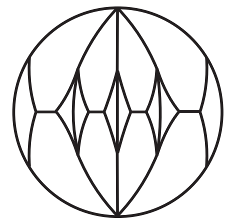
21



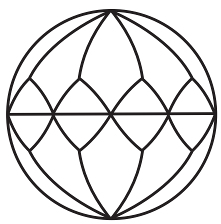
22



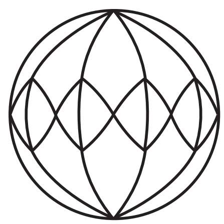
23



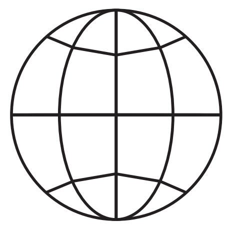
24



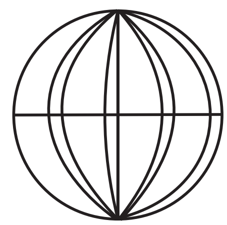
25



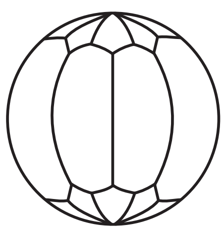
26



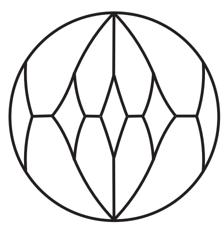
27



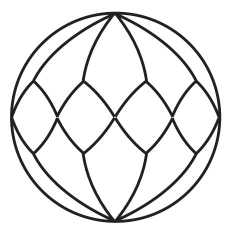
28



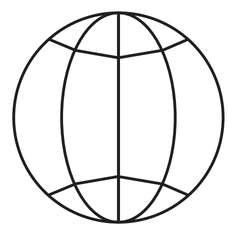
29



30

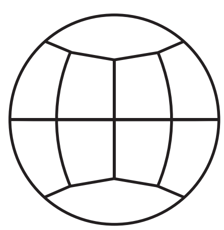


31

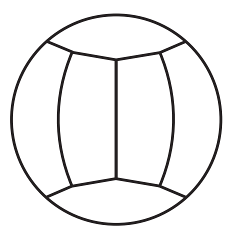


32

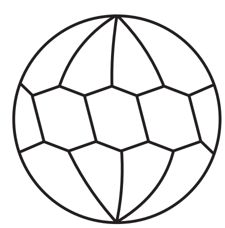
33



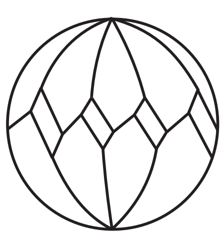
34



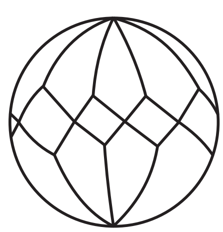
35



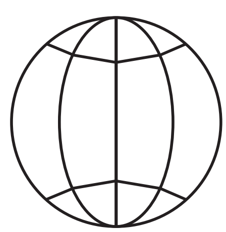
36



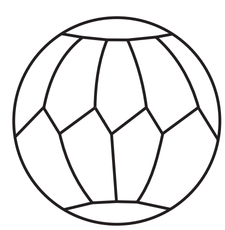
37



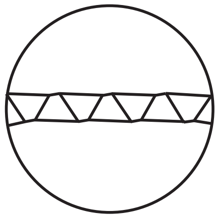
38

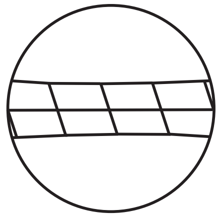


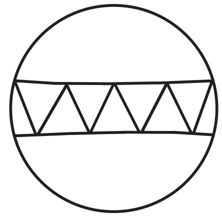
39

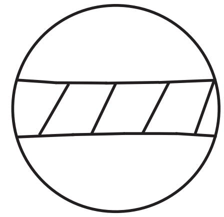


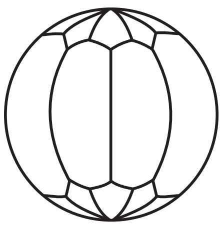
40

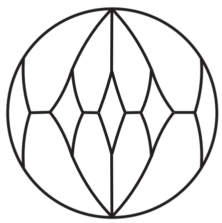

41

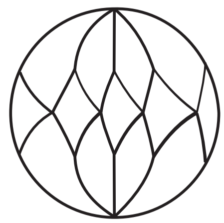

42

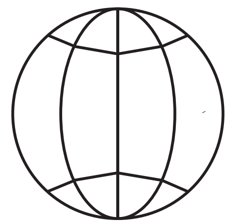

43

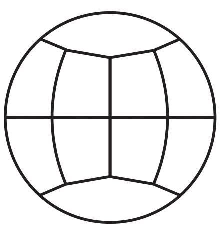

44

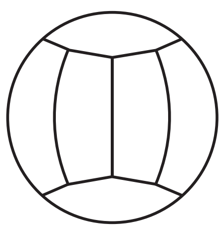

45


46

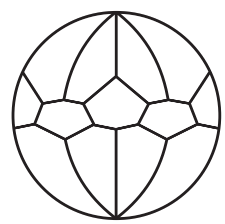

47

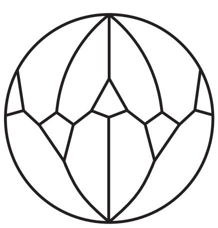

48

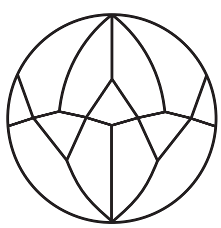

49

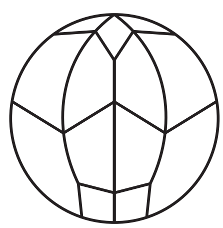

50

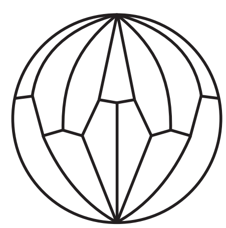

51

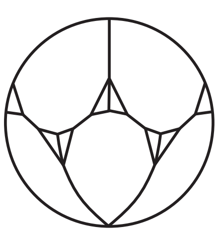

52

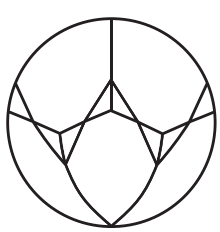

53

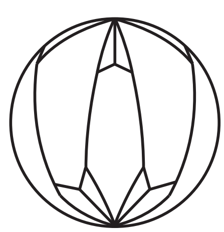

54

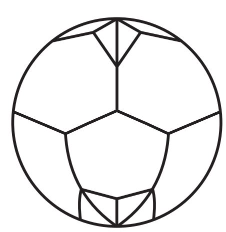

55

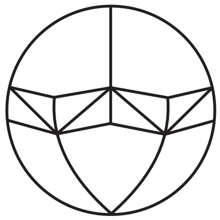

56

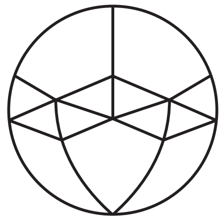

57

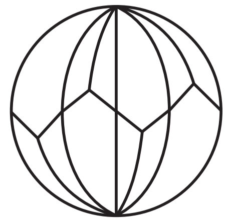

58

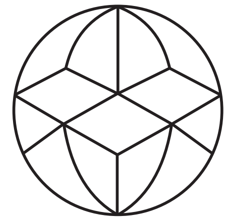

59

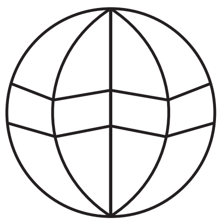

60

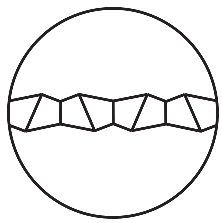

61

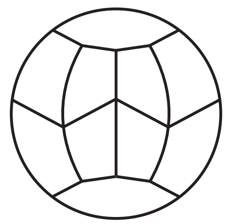

62

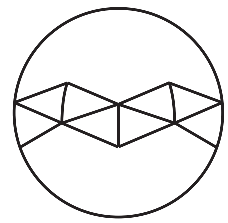

63

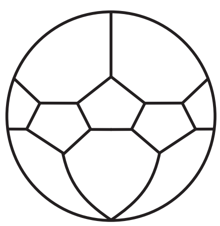

64


65

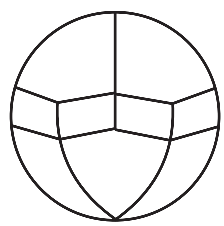

66

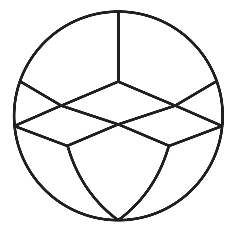

67

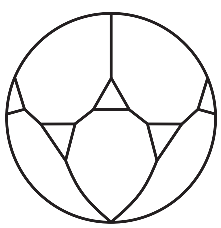

68

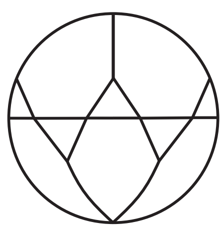

69

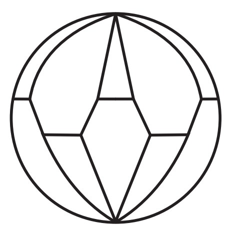

70

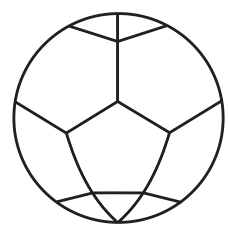

71

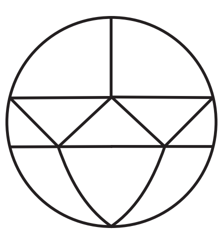

72

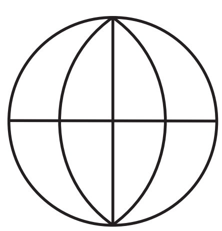

73


74

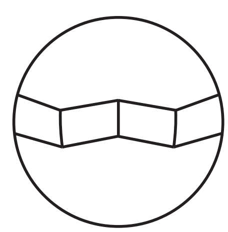

75

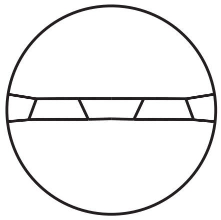

76


77

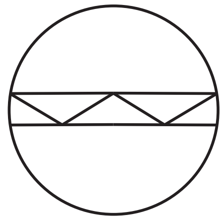

78


79

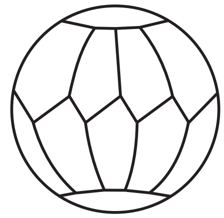

80



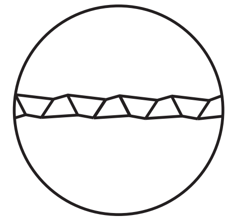
81



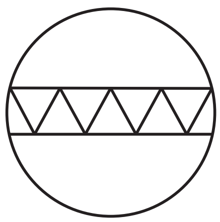
82



83



84



85

(2,4)-transitive tilings containing digons; 2 series of (2,2n)-transitive normal tilings (Fig. 33, 34), 2 series of (2, 2n)-transitive non-normal tilings without digonal disks and 5 series of (2,2n)-transitive tilings containing digons; 2 series of 4-transitive tilings containing digonal disks; 1 series of (4,2n)-transitive normal tilings (Fig. 35) and 2 series of (4,2n)-transitive tilings containing digons; altogether there are 105 series of tilings (including 29 series of normal ones).

For the series of groups 22n there are 4 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 36–39), 2 series of (1,2)-transitive non-normal tilings without digonal disks and 11 series of (1,2)-transitive tilings containing digons; 4 series of (1,n)-transitive normal tilings (Fig. 40–43), 2 series of (1,n)-transitive non-normal tilings without digonal disks and 5 series of (1,n)-transitive tilings containing digons; 2 series of 2-transitive tilings containing digonal disks; 1 series of (2,n)-transitive normal tilings (Fig. 44) and 2 series of (2,n)-transitive tilings containing digons; altogether there are 33 series of tilings (including 9 series of normal ones).

For the series of groups $n*$ there are 4 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 45–48), 2 series of (1,2)-transitive non-normal tilings without digonal disks and 7 series of (1,2)-transitive tilings containing digons; 1 series of (1,n)-transitive normal tilings (Fig. 49) and 2 series of (1,n)-transitive tilings containing digonal disks; 2 series of 2-transitive tilings containing digons; 1 series of (2,n)-transitive normal tilings (Fig. 50) and 2 series of (2,n)-transitive tilings containing digonal disks; altogether there are 21 series of tilings (including 6 series of normal ones).

For the series of groups $2{*}n$ there are 15 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 51–65), 4 series of (1,2)-transitive non-normal tilings without digonal disks and 28 series of (1,2)-transitive tilings containing digons; 3 series of (1,2n)-transitive normal tilings (Fig. 66–68), 4 series of (1,2n)-transitive non-normal tilings without digonal disks and 6 series of (1,2n)-transitive tilings containing digons; 10 series of 2-transitive normal tilings (Fig. 69–78) and 10 series of 2-transitive tilings containing digonal disks; 4 series of (2,$2n$)-transitive normal tilings (Fig. 79–82) and 4 series of (2,$2n$)-transitive tilings containing digons; altogether there are 88 series of tilings (including 32 series of normal ones).

For the series of groups $n\times$ there are 3 series of (1,n)-transitive Delone classes of normal 2-isohedral tilings (Fig. 83–85) and 2 series of (1,n)-transitive tilings containing digonal disks; altogether there are 5 series of tilings (including 3 series of normal ones).

For the group $*332$ there are 12 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 2 (1,2)-transitive non-normal tilings without digonal disks and 19 (1,2)-transitive tilings containing digons; 3 (1,4)-transitive normal tilings and 3 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 12 2-transitive normal tilings, 2 2-transitive non-normal tilings without digonal disks and 14 2-transitive tilings containing digons; 4 (2,4)-transitive normal tilings and 5 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 2 6-transitive normal tilings; altogether there are 105 tilings (including 44 normal ones).

For the group 332 there are 4 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 7 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 1 (2.3)-transitive normal tilings and 2 (2,3)-transitive tilings containing digonal disks; 2 3-transitive normal tilings; altogether there are 33 tilings (including 16 normal ones).

For the group $*432$ there are 23 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 4 (1,2)-transitive non-normal tilings without digonal disks and 36 (1,2)-transitive tilings containing digons; 5 (1,4)-transitive normal tilings and 4 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 3 (1,8)-transitive normal tilings, 2 (1,8)-transitive non-normal tilings without digonal disks and 4 (1,8)-transitive tilings containing digons; 22 2-transitive normal tilings, 4 2-transitive non-normal tilings without digonal disks and 28 2-transitive tilings containing digons; 7 (2,4)-transitive normal tilings and 8 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 7 (2,8)-transitive normal tilings, 2 (2,8)-transitive non-normal tilings without digonal disks and 6 (2,8)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 1 (4,8)-transitive normal tiling and 2 (4,8)-transitive tilings containing digons; 3 (6,8)-transitive normal tilings; altogether there are 198 tilings (including 82 normal ones).

For the group 432 there are 7 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 10 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 9 (1,4)-transitive normal tilings, 2 (1,4)-transitive non-normal tilings without digonal disks and 6 (1,4)-transitive tilings containing digons; 1 (2,3)-transitive normal tiling and 2 (2,3)-transitive tilings containing digonal disks; 1 (2,4)-transitive normal tiling and 2 (2,4)-transitive tilings containing digonal disks; 3 (3,4)-transitive normal tilings; altogether there are 60 tilings (including 30 normal ones).

For the group $3*2$ there are 19 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 2 (1,2)-transitive non-normal tilings without digonal disks and 19 (1,2)-transitive tilings containing digons; 4 (1,3)-transitive normal tilings and 3 (1,3)-transitive tilings containing digonal disks; 7 (1,4)-transitive normal tilings, 2 (1,4)-transitive non-normal tilings without digonal disks and 4 (1,4)-transitive tilings containing digons; 5 2-transitive normal tilings, 2 2-transitive non-normal tilings without digonal disks and 6 2-transitive tilings containing digons; 4 (2,3)-transitive normal tilings and 3 (2,3)-transitive tilings containing digonal disks; 1 (2,4)-transitive normal tiling and 4 (2,4)-transitive tilings containing digons; 3 (3,4)-transitive normal tilings; altogether there are 88 tilings (including 43 normal ones).

For the group $*532$ there are 23 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 4 (1,2)-transitive non-normal tilings without digonal disks and 36 (1,2)-transitive tilings containing digons; 5 (1,4)-transitive normal tilings and 4 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 3 (1,10)-transitive normal tilings, 2 (1,10)-transitive non-normal tilings without digonal disks and 4 (1,10)-transitive tilings containing digons; 22 2-transitive normal tilings, 4 2-transitive non-normal tilings without digonal disks and 28 2-transitive tilings containing digons; 7 (2,4)-transitive normal tilings and 8 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 7 (2,10)-transitive normal tilings, 2 (2,10)-transitive non-normal tilings without digonal disks and 6 (2,10)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 1 (4,10)-transitive normal tiling and 2 (4,10)-transitive tilings containing digons; 3 (6,10)-transitive normal tilings; altogether there are 198 tilings (including 82 normal ones).

For the group 532 there are 7 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 10 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 9 (1,5)-transitive normal tilings, 2 (1,5)-transitive non-normal tilings without digonal disks and 6 (1,5)-transitive tilings containing digons; 1 (2,3)-transitive normal tiling and 2 (2,3)-transitive tilings containing digonal disks; 1 (2,5)-transitive normal tiling and 2 (2,5)-transitive tilings containing digonal disks; 3 (3,5)-transitive normal tilings; altogether there are 60 tilings (including 30 normal ones).

Remark that in the pictures of tilings for simplicity straight-line segments are drawn instead of some arcs.

Acknowledgements. I would like to thank N. Vulpe for help in preparing the figures by computer.

References

[1] Grňbaum B., Shephard G.C. *Spherical tilings with transitivity properties.* The Geometric Vein. The Coxeter Festschrift. Springer–Verlag, New York–Heidelberg–Berlin, 1982, 65–98.

[2] Grňbaum B., Shephard G.C. *The 2-homeotoxal tilings of the plane and 2-sphere.* J. Combin. Theory, Ser. B, 1983, **34**, No. 3, 113–150.

[3] Zamorzaeva E.A. *On sorts of two-dimensional multiregular tilings.* Izv. Akad. Nauk Respub. Moldova. Matematika, 1992, No. 1, 59–66 (in Russian).

[4] Zamorzaeva E.A. *Classification of 2-isohedral tilings of the sphere.* Bul. Acad. Şt. Rep. Moldova. Matematica, 1997, No. 3, 74–85 (in Russian).

[5] Zamorzaeva E. *Enumeration of 2-isohedral tilings on the sphere.* Visual Mathematics (electronic journal), 2004, **6**, No. 1.

[6] Huson D.H. *The generation and classification of tile-k-transitive tilings of the Euclidean plane, the sphere and the hyperbolic plane*, Geom. Dedicata, 1993, **47**, 269–296.

[7] Delgado O., Huson D., Zamorzaeva E. *The classification of 2-isohedral tilings of the plane*, Geom. Dedicata, 1992, **43**, 43–117.

[8] Balke L., Huson D.H. *Two-dimensional groups, orbifolds and tilings*, Geom. Dedicata, 1996, **60**, 89–106.

[9] Delone B.N., Dolbilin N.P., Shtogrin M.I. *Combinatorial and metrical theory of planigons.* Tr. Mat. Inst. Steklov Akad. Nauk SSSR, 1978, **148**, 109–140 (in Russian). English translation: Proc. Steklov Inst. Math., 1980, **4**, 111–141.

State University of Moldova
str. A. Mateevici, MD-2009 Chişinău
Moldova

E-mail: *zamorzaeva@yahoo.com*

Received September 22, 2007