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Fuzzy subquasigroups with respect to a *s*-norm

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Abstract. In this paper the notion of idempotent fuzzy subquasigroups with respect to a s-norm is introduced and some related properties are investigated. Then properties of homomorphic image and inverse image of fuzzy subquasigroups respect to a s-norm are discussed. Next some properties of direct product of fuzzy subquasigroups with respect to a s-norm are presented. Finally abnormalization of fuzzy subquasigroups with respect to a s-norm is studied.

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1 Introduction

During the last decade, there have been many applications of quasigroups in different areas, such as cryptography [12], modern physics [13], coding theory, cryptology [17], geometry [11].

The notion of fuzzy sets was first introduced by Zadeh [19]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics, and also there have been wide-ranging applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning. Rosenfeld [14] introduced the fuzzy sets in the realm of group theory. Since then many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader framework of the fuzzy settings. Triangular norms were introduced by Schweizer and Sklar [15, 16] to model the distances in probabilistic metric spaces. In fuzzy sets theory triangular norm (t-norm) and triangular co-norm (t-conorm or s-norm) are extensively used to model the logical connectives: conjunction (AND) and disjunction (OR) respectively. There are many applications of triangular norms in several fields of Mathematics, and Artificial Intelligence [10]. Dudek [6] introduced the notion of a fuzzy subquasigroup and studied some of its properties. Dudek and Jun [7] introduced the notion of an idempotent fuzzy subquasigroup with respect to a t-norm and discussed some of its properties. In this paper the notion of idempotent fuzzy subquasigroups with respect to a s-norm is introduced, and some related properties are investigated. Relationship between T-fuzzy subquasigroups and S-fuzzy subquasigroups of quasigroups is given. Some

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properties of direct product of fuzzy subquasigroups with respect to a s-norm are also discussed.

2 Preliminaries

In this section we first review some elementary aspects that are necessary for this paper:
A groupoid $(G, \cdot)$ is called a *quasigroup* if for any a , $b \in G$ each of the equations $a \cdot x = b$, $x \cdot a = b$ has a unique solution in G. A quasigroup may be also defined as an algebra $(G, \cdot, \backslash, /)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x \cdot y)/y = x, \quad x \backslash (x \cdot y) = y,$$
$$(x/y) \cdot y = x, \quad x \cdot (x \backslash y) = y.$$

The operations $\backslash$ and $/$ are called *left and right division*. In abstract algebra, a quasigroup is a algebraic structure resembling a group in the sense that "division" is always possible. Quasigroups differ from groups mainly in that they need not be associative. A nonempty subset S of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *subquasigroup* if it is closed with respect to these three operations, that is, if $x * y \in S$ for all x, $y \in S$ and $* \in \{\cdot, \backslash, /\}$.

The class of all equasigroups forms a variety. This means that a homomorphic image of an equasigroup is an equasigroup. Also every subset of an equasigroup closed with respect to these three operations is an equasigroup.

For the general development of the theory of quasigroups the unipotent quasigroups, i.e., quasigroups with the identity $x \cdot x = y \cdot y$, play an important role. These quasigroups are connected with Latin squares which have one fixed element in the diagonal [5]. Such quasigroups may be defined as quasigroups G with the special element θ satisfying the identity $x \cdot x = \theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

To avoid repetitions we use the following convention: *a quasigroup* $\mathcal{G}$ always denotes an equasigroup $(G, \cdot, \backslash, /)$; G always denotes a nonempty set.

A mapping $\mu : G \to [0, 1]$ is called a *fuzzy set* in a quasigroup $\mathcal{G}$. For any fuzzy set μ in G and any $t \in [0, 1]$ we define set

$$L(\mu; t) = \{x \in G \mid \mu(x) \leq t\},$$

which is called *lower t-level cut* of μ.

Definition 1. [6] A fuzzy set μ in G is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geq \min(\mu(x), \mu(y))$$

for all x, $y \in G$ and $* \in \{\cdot, \backslash, /\}$.

Proposition 1. *[6] A fuzzy set μ of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a fuzzy subquasigroup if and only if for every $\alpha \in [0, 1]$, μ_α is empty or a subquasigroup of G.*

Proposition 2. *[6] If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G, \cdot, \backslash, /, \theta)$, then $\mu(\theta) \geq \mu(x)$ for all $x \in G$.*

Definition 2. [15] A s-norm is a mapping $S : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions:

(S1) $S(x, 0) = x$,

(S2) $S(x, y) = S(y, x)$,

(S3) $S(x, S(y, z)) = S(S(x, y), z)$,

(S4) $S(x, y) \leq S(x, z)$ whenever $y \leq z$

for all x, y, $z \in [0, 1]$.

Definition 3. Given a t-norm T and a s-norm S, T and S are *dual (with respect to the negation ′)* if and only if $(T(x, y))' = S(x', y')$.

Proposition 3. *Conjunctive(AND) operator is a t-norm T and disjunctive(OR) operator is its dual s-norm S.*

3 Fuzzy subquasigroups with respect to a s-norm

Definition 4. The set of all idempotents with respect to S, i.e., the set $E_S = \{x \in [0; 1] \mid S(x, x) = x\}$, is a subsemigroup of $([0, 1], S)$. If $Im(\mu) \subseteq E_S$, then a fuzzy set μ is called *an idempotent with respect to a s-norm S* (briefly, a S-idempotent).

Definition 5. Let S be a s-norm. A fuzzy set μ in G is called a *fuzzy subquasigroup of $\mathcal{G}$ with respect to a s-norm S* (briefly, S-fuzzy subquasigroup) if

$$\mu(x * y) \leq S(\mu(x), \mu(y))$$

for all x, $y \in G$ and $* \in \{\cdot, \backslash, /\}$. If a S-fuzzy subquasigroup μ of $\mathcal{G}$ is an idempotent, we say that μ is an idempotent S-fuzzy subquasigroup of $\mathcal{G}$.

Example 1. Let $G = \{0, a, b, c\}$ be a quasigroup with the following Cayley Table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let S_m be a s-norm defined by $S_m(x, y) = \min\{x + y, 1\}$ for all x, $y \in [0, 1]$. Define a fuzzy set μ in $\mathcal{G}$ by

$$\mu(x) = \begin{cases} 1, & \text{if x} \in \{\text{0, a, b}\} \\ 0, & otherwise. \end{cases}$$

It is easy to see that μ satisfies

$$\mu(x * y) \leq S_m(\mu(x), \mu(y))$$

for all x, $y \in G$, and $Im(\mu) \subseteq E_{S_m}$. Hence μ is an idempotent fuzzy subquasigroup of $\mathcal{G}$ with respect to S_m.

The following three propositions are obvious.

Proposition 4. *If a fuzzy set μ is an idempotent with respect to a s-norm S, then $S(x, y) = \max\{x, y\}$ for all x, $y \in Im(\mu)$.*

Proposition 5. *Let a fuzzy set μ on a quasigroup $\mathcal{G}$ be an idempotent with respect to a s-norm S. If each nonempty level set μ_α is a subquasigroup of $\mathcal{G}$, then μ is a S-idempotent fuzzy subquasigroup.*

Proposition 6. *Let μ be a S-fuzzy subquasigroup of $\mathcal{G}$ and $\alpha \in [0, 1]$.*

(a) *If $\alpha = 0$, then $L(\mu; \alpha)$ is either empty or a subquasigroup of $\mathcal{G}$.*

(b) *If $S = \max$, then $L(\mu; \alpha)$ is either empty or a subquasigroup of $\mathcal{G}$.*

Theorem 1. *Let S be a s-norm. If each nonempty level subset $L(\mu; \alpha)$ of μ is a subquasigroup of $\mathcal{G}$, then μ is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Assume that every nonempty level subset $L(\mu; \alpha)$ of μ is a subquasigroup of $\mathcal{G}$. If there exist x, $y \in \mathcal{G}$ such that $\mu(x * y) >$ S $(\mu(x), \mu(y))$, then by taking $t_0 := \frac{1}{2}\{\mu(x * y) + S(\mu(x), \mu(y))\}$, we have $x \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$. Since μ is a subquasigroup of $\mathcal{G}$, $x * y \in L(\mu; t_0)$, $\mu(x * y) \leq t_0$, a contradiction. Hence μ is a S-fuzzy subquasigroup of $\mathcal{G}$. □

Definition 6. Let $\mathcal{G}$ be a quasigroup and a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a quasigroup $\mathcal{G}$. Then the union $\bigvee_{i \in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by

$$(\bigvee_{i \in I} \mu_i)(x) = \sup\{\mu_i(x) \mid i \in I\},$$

for each $x \in \mathcal{G}$.

Theorem 2. *If $\{\mu_i \mid i \in I\}$ is a family of fuzzy subquasigroups of a quasigroup $\mathcal{G}$ with respect to S, then $\bigvee_{i \in I} \mu(x_i)$ is a fuzzy subquasigroup of $\mathcal{G}$ with respect to S.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy subquasigroups of $\mathcal{G}$ with respect to S. For x, $y \in \mathcal{G}$, we have

$$\begin{aligned}
(\bigvee_{i \in I} \mu_i)(x * y) &= \sup\{\mu_i(x * y) \mid i \in I\} \\
&\leq \sup\{S(\mu_i(x), \mu_i(y)) \mid i \in I\} \\
&= S(\sup\{\mu_i(x) \mid i \in I\}, \sup\{\mu_i(y) \mid i \in I\}) \\
&= S(\bigvee_{i \in I} \mu_i(x), \bigvee_{i \in I} \mu_i(y)).
\end{aligned}$$

Hence $\bigvee_{i \in I}$ is a fuzzy subquasigroup of $\mathcal{G}$ with respect to S. □

Definition 7. Let f be a mapping on $\mathcal{G}$. If v is a fuzzy set in $f(\mathcal{G})$, then the fuzzy set $\mu = v \circ f$ (i.e., $(v \circ f)(x) = v(f(x))$) in $\mathcal{G}$ is called the *preimage* of v under f.

Theorem 3. *An onto homomorphism preimage of a S-fuzzy subquasigroup of $\mathcal{G}$ is a S-fuzzy subquasigroup.*

Proof. Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an onto homomorphism of quasigroups. If v is a S-fuzzy subquasigroup of $\mathcal{G}_2$ and μ is the preimage of v under f, then for any x, $y \in \mathcal{G}_1$, we have

$$\begin{aligned} \mu(x * y) &= (v \circ f)(x * y) = v(f(x * y)) \\ &\leq S(v(f(x)), v(f(y))) \\ &= S((v \circ f)(x), (v \circ f)(y)) \\ &= S(\mu(x), \mu(y)). \end{aligned}$$

This shows that μ is a fuzzy subquasigroup of $\mathcal{G}_1$ with respect to a s-norm S. □

Definition 8. Let μ be a fuzzy set in a quasigroup $\mathcal{G}$ and let f be a mapping defined on $\mathcal{G}$. Then the fuzzy set μ^f in $f(\mathcal{G})$ defined by

$$\mu^f(y) = \inf_{x \in f^{-1}(y)} \mu(x) \quad \forall y \in f(\mathcal{G})$$

is called the *image* of μ under f. A fuzzy set μ in $\mathcal{G}$ has the inf *property* if for any subset $A \subseteq \mathcal{G}$, there exists $a_0 \in A$ such that $\mu(a_0) = \inf_{a \in A} \mu(a)$.

Theorem 4. *An onto homomorphism image of a fuzzy subquasigroup with the* inf *property is a fuzzy subquasigroup.*

Proof. Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an epimorphism of $\mathcal{G}_1$ and μ a fuzzy subquasigroup of $\mathcal{G}_1$ with the inf property. Consider $f(x)$, $f(y) \in \mathrm{f}(\mathcal{G}_1)$. Now, let x_0, $\mathrm{y}_0 \in \mathrm{f}^{-1}(f(x))$ be such that

$$\mu(x_0) = \inf_{t \in f^{-1}(f(x))} \mu(t)$$

and

$$\mu(y_0) = \inf_{t \in f^{-1}(f(y))} \mu(t)$$

respectively. Then we can deduce that

$$\begin{aligned} \mu^f(f(x) * f(y)) &= \inf_{t \in f^{-1}(f(x) * f(y))} \mu(t) \\ &\leq \max\{\mu(x_0), \mu(y_0)\} \\ &= \max\{\inf_{t \in f^{-1}(f(x))} \mu(t), \inf_{t \in f^{-1}(f(y))} \mu(t)\} \\ &= \max\{\mu^f(f(x)), \mu^f(f(y))\}. \end{aligned}$$

Consequently, μ^f is a fuzzy subquasigroup of $\mathcal{G}_2$. □

Definition 9. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasigroups and let f be a function from $\mathcal{G}_1$ into $\mathcal{G}_2$. If ν is a fuzzy set in $\mathcal{G}_2$, then the *preimage* of ν under f is the fuzzy set in $\mathcal{G}_1$ defined by

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \forall \;\; x \in \mathcal{G}_1.$$

Theorem 5. *Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an epimorphism of quasigroups. If ν is a S-fuzzy subquasigroup in $\mathcal{G}_2$, then $f^{-1}(\nu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_1$.*

Proof. Let x, $y \in \mathcal{G}_1$, then

$$\begin{aligned} f^{-1}(\nu)(x*y) &= \nu(f(x*y)) \\ &\leq S(\nu(f(x), f(y)) \\ &= S(\nu(f(x)), \nu(f(y))) \\ &= S(f^{-1}(\nu)(x), f^{-1}(\nu)(y)). \end{aligned}$$

Hence $f^{-1}(\nu)$ is a S-fuzzy quasigroup in $\mathcal{G}_1$. □

Definition 10. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be quasigroups and f a function from $\mathcal{G}_1$ into $\mathcal{G}_2$. If ν is a fuzzy set in $\mathcal{G}_2$, then the *image* of μ under f is the fuzzy set in $\mathcal{G}_1$ defined by $f(\mu)(x) = \begin{cases} \inf_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset \;, \\ 0, & \text{otherwise,} \end{cases}$
for each $y \in \mathcal{G}_2$.

Theorem 6. *Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an onto homomorphism of quasigroups. If μ is a S-fuzzy subquasigroup in $\mathcal{G}_1$, then $f(\mu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_2$.*

Proof. Let y_1, $y_2 \in \mathcal{G}_2$, then

$$\{x|\; x \in f^{-1}(y_1 * y_2)\} \subseteq \{x_1 * x_2) | x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\},$$

and hence

$$\begin{aligned} f(\mu)(y_1 * y_2) &= \inf\{\mu(x) \mid f^{-1}(y_1 * y_2)\} \\ &\leq \inf\{S(\mu(x_1), \mu(x_2)) \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= S(\inf\{\mu(x_1) \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &, \;\; \inf\{\mu(x_2) \mid x_2 \in f^{-1}(y_2)\}) \\ &= S(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

Hence $f(\mu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_2$. □

Definition 11. A s-norm S on [0,1] is called a continuous s-norm if S is a continuous function which maps [0,1]×[0,1] to [0, 1] with respect to the usual topology. Obviously, the function "max" is a continuous s-norm.

Theorem 7. *Let S be a continuous s-norm and f be a homomorphism on $\mathcal{G}$. If μ is a S-fuzzy subquasigroup of $\mathcal{G}$, then μ^f is a S-fuzzy subquasigroup of $f(\mathcal{G})$.*

Proof. The proof is obtained dually by using the notion of s-norm S instead of t-norm T in [7]. □

Lemma 1. *Let T be a t-norm. Then s-norm S can be defined as*

$$S(x, y) = 1 - T(1 - x, 1 - y).$$

Proof. Straightforward. □

Theorem 8. *A fuzzy set μ of a quasigroup $\mathcal{G}$ is a T-fuzzy subquasigroup of $\mathcal{G}$ if and only if its complement μ^c is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let μ be a T-fuzzy subquasigroup of $\mathcal{G}$. For x, $y \in \mathcal{G}$, we have

$$\begin{aligned} \mu^c(x * y) &= 1 - \mu(x * y) \\ &\leq 1 - T(\mu(x), \mu(y)) \\ &= 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) \\ &= S(\mu^c(x), \mu^c(y)). \end{aligned}$$

Hence μ^c is a S-fuzzy subquasigroup of $\mathcal{G}$. The converse is similar. □

Theorem 9. *Let S be a s-norm. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be quasigroups and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ be the direct product quasigroup of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let λ be a fuzzy subquasigroup of a quasigroup $\mathcal{G}_1$ with a s-norm S and μ a fuzzy quasigroup of a quasigroup $\mathcal{G}_2$ also with the s-norm S. Then $\nu = \lambda \times \mu$ is a fuzzy quasigroup of $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ with the s-norm S which is defined by*

$$\nu(x_1, x_2) = (\lambda \times \mu)(x_1, x_2) = S(\lambda(x_1), \mu(x_2)).$$

Moreover, if λ and μ are S-idempotent, then $\lambda \times \mu$ is also S-idempotent.

Proof. The proof is obtained dually by using the notion of s-norm S instead of t-norm T in [7]. □

Theorem 10. *Let λ and μ be fuzzy sets in a unipotent quasigroup $\mathcal{G}$ such that $\lambda \times \mu$ is a S-fuzzy subquasigroup of $\mathcal{G} \times \mathcal{G}$, then*

(i) *either $\lambda(\theta) \leq \lambda(x)$ or $\mu(\theta) \leq \mu(x)$ $\forall x \in \mathcal{G}$.*

(ii) *If $\lambda(\theta) \leq \lambda(x)$ $\forall x \in \mathcal{G}$, then either $\mu(\theta) \leq \lambda(x)$ or $\mu(\theta) \leq \mu(x)$.*

(iii) *If $\mu(\theta) \leq \mu(x)$, $\forall x \in \mathcal{G}$, then either $\lambda(\theta) \leq \lambda(x)$ or $\lambda(\theta) \leq \mu(x)$.*

Proof. (i) We prove it using reductio ad absurdum.

Assume $\lambda(x) < \lambda(\theta)$ and $\mu(y) < \mu(\theta)$ for some x, y $\in \mathcal{G}$. Then $(\lambda \times \mu)(x, y) = \mathrm{S}(\lambda(x), \mu(y)) < \mathrm{S}(\lambda(\theta), \mu(\theta)) = (\lambda \times \mu)(\theta, \theta)$.

This implies $(\lambda \times \mu)(x, y) < (\lambda \times \mu)(\theta, \theta)$ $\forall$ x, y $\in \mathcal{G}$.

which is a contradiction. Hence (i) is proved.

(ii) Again, we use reduction to absurdity.

Assume $\mu(\theta) > \lambda(x)$ and $\mu(\theta) > \mu(y)\ \forall\ x,\ y \in \mathcal{G}$. Then,
$(\lambda \times \mu)(\theta, \theta)$= S $(\lambda(\theta), \mu(\theta)) = \mu(\theta)$

and $(\lambda \times \mu)(x,\ y) =$ S $(\lambda(x),\ \mu(y)) < \mu(\theta) = (\lambda \times \mu)(\theta, \theta)$

$\Rightarrow (\lambda \times \mu)(x,\ y) < (\lambda \times \mu)(\theta, \theta)$,

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

□

Theorem 11. *Let μ and ν be fuzzy sets in a unipotent quasigroup $\mathcal{G}$ such that $\mu \times \nu$ is a S-fuzzy subquasigroup of $\mathcal{G} \times \mathcal{G}$. Then*

(a) *If $\nu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$, then ν is a S-fuzzy subquasigroup of $\mathcal{G}$.*

(b) *If $\mu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$ and $\nu(y) < \mu(\theta)$ for some $y \in \mathcal{G}$, then μ is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof.

(a) If $\nu(x) \geq \mu(\theta)$ for any $x \in \mathcal{G}$, then

$$\begin{aligned}
\nu(x * z) &= S(\mu(\theta), \nu(x * y)) \\
&= (\mu \times \nu)(\theta, x * y) \\
&\leq S((\mu \times \nu)(\theta, x), (\mu \times \nu)(\theta, y)) \\
&= S(S(\mu(\theta), \nu(x)), S(\mu(\theta), \nu(y))) \\
&= S(\nu(x), \nu(y)).
\end{aligned}$$

Hence ν is a S-fuzzy subquasigroup of $\mathcal{G}$.

(b) Assume that $\mu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$ and $\nu(y) < \mu(\theta)$ for $y \in \mathcal{G}$. Then $\nu(\theta) \leq \nu(y) < \mu(\theta)$. since $\mu(\theta) \leq \mu(x)$ for all $x \in \mathcal{G}$, it follows that $\nu(\theta) < \mu(x)$ for any $x \in \mathcal{G}$. Thus

$$(\mu \times \nu)(x, \theta) = S(\mu(x), \nu(\theta)) = \mu(x)\ for\ all\ x \in \mathcal{G}.$$

Thus

$$\begin{aligned}
\mu(x * y) &= (\mu \times \nu)(x * y, \theta) \\
&\leq S((\mu \times \nu)(x), \theta), (\mu \times \nu)(y, \theta)) \\
&= S(S(\mu(x), \nu(\theta)), S(\mu(y), \nu(\theta))) \\
&= S(\mu(x), \mu(y)).
\end{aligned}$$

Hence μ is a S-fuzzy subquasigroup of $\mathcal{G}$. □

Definition 12. Let S be a s-norm. If ν is a fuzzy set in a set A , the *weakest S-fuzzy relation* on A that is S-fuzzy relation on ν is μ_ν given by $\mu_\nu(x, y)= S(\nu(x), \nu(y))$ for all x, $y \in A$.

Theorem 12. *Let ν be a fuzzy set in a quasigroup $\mathcal{G}$ and let μ_ν be the weakest S-fuzzy relation on $\mathcal{G}$. Then ν is a S-fuzzy subquasigroup of $\mathcal{G}$ if and only if μ_ν is a S-fuzzy subquasigroup of $\mathcal{G} \times \mathcal{G}$.*

Proof. Suppose that ν is a fuzzy subquasigroup of $\mathcal{G}$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ $\in \mathcal{G} \times \mathcal{G}$. Then

$$\begin{aligned}
\mu_\nu(x * y) &= \mu_\nu((x_1, x_2) * (y_1, y_2)) = \mu_\nu((x_1 * y_1, x_2 * y_2)) \\
&= S(\nu(x_1 * y_1), \nu(x_2 * y_2)) \\
&\leq S(S(\nu(x_1), \nu(y_1)), S(\nu(x_2), \nu(y_2))) \\
&= S(S(\nu(x_1), \nu(x_2), S(\nu(y_1), \nu(y_2))) \\
&= S(\mu_n(x_1), x_2), \mu_\nu(y_1, y_2)) \\
&= S(\mu_\nu((x_1, x_2)), \mu_\nu((y_1, y_2))) \\
&= S(\mu_\nu(x), \mu_\nu(y)).
\end{aligned}$$

Thus μ_ν is a fuzzy subquasigroup of $\mathcal{G} \times \mathcal{G}$.
The converse is proved similarly. □

Definition 13. A S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ is said to be *abnormal* if there exist $x \in G$ such that $\mu(x) = 0$. Note that if S-fuzzy subquasigroup μ of $\mathcal{G}$ is abnormal, then $\mu(\theta) = 0$, and hence μ is an abnormal if and only if $\mu(\theta) = 0$.

Theorem 13. *Let μ be a S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ and μ^+ be a fuzzy set in G defined by $\mu^+(x) = \mu(x) - \mu(\theta)$ for all $x \in G$. Then μ^+ is an abnormal S-fuzzysubquasigroup of $\mathcal{G}$ containing μ.*

Proof. We have $\mu^+(x) = \mu(\theta) - \mu(\theta) = 0 \leq \mu^+(x)$ for all $x \in G$. For any x, $y \in G$, we have

$$\begin{aligned}
\mu^+(x * y) &= \mu(x * y) - \mu(\theta) \\
&\leq S(\mu(x), \mu(y)) - \mu(\theta) \\
&= S(\mu(x) - \mu(\theta), \mu(y) - \mu(\theta)) \\
&= S(\mu^+(x), \mu^+(y)).
\end{aligned}$$

This shows that μ^+ is a S-fuzzy subquasigroup of a unipotent quasigroup. Clearly, $\mu \subset \mu^+$. This ends the proof. □

Corollary 1. *If μ is a S-fuzzy subquasigroup of a unipotent quasigroup satisfying $\mu^+(x) = 1$ for some $x \in G$, then $\mu(x) = 1$.*

Theorem 14. *Let μ and ν be S-fuzzy subquasigroups of a unipotent quasigroup $\mathcal{G}$. If $\nu \subset \mu$ and $\mu(\theta) = \nu(\theta)$, then $\mathcal{G}_\mu \subset \mathcal{G}_\nu$.*

Proof. Assume that $\nu \subset \mu$ and $\mu(\theta) = \nu(\theta)$. If $x \in \mathcal{G}_\mu$, then $\nu(x) \leq \mu(x) = \mu(\theta) = \nu(\theta)$. Noticing that $\nu(\theta) \leq \nu(x)$ for all $x \in G$, we have $\nu(x) = \nu(\theta)$, i.e., $x \in \mathcal{G}_\nu$. The proof is complete. □

Corollary 2. *If μ and ν are abnormal S-fuzzy subquasigroups of a unipotent quasigroup $\mathcal{G}$ satisfying $\nu \subset \mu$, then $\mathcal{G}_\mu \subset \mathcal{G}_\nu$.*

Theorem 15. *A S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ is abnormal if and only if $\mu^+ = \mu$.*

Proof. The sufficiency is obvious. Assume that μ is an abnormal S-fuzzy subquasigroup of a quasigroup $\mathcal{G}$ and $x \in G$. Then $\mu^+(x) = \mu(x) - \mu(\theta) = \mu(x)$, hence $\mu^+ = \mu$. □

Theorem 16. *If μ is a S-fuzzy subquasigroup of a unipotent quasigroup, then $(\mu^+)^+ = \mu^+$.*

Proof. For any $x \in G$, we have $(\mu^+)^+(x) = \mu^+(x) - \mu^+(\theta)$, completing the proof. □

Corollary 3. *If μ is an abnormal S-fuzzy subquasigroup of a unipotent quasigroup, then $(\mu^+)^+ = \mu$.*

Theorem 17. *Let μ be a non-constant abnormal S-fuzzy subquasigroup of a unipotent quasigroup, which is minimal in the poset of abnormal S-fuzzy subquasigroup under set inclusion. Then clearly μ takes only two values 0 and 1.*

Proof. Note that $\mu(\theta) = 0$. Let $x \in G$ be such that $\mu(x) \neq 0$. It is sufficient to show that $\mu(x) = 1$. Assume that then there exists $a \in G$ such that $0 < \mu(a) < 1$. Define on G a fuzzy set ν by putting $\nu(x) = \frac{1}{2}(\mu(x) + \mu(a))$ for each $x \in G$. Then clearly ν is well-defined and for all $x, y \in G$, we have

$$\begin{aligned}
\nu(x * y) &= \frac{1}{2}\mu(x * y) + \frac{1}{2}\mu(a) \\
&\leq \frac{1}{2}(S(\mu(x), \mu(y) + \mu(a))) \\
&= S(\frac{1}{2}(\mu(x) + \mu(a)), \frac{1}{2}(\mu(y) + \mu(a))) \\
&= S(\nu(x), \nu(y)).
\end{aligned}$$

Hence ν^+ is an abnormal S-fuzzy subquasigroup of a unipotent quasigroup. Noticing that $\nu^+(\theta) = 0 < \nu^+(a) = \frac{1}{2}\mu(a) < \mu(a)$, we know that ν^+ is non-constant. From $\nu^+(a) < \mu(a)$, it follows that μ is not minimal. This proves the theorem. □

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